Mathematical modelling, finance, and the recession

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1 Introduction

1.1 Aim of the work

- Trying to set up a mathematical model of the crisis-stricken economy (including the influence of the financial sector)
- Looking at this as an applied mathematician
- Policy implications, if any

1.2 Why not leave it to the economists?

- 1. Bad forecasts
- 2. Ludicrous mathematical models

1.2.1 2007 Treasury forecasts of GDP growth

- 2008 growth: forecast between 2 and 2.5%, actual 0%
- 2009 growth: forecast between 2.5 and 3%, actual - 2%

Admittedly these were only forecasts, not firm predictions; nevertheless the fact that the actual result was so far outside the putative error bounds of the forecast suggests that there was something wrong with the forecasting method that was being used.

1.2.2 Bank of England Quarterly Model : core equations

```
                  c   =   mpc*wealth/pc;
                mpc   =   1/(1+duser*xid/pc+1/psimon*ximonˆ(-sigmac)*
                          *(1-gamma/(1+rg))ˆ(1-sigmac)+1/mpc(+1)*gamma*
                          *(beta*ximon(+1)*(1+ydot(+1))ˆ(psihab*
                          *(1-sigmac)/sigmac)/ximon)ˆsigmac*((1+rg)*
                          *pc/pc(+1)/(1+pdot(+1)))ˆ(sigmac-1));
             wealth   =   (1+rg(-1))*bg(-1)*pg(-1)/((1+ydot)*(1+pdot))+
                          +(1+rk(-1))*bk(-1)/((1+ydot)*(1+pdot))+(1+rf(-1))*
                          *bf(-1)*pc/((1+ydot)*(1+pdotf)*q)+v+dv+pc(-1)*
                          *mon(-1)/((1+ydot)*(1+pdot))+hw+trw+dw+xigain;
bf*pc/q+bg*pg+v+bk   =   pc(-1)*mon(-1)/((1+ydot)*(1+pdot))-
                          -pc*mon+(1+rf(-1))*bf(-1)*pc/((1+ydot)*(1+pdotf)*q)+
                          +(1+rg(-1))*bg(-1)*pg(-1)/((1+ydot)*(1+pdot))+v+dv+
                          +(1+rk(-1))*bk(-1)/((1+ydot)*(1+pdot))+wl*l-pc*c+
                          +transc+transkc+transkp+transfp+rfprem+rgprem-
                          taxlumpc+transec-pdv*(d-(1-deltad)*d(-1)/(1+ydot))-
                          -taud*pdv*d;
```

and roughly **160 more equations** in similar vein ...

1.3 Some mathematical modelling principles (or aspirations)

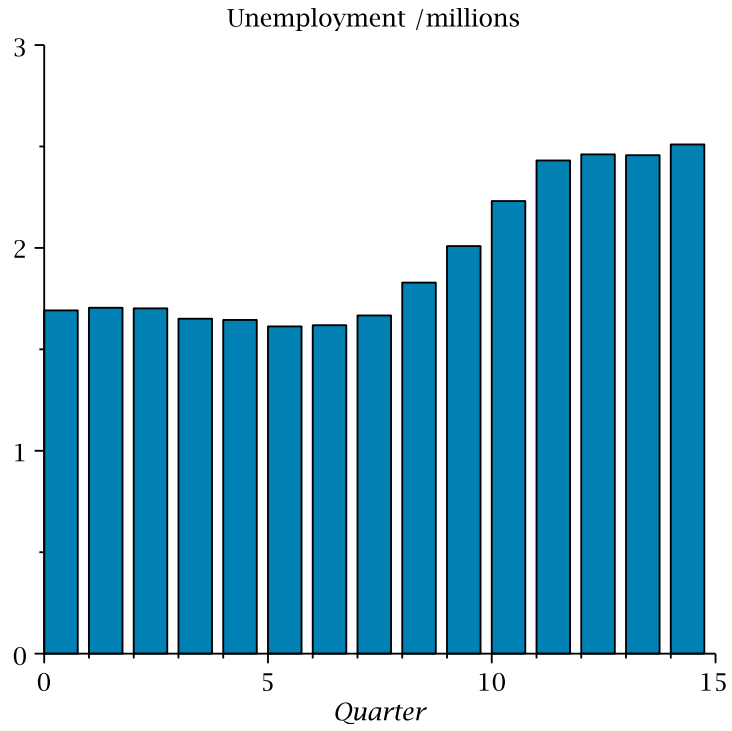
- Identify and define a **small** number of variables which describe the salient features of the system under study
- Look for **simple** relations connecting them which
  - (i) are backed up by some theoretical rationale
  - and (ii) can be tested empirically
  - and (iii) lead towards a coherent mathematical structure

Difficulties:

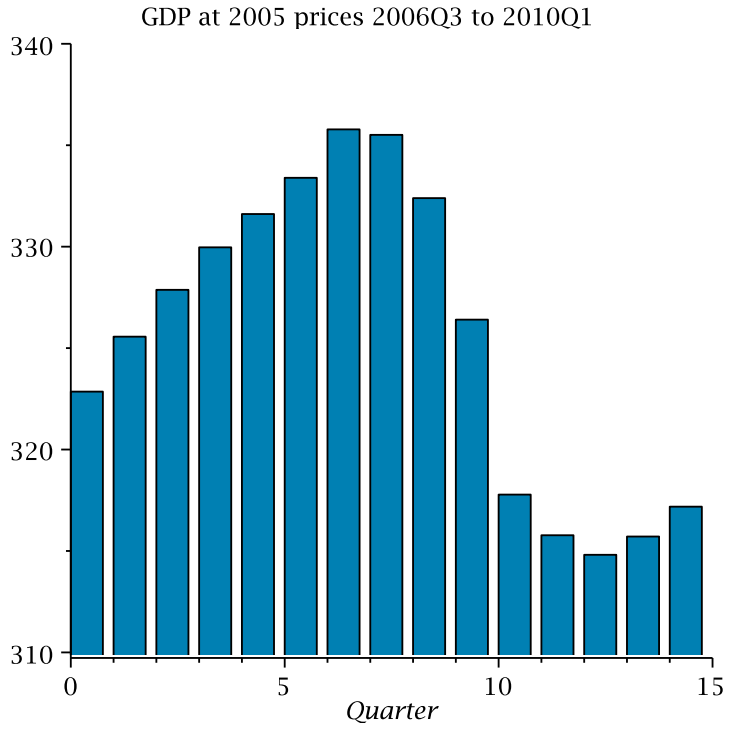
- The simple relations are inaccurate and may lose their validity when conditions change
- People are not molecules: they compete and are unpredictable

1.4 Some salient features of today's economic situation

1.4.1 UK unemployment 2006Q3 to 2010Q1



1.4.2 UK GDP (at 2005 prices) mid-2006 to end-2009



1.5 Some ingredients for a model

- Gross Domestic Product
  - GDP as expenditure
  - GDP as income
  - Budget deficit
- Money, finance
  - The demand for money
  - The supply of money
- International trade
  - The exchange rate
  - The balance of payments

2 Gross Domestic Product

2.1 GDP as expenditure

2.1.1 The expenditure components of GDP

The GDP Y can be defined as the rate at which money is being spent for providing goods and services produced in the UK

The UK Office of National Statistics breaks down this total into various sub-totals:

- Y := C + I + G + X - M
- C := spending by consumers (households and non-profit institutions) on goods and services
- I := private sector new investment (capital formation plus changes in inventories)
- G := government spending on goods and services (excludes interest on Gov't bonds, pensions, unemployment benefit)
- X := exports
- M := imports

The three terms $C+I+G$ are the rate at which UK inhabitants are spending money on goods and services. Subtract from that M which is the part of that money which is spent on goods and services produced outside the UK, and add X which is the money spent by foreigners on goods and services produced in the UK, and the result is the GDP as defined.

2.1.2 GDP as expenditure: UK data /£10^9 per annum

year	2007	2008	2009
Consumption by households and NPIs	893	928	911
Investment: capital formation, inventory changes	255	244	194
Gov't spending on goods and services	295	314	330
e**X**ports	372	422	387
i**M**ports	416	460	421
$\mathbf{Y} = C + I + G + X - M$	1399	1448	1401

2.2 GDP evaluated as income

2.2.1 The income components of GDP

Alternatively, the GDP Y can be defined as the rate at which money is being *received* in exchange for providing goods and services

- $Y = W + P + T$
- W := wages and salaries (less taxes)
- P := profits and rents
- T := taxes less subsidies

2.2.2 GDP as income: UK data / £10^9 per annum

year	2007	2008	2009
Wages and salaries of employees	746	772	764
Profits of corporations and businesses	484	510	480
Taxes less subsidies	168	166	152
$\mathbf{Y} = W + P + T$	1399	1448	1396

2.3 The accounting identity

The expenditure and income GDPs are equal:

$$Y = C + I + G + X - M = W + P + T$$

This formula can be rearranged in various ways, e.g.:

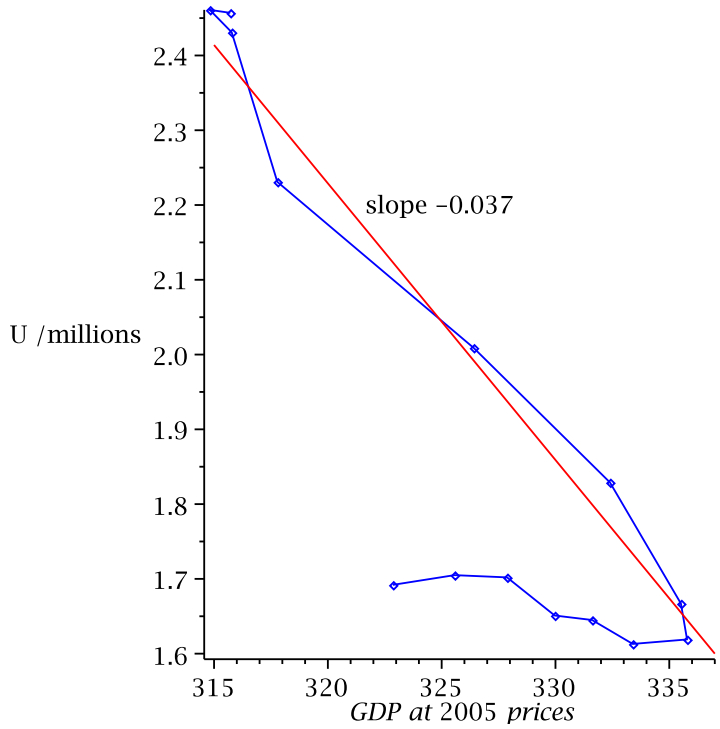
$$(G - T) + I = (W + P - C) + (M - X)$$

i.e. deficit + investment = saving + net imports

People sometimes make statements about policy which can be expressed in terms of the components of this rearrangement, e.g. that if people could be encouraged to save more, or if the deficit were reduced, then investment would rise. However the terms in the formula are not independent: there are relations between them, and so it is not immediately clear what the effect of a change in policy will be. In the next subsection we look at some of these relations.

2.4 Some empirical relations

2.4.1 GDP and unemployment: the phase transition



The graph begins in 2006Q3 (third quarter of 2006) at the bottom left-hand point. It moves more or less horizonatally to the right (i.e. GDP increases and unemployment stays constant) to the seventh point, which corresponds to 2008Q1. But then there is a "phase change" and from then on the graph approximately follows the red line: unemployment rises while GDP falls, roughly in proportion to one another.

The two branches of the graph correspond to two distinct "phases", which can be understood using the equation

$$Y = wN$$

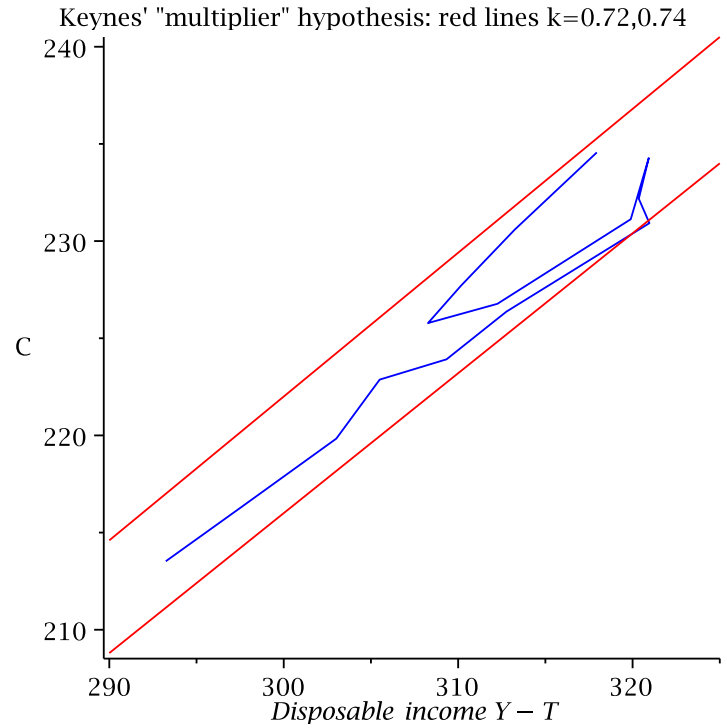
where $w :=$ average wage, $N :=$ number employed, if we assume (i) that there is an upper bound N_{max} on the level of employment and (ii) that wages do not decrease.

- Full employment phase: GDP increases and employment N stays constant at its upper bound N_{max}. Average wage w rises slowly.
  If GDP increases too fast, the result is an inflationary increase in wages and prices

- Partial employment phase: employment is below the upper bound and varies in proportion to GDP. Wages and prices do not increase
  This phase begins at the time when GDP turns from increasing to decreasing. As Y decreases from its maximum, w does not decrease, so N must decrease and goes below its upper bound.

Price level p assumed proportional (in the short term) to w. The ratio w/p is a measure of productivity.

2.4.2 "Keynes' hypothesis": that C depends mainly on $Y - T$



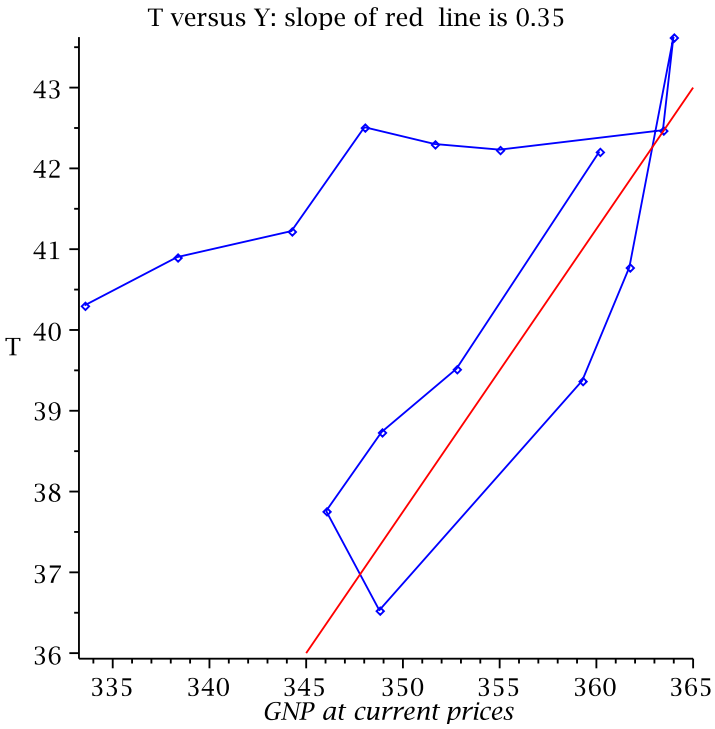
The bottom left-hand point on the blue graph corresponds to 2008Q3 and the rest of it to later times. The data can be described by the formula

$$C \approx 0.73(Y - T)$$

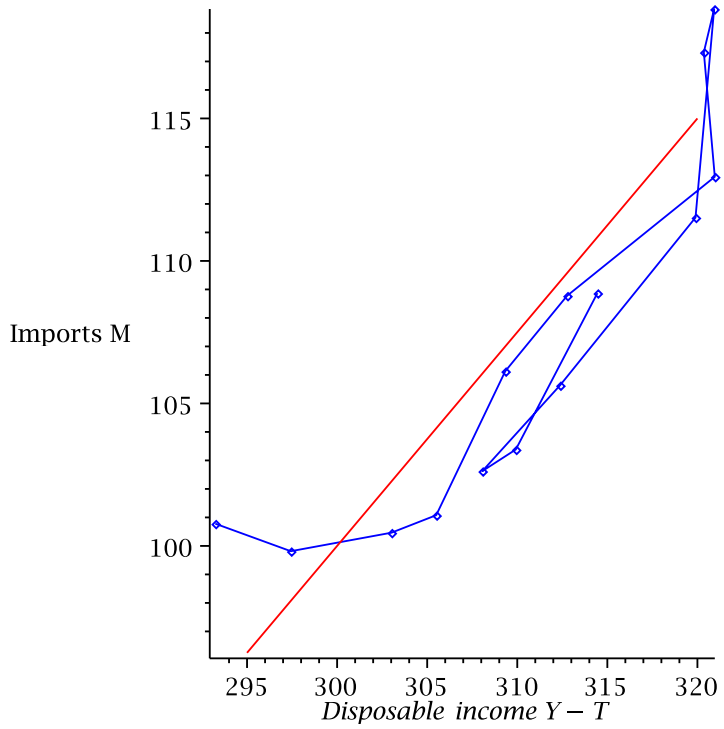
or, more precisely, by

$$C = k(Y - T) \quad \text{with} \quad |k - 0.73| < 0.1$$

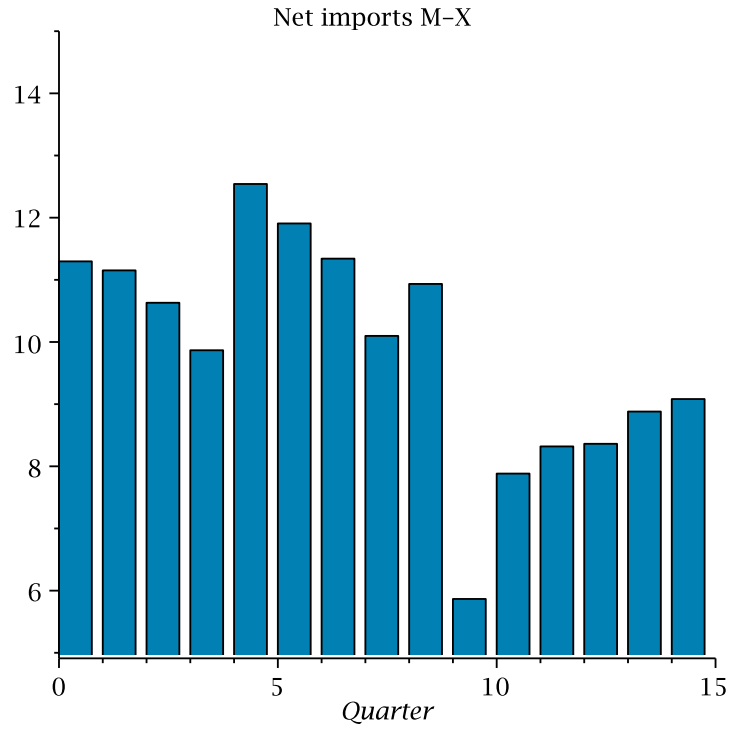
2.4.3 How tax revenue varies with GDP



2.4.4 How imports depend on disposable income



2.4.5 Net imports



2.5 Ways to influence the GDP

- $Y = C + I + G + X - M$
- increase C : persuade householders to spend more
  (reduce taxes : 'fiscal stimulus')
- increase I : persuade firms to invest more
  (reduce interest rate)
- increase G : spend more on goods and services
- increase $X - M$: export more and/or import less

2.5.1 Using Keynes' hypothesis: the 'fiscal simulus'

The data on C vs $Y - T$ can be summarized by the formula

$$C \approx k(Y - T)$$

where k is about 3/4. i.e. people spend about 3/4 of their income and save about 1/4. Using this in the expenditure equation,

$$Y \approx 0.75(Y - T) + I + G + X - M$$

i.e.

$$0.25Y \approx -0.75T + I + G + X - M$$

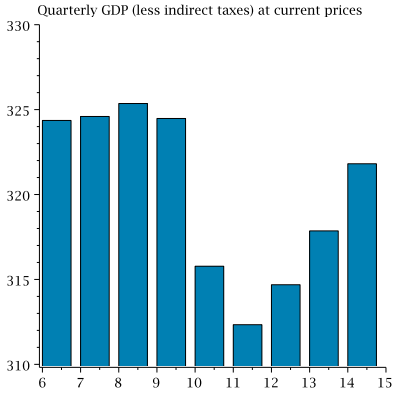
Whence, if I, E and M don't change,

$$\partial Y/\partial T = -3 \quad \text{and} \quad \partial Y/\partial G = 4$$

So a tax cut should increase Y by 3 times as much;
an increase of G should increase Y by 4 times as much.

2.5.2 Efficacy of a tax cut in increasing Y: observations

quarter	Y/4	Y'/4
2008 Q1	363.4	324.4
2008 Q2	364.0	324.6
2008 Q3	361.7	325.4
2008 Q4	359.3	324.5
2009 Q1	348.8	315.8
2009 Q2	346.0	312.3
2009 Q3	348.9	314.7
2009 Q4	352.8	317.9
2010 Q1	360.2	321.8



The ordinate $Y'/$ in the column graph is the (quarterly) "GDP at basic prices", i.e. with indirect taxes such as VAT deducted, so as to make the columns for 2009, when VAT was temporarily reduced to 15 per cent, comparable with the others.

During 2009, VAT was cut by 12 £10^9, i.e. $dT = -12$ so the "fiscal stimulus" theory would predict

$$dY = 3 \times 12 = 36, \quad d(Y/4) = 9$$

(assuming no change in the other components of GDP, namely I, G and $X - M$

2.5.3 Spending cuts and the Deficit

The Deficit D is the excess of government expenditure over taxation. In this model the deficit is $D = G - T$ However, one should be careful because G does not include all expenditure, only the expenditure on goods and services; thus it excludes pensions, payments of interest on the government debt, and social services payments. Likewise T does not include all tax payments but only those that are deducted from payments on goods and services. [elaborate on this]

With this caveat we can estimate the effect of a cut in G on the deficit. We have the equations

$$\begin{aligned} dY &= dC + dG + dI + d(X - M) \\ dC &\approx 0.75d(Y - T) \\ dT &\approx 0.35dY \end{aligned}$$

whence $dC \approx 0.75(1-0.35)dY \approx 0.5dY$ so that the first and third lines above yield (neglecting dI and $d(X-M)$,

$$dY \approx 2(dG + dI + d(X-M)) \qquad dT \approx 0.7(dG + dI + d(X-M)).$$

so that

$$dD = dG - dT \approx 0.3G - 0.7(dI + d(X-M))$$

For example the cut of 6 billions in Government spending announced on 24 May 2010 might be expected to reduce the deficit by $0.3 \times 6 \approx 2$ billions, if I and $X-M$ do not change.

2.5.4 fiscal stimulus and budget Deficit

$$D := G - T$$

$$\begin{bmatrix} dY \\ dD \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} dG \\ dT \end{bmatrix}$$

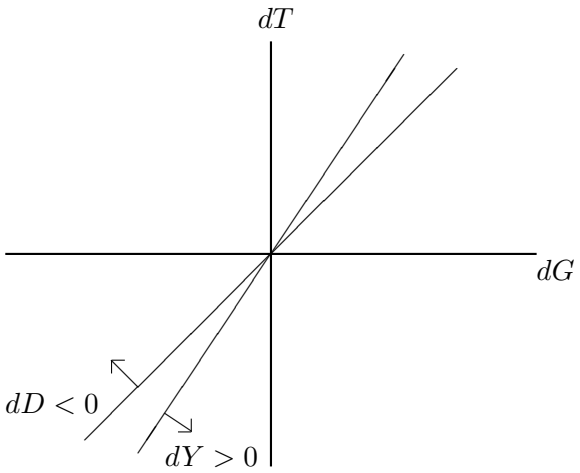
whence

$$\begin{bmatrix} dG \\ dT \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} dY \\ dD \end{bmatrix}$$

To increase GDP without changing deficit (i.e. $dY > 0, dD = 0$) make equal increases in G and T To reduce deficit without changing GDP (i.e. $dD < 0, dG = 0$) increase T and increase G by $3/4$ times as much. Pairs (dG, dT) which satisfy

$$dG < dT < (4/3)dG$$

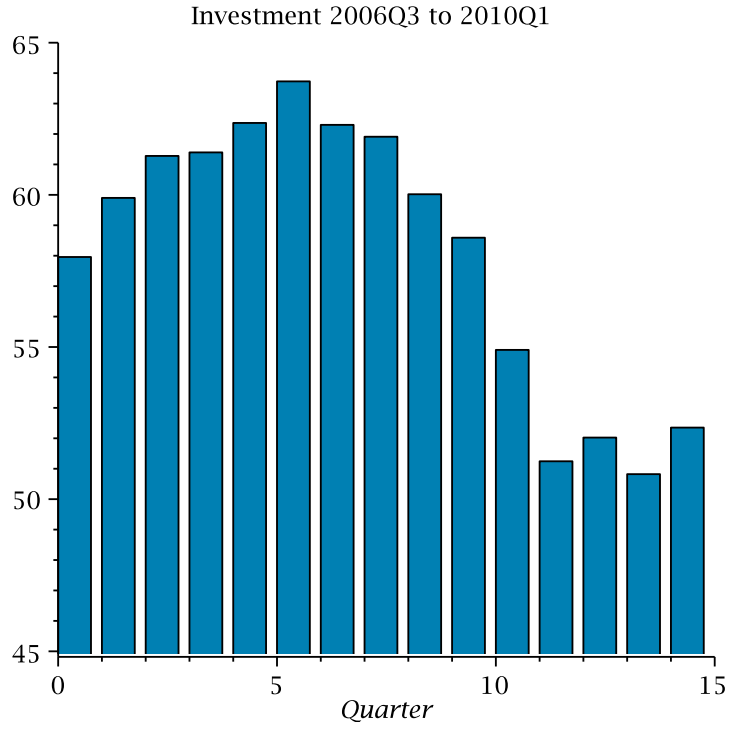
should simultaneously reduce the deficit and increase GDP. They are the points in the wedge-shaped regioun above the origin in the diagram.



2.5.5 investment

The remaining component of GDP expenditure is investment. Since our future depends on it, one could argue that it is the most important; but it is also the most difficult to control. Assuming that businesses use borrowed money to pay for their investments, they will presumably make an investment if they estimate that the return on the investment will exceed the interest they must pay on the loan. Thus, investment can be encouraged by reducing the interest rate on loans to businesses. The factors that determine the interest rate will be the subject of the next part of this document.

[Another factor that affects the level of investment is the confidence of potential investors. A plausible formula to use in the mathematical model might be $I = i(r^* - r)$ where r is the rate of interest and r^* is a measure of confidence, say the average rate of return that a typical businessman expects to receive on a typical investment. The question is, how to test this hypothesis against real data.]



3 Money and finance

3.1 The quantity of money in the economy

The quantity of money, Q, can be defined as the sum of all cash + readily-available deposits in banks (including Bank of England) and similar institutions.

The quantity of money is not constant. It can be created, and sometimes destroyed, by banks. When a bank makes a loan, it creates money. If a bank loan is repaid, or if the bank refuses to renew a loan that has reached the end of its term, money is destroyed. [During the talk there was a discussion about whether there is any conserved quantity here. Thus, when the bank creates money, a compensating indebtedness is also created, so it might be that "(total money) - (total

indebtendness)" is conserved.]

The rate of interest can be thought of as the price of a loan, i.e. the price that the bank charges for creating money. Just like the price of a commodity such as coffee, the price of money can be thought of as reflecting an equilibrium between supply and demand. For coffee, the supply is an increasing function of price (because when the price goes up, more producers find it profitable to produce), whereas the demand is a decreasing function of price (because when the price goes up, consumers will buy less); the graphs of these these two functions intersect at a point representing the price at which supply and demand are equal (i.e. the price at which consumers buy all the coffee that is produced but no more. A similar analysis can be used for money, as we now show.

3.1.1 The demand for money

A simple model for the amount of money needed to support a given GDP is Irving Fisher's (1911) equation

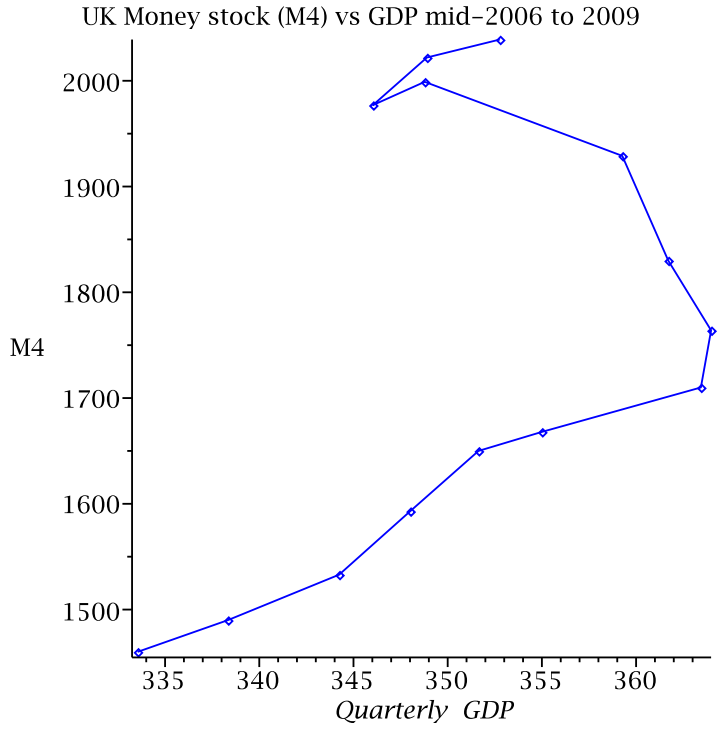
$$Y = QV$$

where V is the velocity of money (average ratio of people's spending rate to the amount of money they hold).
The presumption is that V is a constant or at least varies much mores slowly than Y and Q. So, on this model the demand for money is

$$Q_{demand} = Y/V$$

independent of its price (i.e. the rate of interest).

3.1.2 Does Irving Fisher's equation describe the recent UK data?



If Fisher's equation is true with a constant V, then Q should decrease whenever Y decreases; but according to the data Q continued to increase regardless of whether Y was increasing or decreasing. It would seem that his equation will only fit the data if the assumption of an approximately constant V is abandoned, but if that is done it is questionable what use the equation is.

A (possibly) better model is

$$Q = Y/V + L(-r)$$

Following Keynes, we could call L the "speculative demand". It is money that people keep in the form of cash while they await favourable opportunities to invest it in things like Government bonds. It depends on interest rates, and on people's expectations about future interest rates. The $-r$ means that L decreases when r increases.

3.1.3 The supply of money

To meet the demand for money, banks can supply money in the form of loans. Indeed, banks (including but not only the central bank) can create money. They do this by opening new accounts or expanding credit limits on existing ones. In effect they buy financial assets (transferrable promises of future payments) in exchange for newly created money. The amount of money created by the banking system depends on the rate of interest r: the more money that is asked from them, the higher price they will charge.

Denote by $q_{supply}(r)$ the amount of money the banking system (including the central bank) is willing to provide at interest rate r. It is an increasing function of r.

3.1.4 Equality of supply and demand

$$q_{supply}(r) = q_{demand}(-r) := Y/V + L(-r)$$

provides an equation from which Q and r could be determined — if we knew the two functions involved. Frequently q_{supply} is treated as a constant whose value can be controlled by the central bank.

Unfortunately, time ran out at this point, though there is much more to say.