Statistical mechanics of two-dimensional turbulence

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1 Introduction

Two-dimensional flow is much simpler than three. Nevertheless it has some relevance to real life — the flows in the atmosphere that are responsible for the weather are said to be essentially two-dimensional.But we also study it for its own interest and the possibility that some of the ideas may be helpful for three-dimensional turbulence.

The main formula arrived at in this paper, eqn (48) is not new: an equivalent formula was obtained by Miller[4] based on a discrete-space approximation, and Robert[5] gave a derivation of the formula which does not make this approximation. Robert's derivation, however, depends upon a principle (axiom) of 'conditional concentration', the rationale for which is not entirely clear, at least to this author. The purpose of the present paper is to provide a derivation of the Miller-Robert formula which is derived from principles more deeply rooted in statistical mechanics. These principles are contained in three postulates or hypotheses. The first is that at large times the fluid approaches a state of statistical equilibrium, i.e. that it can be described by a stationary probability measure (represented mathematically in the theory as a Young measure). The second is that in this state of statistical equilibrium the fluctuations at different places are uncorrelated (statistically independent). The third hypothesis is that the only invariants of motion are the known ones, namely the energy and the overall distribution of values for the vorticity.

This work arose out of discussions with Sergei Kuksin who has shown[3] that in a certain limit the two-dimensional incompressible Navier-Stokes equation with random forcing approaches a stationary statistical solution of the Euler equation. This paper describes a different approach to these statistical solutions, based on the application of some ideas from statistical mechanics to the long-time behaviour of solutions of the Euler equations from an arbitrary initial condition.

2 The vorticity evolution equation

We are going to consider flow of a two-dimensional inviscid fluid in a region X which may be either the plane R^2, or a finite simply-connected region in in that plane with smooth boundary, or a two-dimensional torus (i.e. a square or rectangular region with periodic boundary conditions). If X is a finite region, the condition at its boundary is that the velocity of the fluid must be tangential.The velocity field is a (vector) function $\mathbf{u} : X \to R^2$. It satisfies the Euler equation

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \qquad \boxed{\text{Euler}} \tag{1}$$

where the pressure field p is determined by the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \qquad \boxed{\text{incompr}} \tag{2}$$

For two-dimensional hydrodynamics, the (scalar) vorticity field w can be defined by

$$\begin{aligned} \mathbf{k}w &= \mathbf{curl\ u} \\ \text{i.e.}\quad w(\mathbf{x}) &= \partial u_1/\partial x_2 - \partial u_2/\partial x_1 \qquad \boxed{\text{bwe}} \end{aligned} \tag{3}$$

where $\mathbf{x} := (x_1, x_2)$ and $\mathbf{k}$ is a unit vector perpendicular to the (x_1, x_2) plane. The vorticity is equal to twice the local angular velocity of the fluid. The two-dimensional Euler equation implies[1]

$$\frac{dw}{dt} = 0 \qquad \boxed{\text{dw/dt}} \tag{4}$$

where dw/dt denotes the convective derivative, defined by

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w \qquad \boxed{\text{convd}} \tag{5}$$

Equation (4) is not a complete time evolution equation for the field $w(\cdot)$ since it involves a second field $\mathbf{u}(\cdot)$, which is related to $w(\cdot)$ through eqn (3). To find the velocity field $\mathbf{u}$ we need to solve eqn (3), subject to the boundary condition that the normal component of $\mathbf{u}$ on the boundary of the region X occupied by the fluid is zero (or periodic boundary conditions in the case where X is the torus T^2). To solve (3), define the stream function ψ as the solution of

$$-\nabla^2 \psi = w(\mathbf{x}') \qquad \boxed{\text{psi}} \tag{6}$$

[1]The three-dimensional version of (4) is given in, for example, eq (1.9) of Chorin [1].

with Dirichlet boundary conditions in the case where X is a finite region in R^2. In the case where X is the torus T^2, eqn (6) is soluble because Stokes' theorem and the definition of $\mathbf{w}$ imply that $\int_X \mathbf{w}\,\mathrm{d}^2\mathbf{x} = 0$; in that case the solution of (6) is unique apart from an additive constant. Since $\nabla \cdot \psi = 0$, we have

$$\mathbf{curl}\ \mathbf{curl}\ (\psi\mathbf{k}) = -\nabla^2\psi\mathbf{k} = w(\mathbf{x}')\mathbf{k} \qquad \boxed{\text{curlcurl}} \tag{7}$$

so that

$$\begin{aligned} \mathbf{u} &= \mathbf{curl}\ (\psi\mathbf{k}) = (\mathbf{grad}\psi) \times \mathbf{k} \\ \text{i.e.}\quad (u_1, u_2) &= (\partial\psi/\partial x_2, -\partial\psi/\partial x_1) \qquad \boxed{\text{u}} \end{aligned} \tag{8}$$

solves eqn (3) for given $\mathbf{w}(\cdot)$. This solution satisfies the boundary conditions (if any), since in the case where X is a finite region the boundary condition on G implies that $\psi = 0$ on ∂X, and hence the the normal component of $\mathbf{u}$ on ∂X is zero, as required. The solution is unique for simply-connected X, but in the case where X is T^2 an arbitrary additive constant added to the solution (8) gives another solution.

3 Invariants of the motion

The inviscid time evolution (in either two or three dimensions) has the usual energy invariant

$$E := \frac{1}{2}\int_X \mathbf{u}^2\,\mathrm{d}^2\mathbf{x} = \text{invariant.} \qquad \boxed{\text{E}} \tag{9}$$

To express this in terms of w we use the explicit solution of (6),

$$\psi(\mathbf{x}) = \int_X G(\mathbf{x}, \mathbf{x}')w(\mathbf{x}')\,\mathrm{d}^2\mathbf{x}' \qquad \boxed{\text{G}} \tag{10}$$

where G is the negative of the Green's function for Poisson's equation in X, with Dirichlet boundary conditions if X is a finite region. By virtue of (8) we can then write (9) as

$$\begin{aligned} E &= \frac{1}{2}\int_X (\mathbf{curl}\ (\psi\mathbf{k}))^2\,\mathrm{d}^2\mathbf{x} \\ &= \frac{1}{2}\int_X \psi \cdot \mathbf{curl}\ \mathbf{curl}\ (\psi\mathbf{k})\,\mathrm{d}^2\mathbf{x} \\ &= \frac{1}{2}\int_X \psi(\mathbf{x})w(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \quad \text{by (7)} \\ &= \frac{1}{2}\int_X\int_X w(\mathbf{x})G(\mathbf{x}, \mathbf{x}')w(\mathbf{x}')\,\mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{x}' \qquad \boxed{\text{quadform}} \end{aligned} \tag{11}$$

If X is the whole of space or the torus T^2, the linear momentum is another invariant :

$$\frac{d}{dt}\int_X \mathbf{u}\,\mathrm{d}^2\mathbf{x} = 0. \qquad \boxed{\text{linmom}} \tag{12}$$

If X is a circle, then the angular momentum is also an invariant of the motion.

For two-dimensional inviscid flow, there is another class of invariants arising from eqn (4), which implies that each particle of fluid carries a particular value of $\mathbf{w}$ along with it. So, for example, if initially only two different values of $\mathbf{w}$ are present, then at later times those two values are still the only ones present. Moreover, since we are assuming incompressibility (eqn (2)) it follows that the area of the region in which $\mathbf{w}$ takes any particular value is an invariant of the motion.

To simplify the exposition I am going to assume that the distribution of vorticity is restricted to a finite set of values $W := w_1, w_2, \ldots, w_N$. I'll denote by P_i the probability that the value of $w(\mathbf{x})$ at a randomly chosen point $\mathbf{x}$ in X is equal to w_i, i.e.

$$P_i = m(\Gamma_i)/m(X) \qquad \boxed{\text{P}} \tag{13}$$

where the set function $m(\cdot)$ denotes area (measure) and

$$\Gamma_i := \{\mathbf{x} : w(\mathbf{x}) = w_i\} \qquad \boxed{\text{Gammai}} \tag{14}$$

Then it follows from the convection of vorticity that the numbers P_i are invariants of the motion, i.e. they do not depend on the time t.

In some theories of two-dimensional turbulence[2, 3] an important part is played by and invariant called the "enstrophy", which is defined as $\int_X w^2(\mathbf{x})\,\mathrm{d}^2\mathbf{x}$. It follows from the definitions that the enstrophy is equal to $m(X)\sum_i w_i^2 P_i$, so its invariance need not be taken into account separately here.

4 Hypotheses about the equilibrium behaviour

hypoth

Consider the time evolution under the Euler equation (1), with some given initial velocity field. As the time t increases, it is to be expected that the velocity and vorticity distributions will get more and more complicated, while preserving the values of the invariants E and $P_1, P_2, \ldots$ defined in the preceding section. The treatment in this paper is based on certain hypotheses about the long-time behaviour of the hydrodynamic field, which will now be formulated.

We don't expect $\mathbf{u}$ or w to approach limits, but it is plausible that there is a unique equilibrium state in some statistical sense. The first hypothesis makes this assumption precise: it is that *the limiting behaviour of w at large times corresponds to a unique Young measure*[2]. The same hypothesis is used by Robert [5].

From this hypothesis it follows immediately[3] that for each i and each region A in X, the event that at a randomly chosen point $\mathbf{x} \in A$ the function $w(\mathbf{x})$ takes the value w_i has probability

$$\lim_{t\to\infty} \frac{m(\Gamma_i(t)\cap A)}{m(A)} = \frac{1}{m(A)} \int_A p_i(\mathbf{x})\ \mathrm{d}\mathbf{x} \qquad \boxed{\text{p}} \qquad (15)$$

where $p_i(\mathbf{x})$ is a function which can be thought of as the probability that w takes the value w_i at the point $\mathbf{x}$ (or, perhaps, as the long-term fraction of time that the equation $w(\mathbf{x},t) = w_i$ is true). The formula (15) is self-consistent in its dependence on A, since after multiplication by $m(A)$ both sides are additive set functions.

It will be convenient to use the notation $\bar{p}_i(A)$ for the average of $p_i(\mathbf{x})$ which appears on the right side of (15):

$$\bar{p}_i(A) := \frac{1}{m(A)} \int_A p_i(\mathbf{x})\, \mathrm{d}^2\mathbf{x} \qquad \boxed{\text{barpi}} \qquad (16)$$

As an example, if we choose A to be X and use (13), we see that

$$\bar{p}_i(X) = \frac{1}{m(X)} \int_X p_i(\mathbf{x})\ \mathrm{d}\mathbf{x} = P_i : \qquad \boxed{\text{barpiX}} \qquad (17)$$

the average of $p_i(\mathbf{x})$ over the whole of X is the invariant P_i.

[2]The defining property of a Young measure $\nu_{\mathbf{x}}$ is (as told to me by Jan Kristensen and probably misunderstood by me) is that, for any measurable set A and any continuous function φ which goes to zero at $\pm\infty$ we have $\int_A \varphi(w(\mathbf{x},t))\,\mathrm{d}^2\mathbf{x} \to \int_A [\int_{-\infty}^{\infty} \varphi(\varpi) d\nu_{\mathbf{x}}(\varpi)]\,\mathrm{d}^2\mathbf{x}$ on an unbounded increasing sequence of values of t. In our case the Young measure is concentrated on a discrete set of w-values, so that it can be written $\nu_{\mathbf{x}} = \sum_i p_i(\mathbf{x})\delta_{w_i}$ and hence the defining formula becomes $\int_A \varphi(w(\mathbf{x},t))\,\mathrm{d}^2\mathbf{x} \to \int_A [\sum_i \varphi(w_i)p_i(\mathbf{x})]\,\mathrm{d}^2\mathbf{x}$. Our hypothesis is that the Young measure is unique, i.e. that the limit is the same whatever increasing sequence of t-values is used.

[3]In the defining formula for the Young measure given in the preceding footnote, choose $\varphi(w)$ to be equal to 1 for $w = w_1$ and to 0 for all the other possible values that $w(x)$ can take; this gives the formula (15)

It follows (I think) from the hypothesis (15) and the integral formula (10) for the stream function that

$$\lim_{t\to\infty} \psi(\mathbf{x}) = \sum_i w_i \int_X G(\mathbf{x},\mathbf{x}')p_i(\mathbf{x}')\,\mathrm{d}^2\mathbf{x}' \qquad \boxed{\text{psip}} \tag{18}$$

Likewise, using the formula (11) for the energy we obtain:

$$\begin{aligned} E &= \frac{1}{2}\sum_{i,j} w_i w_j \int_X \int_X G(\mathbf{x},\mathbf{x}')p_i(\mathbf{x}')p_j(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \\ &= \frac{1}{2}\sum_i w_i \int_X p_i(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \qquad \boxed{\text{Ep}} \end{aligned} \tag{19}$$

In order to get the results I want, I shall need a strengthened form of the hypothesis, which extends (15) to simultaneous events in different places. Let A and B be two disks in X of equal radius and let T be the translation operator such that $TA = B$; then the hypothesis concerns the joint probability that the value of $w(\mathbf{x})$ a randomly chosen point $\mathbf{x}$ in A is w_i and that simultaneously the value of $w(T\mathbf{x})$ is w_j. If this event occurs, then $\mathbf{x}$ lies in $\Gamma_i \cap A$ and at the same time $T\mathbf{x}$ lies in $\Gamma_j \cap TA$, that is to say $\mathbf{x}$ lies in $T^{-1}\Gamma_j \cap A$; so the joint event occurs if and only if $\mathbf{x} \in \Gamma_i \cap T^{-1}\Gamma_j \cap A$. Our hypothesis is that in the limit $t \to \infty$ the events in A and TA are statistically independent, i.e. that the probability of the joint event equals the product of the individual probabilities. In symbols, this hypothesis is

$$\lim_{t\to\infty} \frac{m(\Gamma_i \cap T^{-1}\Gamma_j \cap A)}{m(A)} = \bar{p}_i(A)\bar{p}_j(B) \qquad \boxed{\text{pp}} \tag{20}$$

where (15) and (16) have been used on the right-hand side. It will be assumed that the equations corresponding to (20) for more than two regions are also true, in particular the one for four regions, which will be crucial later on.

Our second hypothesis concerns the nature of the equilibrium Young measure. Obviously it will depend on the invariants of the motion. The ones we know are the energy and the invariants P_i. We shall assume that there are no other invariants so that, for example, the functions $p_i(\mathbf{x})$ are fully determined by the values of these invariants[4]. This is analogous to the standard assumption of (microcanonical) statistical mechanics that the probability distribution in phase space depends only on the energy, and

[4]Sometimes there are others, for example the angular momentum if the region X is a disk

other invariants such as angular momentum if they exist. Our hypothesis is that *the Young measure is invariant under any transformation R of the hydrodynamic states that leaves the energy and the P_i invariants unchanged.* By a hydrodynamic state I mean a function $w(\cdot)$.

5 A four-point theorem

Let ϵ be a small number and let $D_1, D_2, D_3, D_4 \subset X$ be four non-overlapping disks of area ϵ, centred at points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. For each pair (m,n) with $m,n \in \{1,2,3,4\})$ let $T_{mn})$ the translation operator such that $T_{mn}D_n = D_m$. The independence hypothesis analogous to (20) four four regions asserts that, for a point $\mathbf{x}$ chosen at random in D_1, so that $T_{21}\mathbf{x} \in D_2$,etc, the limiting joint probability of the four simultaneous events $w(\mathbf{x}) = w_i, w(T_{21}\mathbf{x}) = w_j, w(T_{31}\mathbf{x}) = w_k, w(T_{41}\mathbf{x}) = w_l$ equals the product of their separate probabilities. Since $T_{21}^{-1} = T_{12}$, etc, the equations defining the four events are equivalent to $\mathbf{x} \in \Gamma_i, \mathbf{x} \in T_{12}\Gamma_j, \mathbf{x} \in T_{13}\Gamma_k, \mathbf{x} \in T_{14}\Gamma_k$ and so the statement about the probabilities can be written

$$\lim_{t\to\infty} \frac{m(\Delta_1)}{m(D_1)} = \bar{p}_i(D_1)\bar{p}_j(D_2)\bar{p}_k(D_3)\bar{p}_l(D_3) \qquad \boxed{\text{pppp}} \qquad (21)$$

where

$$\Delta_1 := D_1 \cap \Gamma_i \cap T_{12}\Gamma_j \cap T_{13}\Gamma_k \cap T_{14}\Gamma_l \qquad \boxed{\text{Delta1}} \qquad (22)$$

and $\bar{p}_i(D_1)$, etc. are defined in (16).

We define a transformation R by

$$\begin{array}{rrcllcl} & R(\mathbf{x}) & := & T_{21}\mathbf{x} & \text{if} & \mathbf{x} & \in & \Delta_1 \\ & \text{and} & := & T_{12}\mathbf{x} & \text{if} & \mathbf{x} & \in & \Delta_2 := T_{21}\Delta_1 \\ & \text{and} & := & T_{43}\mathbf{x} & \text{if} & \mathbf{x} & \in & \Delta_3 := T_{31}\Delta_1 \\ & \text{and} & := & T_{34}\mathbf{x} & \text{if} & \mathbf{x} & \in & \Delta_4 := T_{41}\Delta_1 \\ \text{else} & R(\mathbf{x}) & := & \mathbf{x} & & & & \end{array} \qquad \boxed{\text{R}} \qquad (23)$$

It is easily checked that R^2 is the identity, therefore R is invertible, and being constructed by translating different measurable subsets of X it must preserve measure. Moreover, it has the properties

$$\begin{array}{rcl} R(\Delta_1) & = & \Delta_2 \\ R(\Delta_2) & = & \Delta_1 \\ R(\Delta_3) & = & \Delta_4 \\ R(\Delta_4) & = & \Delta_3 \\ \boxed{\text{RDelta}} & & \end{array} \qquad (24)$$

Its effect is to exchange the two sets Δ_1 and Δ_2, and also to exchange the two sets Δ_3 and Δ_4. By (27), Δ_1 is a subset of Γ_i and hence all points $\mathbf{x}$ in Δ_1 have $w(\mathbf{x}) = w_i$. Again by (27), $\Delta_2 := T_{21}\Delta_1$ is a subset of Γ_j and hence all points $\mathbf{x}$ in Δ_2 have $w(\mathbf{x}) = w_j$. The transformation R exchanges a measurable set of points in which $w(\mathbf{x}) = w_i$ for one of the same measure in which $w(\mathbf{x}) = w_j$. At the same time, it also exchanges the measurable set Δ_3, in which $w(\mathbf{x}) = w_k$, for the set Δ_4, in which $w(\mathbf{x}) = w_l$.

From these considerations, it follows that

$$\lim_{t\to\infty} \frac{m(R\Delta_1)}{m(D_1)} = \bar{p}_j(D_1)\bar{p}_i(D_2)\bar{p}_l(D_3)\bar{p}_m(D_3) \qquad \boxed{\text{Rpppp}}) \tag{25}$$

I want to apply the second hypothesis to show that the right hand sides of (21) and (25) are equal. Let us calculate the change in energy brought about by the transformation R. According to formula (11) the increase in energy is

$$\begin{aligned}
\delta E \;&:=\; \frac{1}{2}\int_X\int_X G(\mathbf{x},\mathbf{x}')[w(R\mathbf{x})w(R\mathbf{x}') - w(\mathbf{x})w(\mathbf{x}')]\,\mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{x}' \\
&=\; \int_X \mathrm{d}^2\mathbf{x}' \int_\Delta \mathrm{d}^2\mathbf{x} G(\mathbf{x},\mathbf{x}')w(\mathbf{x}')[w(R\mathbf{x}) - w(\mathbf{x})] \\
&\quad -\frac{1}{2}\int_\Delta\int_\Delta G(\mathbf{x},\mathbf{x}')[w(R\mathbf{x})w(R\mathbf{x}') - w(\mathbf{x})w(\mathbf{x}')]\,\mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{x}' \qquad \boxed{\text{deltaE}}
\end{aligned} \tag{26}$$

where

$$\Delta := \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \qquad \boxed{\text{Delta}} \tag{27}$$

In the second line of (26) we have used the fact (see (23)) that the integrand is zero unless at least one of $\mathbf{x}, \mathbf{x}'$ is in Δ; we have also used the symmetry of the integrand. The part of the domain of integration where both $\mathbf{x}$ and $\mathbf{x}'$ are in Δ is counted twice in the first double integral on the right, and the second one corrects for this. However, the measure of the domain of integration in the second double integral is $m(\Delta)^2 < [4m(D_1)]^2 = 16(\epsilon^2$ and is therefore small for small ϵ; so we can write (26) more simply as

$$\delta E = \int_\Delta \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})[w(R\mathbf{x}) - w(\mathbf{x})] + c_1 \qquad \boxed{\text{deltaEG}} \tag{28}$$

where ψ is the stream function, as given in eqn (10), and c_1 is an $O(\epsilon^2)$ correction term.

Using the definition (27) of Δ, the integral in (28) can be split into four parts and then in each integral we can approximate $\psi(\mathbf{x})$ by its value at the centre of the disk :

$$
\begin{aligned}
\delta E &= \int_{\Delta_1} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})[w(R\mathbf{x}) - w(\mathbf{x})] + \int_{\Delta_2} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})[w(R\mathbf{x}) - w(\mathbf{x})] \\
&+ \int_{\Delta_3} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})[w(R\mathbf{x}) - w(\mathbf{x})] + \int_{\Delta_4} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})[w(R\mathbf{x}) - w(\mathbf{x})] + c_1 \\
&= \int_{\Delta_1} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})(w_i - w_j) + \int_{\Delta_2} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})(w_j - w_i) \\
&+ \int_{\Delta_3} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})(w_l - w_k) + \int_{\Delta_4} \mathrm{d}^2\mathbf{x}\psi(\mathbf{x})(w_k - w_l) + c_1 \\
&= [m(\Delta_1)\psi(\mathbf{x}_1) - m(\Delta_2)\psi(\mathbf{x}_2)](w_i - w_j) + [m(\Delta_3)\psi(\mathbf{x}_3) - m(\Delta_4)\psi(\mathbf{x}_4)](w_k - w_l) \\
&\quad + c_1 + c_2 \\
&= m(\Delta_1)\{[\psi(\mathbf{x}_1) - \psi(\mathbf{x}_2)](w_i - w_j) + [\psi(\mathbf{x}_3) - \psi(\mathbf{x}_4)](w_k - w_l)\} \\
&\quad + c_1 + c_2 \qquad \boxed{\text{split}}
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
c_2\epsilon^2 &:= \int_{\Delta_1} \mathrm{d}^2\mathbf{x}[\psi(\mathbf{x}) - \psi(\mathbf{x}_1)](w_i - w_j) + \int_{\Delta_2} \mathrm{d}^2\mathbf{x}[\psi(\mathbf{x}) - \psi(\mathbf{x}_2)]\psi(\mathbf{x})(w_j - w_i) \\
&+ \int_{\Delta_3} \mathrm{d}^2\mathbf{x}[\psi(\mathbf{x}) - \psi(\mathbf{x}_3)]\psi(\mathbf{x})(w_l - w_k) + \int_{\Delta_4} \mathrm{d}^2\mathbf{x}[\psi(\mathbf{x}) - \psi(\mathbf{x}_4)]\psi(\mathbf{x})(w_k - w_l)
\end{aligned}
\tag{30}
$$

and in the last line of (29) we used the fact that $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4$. Taylor's theorem gives $\int_{\Delta_n} \mathrm{d}^2\mathbf{x}[\psi(\mathbf{x}) - \psi(\mathbf{x}_n)] \approx \frac{1}{2}\int_{\Delta_1} \mathrm{d}^2\mathbf{x}(\mathbf{x} - \mathbf{x}_n)^2\nabla^2\psi(\mathbf{x}_n) = O(\epsilon^2)$, therefore $c_2 = O(\epsilon^2)$.

By eqn (6), the function ψ is differentiable and therefore continuous, so its range is an interval of the real line. Given any choice of w_i, w_j, w_k, w_l it is therefore possible to find points $\mathbf{x}_1, \ldots \mathbf{x}_4$, all different, such that

$$
(\psi(x_1) - \psi(x_2))(w_i - w_j) + (\psi(x_3) - \psi(x_4))(w_l - w_k) = 0 \qquad \boxed{\text{psixrule}}
\tag{31}
$$

and at the same time $\nabla\psi(\mathbf{x}_n) \neq 0$ $(n = 1, 2, 3, 4)$. With this choice of $\mathbf{x}_1, \ldots \mathbf{x}_4$, eqn (29) leads to $\delta E = c_1 + c_2 = O(\epsilon^2)$. In order to make $\delta E = 0$, we need a slightly different choice. Let $\mathbf{x}'_1$ be a point close to $\mathbf{x}_1$; replacing $\mathbf{x}_1$ by $\mathbf{x}'_1$ then the right side of (29) can be written (again using Taylor's theorem)

$$
\delta E = m(\Delta_1)(w_i - w_j)(\mathbf{x}' - \mathbf{x}) \cdot \nabla\psi(x_1) + O(\mathbf{x} - \mathbf{x}')^2 + c_1 + c_2 \tag{32}
$$

By choosing a suitable $\mathbf{x}'$, which can be within a distance of order ϵ^2 from $\mathbf{x}$, we can arrange to have $\delta E = 0$.

By the second hypothesis in (4), the probabilities under the two distributions are the same. That is to say, the right sides of (21) and (25) are equal:

$$\bar{p}_i(D_1')\bar{p}_j(D_2)\bar{p}_k(D_3)\bar{p}_l(D_4) = \bar{p}_j(D_1')\bar{p}_i(D_2)\bar{p}_l(D_3)\bar{p}_k(D_4) \tag{33}$$

where D_1' is the disk of area ϵ centred at $\mathbf{x}_1'$. Taking the limit $\epsilon \to 0$ and $\mathbf{x}' \to \mathbf{x}$ we obtain

$$p_i(\mathbf{x}_1)p_j(\mathbf{x}_2)p_k(\mathbf{x}_3)p_l(\mathbf{x}_4) = p_j(\mathbf{x}_1)p_i(\mathbf{x}_2)p_l(\mathbf{x}_3)p_k(\mathbf{x}_4) \quad \boxed{\text{probrule}} \tag{34}$$

This equation holds at all points x_1, x_2, x_3, x_4 which satisfy (31) and are points of continuity of the functions $p_i(\cdot), p_j(\cdot), p_k(\cdot), p_l(\cdot)$.

6 Solving eqn (34)

Defining

$$\theta_{ij}(\mathbf{x}) := \ln[p_i(\mathbf{x})/p_j(\mathbf{x})] \quad \boxed{\text{theta}} \tag{35}$$

we can write the result (34) as

$$\theta_{ij}(\mathbf{x}_1) - \theta_{ij}(\mathbf{x}_2) - \theta_{kl}(\mathbf{x}_3) + \theta_{kl}(\mathbf{x}_4) = 0 \quad \boxed{\text{thetarule}} \tag{36}$$

If we vary $\mathbf{x}_1$ while holding $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ fixed, eqns (29) and (36) show that both $\psi(\mathbf{x}_1)$ and $\theta_{ij}(\mathbf{x})$ stay fixed; therefore it would appear that $\theta_{ij}(\mathbf{x}_1)$ depends on $\mathbf{x}_1$ only through the value of $\psi(\mathbf{x}_1)$, so that we may write

$$\theta_{ij}(\mathbf{x}) = \vartheta_{ij}(\psi(\mathbf{x})) \quad \boxed{\text{thetasol}} \tag{37}$$

where ϑ is a real function of one real variable satisfying

$$\vartheta_{ij}(\psi_1) - \vartheta_{ij}(\psi_2) - \vartheta_{kl}(\psi_3) + \vartheta_{kl}(\psi_4) = 0 \quad \boxed{\text{varthetarule}} \tag{38}$$

whenever $\psi_1, \psi_2, \psi_3, \psi_4$ satisfy

$$(\psi_1 - \psi_2)(w_i - w_j) + (\psi_4 - \psi_4)(w_l - w_k) = 0 \quad \boxed{\text{psirule}} \tag{39}$$

Now consider varying ψ_1, ψ_2 while holding ψ_3, ψ_4 fixed. Eqns (39) and (38) imply

$$\vartheta_{ij}'(\psi_1)d\psi_1 - \vartheta_{ij}'(\psi_2)d\psi_2 = 0, d\psi_1 - d\psi_2 = 0 \tag{40}$$

so that

$$\vartheta_{ij}'(\psi_1) = \vartheta_{ij}'(\psi_2) \quad \boxed{\text{varth'}} \tag{41}$$

By suitable choice of ψ_3, ψ_4 we can prove this last result for any pair ψ_1, ψ_2 and it follows that $\vartheta'_{ij}(\psi)$ is a constant, call it a_{ij} and hence that

$$\vartheta_{ij}(\psi) = a_{ij}\psi + b_{ij} \qquad \boxed{\text{varth}} \tag{42}$$

To find out more about the constants a_{ij} and b_{ij}, substitute this into (38), obtaining

$$a_{ij}(\psi_1 - \psi_2) - a_{kl}(\psi_3 - \psi_4) = 0 \qquad \boxed{\text{arule}} \tag{43}$$

which must hold whenever $\psi_1, \psi_2, \psi_3, \psi_4$ satisfy (39). Combining (43) with (39) we find

$$\frac{a_{ij}}{w_i - w_j} = \frac{a_{kl}}{w_k - w_l} \tag{44}$$

and since this has to hold for all i, j, k, l there must be a constant β such that $a_{ij} = -\beta(w_i - w_j)$ (the minus sign is purely conventional). Thus, (42) becomes

$$\vartheta_{ij} = -\beta(w_i - w_j)\psi + b_{ij} \tag{45}$$

from which (37) and (35) give

$$\frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \exp[-\beta(w_i(\mathbf{x}) - w_j(\mathbf{x}))\psi(\mathbf{x}) + b_{ij}] \tag{46}$$

From this it follows that

$$p_i(\mathbf{x}) = p_1(\mathbf{x})\exp[-\beta w_i(\mathbf{x})\psi(\mathbf{x}) + \beta w_1(\mathbf{x})\psi(\mathbf{x}) + b_{i1}] \qquad \boxed{\text{pisol1}} \tag{47}$$

Using the normalization condition $\sum_i p_i(\mathbf{x}) = 1$ we can eliminate $p_1(\mathbf{x})$, so that (47) becomes

$$p_i(\mathbf{x}) = \frac{\lambda_i \exp[-\beta w_i(\mathbf{x})]}{\sum_j \lambda_j \exp[-\beta w_j(\mathbf{x})]} \qquad \boxed{\text{answer}} \tag{48}$$

where $\lambda_i := \exp b_{i1}$. Formula (48) is not new: it was given by Robert.

In the language of statistical thermodynamics, the constants β and λ_i are analogues of the inverse temperature and of the activities of the different types of fluid particle (carrying different vorticities) although with the difference that $1/\beta$ has dimensions of energy pser unit area rather than energy.

To finish finding the equilibrium states, we have to solve eqn (6) relating the stream function and the vorticity, which, using (48), can now be written

$$-\nabla^2\psi(\mathbf{x}) = \frac{\sum_i w_i\lambda_i \exp[-\beta\psi(\mathbf{x})w_i(\mathbf{x})]}{\sum_j \lambda_j \exp[-\beta\psi(\mathbf{x})w_j(\mathbf{x})]} \qquad \boxed{\text{DE}} \tag{49}$$

The existence of solutions of eqn (49) is discussed by Robert[5]; applying the Schauder fixed point theorem he arrives at the conclusion that with Dirichlet boundary conditions on the boundary of X it has a unique solution. The solution depends on the values of the parameters $\beta, \lambda_1, \ldots$. These values are related to the energy E and the initial overall distribution of vorticity values by the following relations, obtained using (13) and (19):

$$\begin{aligned} E &= \int \sum P_i(\mathbf{x}) w_i \psi(\mathbf{x}) = -\frac{\partial Z}{\partial \beta} \\ m(X) P_i &= \int_X p_i(\mathbf{x}) \, \mathrm{d}^2\mathbf{x} = \lambda_i \frac{\partial Z}{\partial \lambda_i} \qquad \boxed{\text{derivs}} \end{aligned} \tag{50}$$

where $Z(\beta, \lambda_1, \ldots, \psi(\cdot))$ is a 'partition functional' defined by

$$Z := \int \mathrm{d}^2\mathbf{x} \ln \sum_j \lambda_j \exp[-\beta w_j(\mathbf{x}) \psi(\mathbf{x}))] \qquad \boxed{\text{Z}} \tag{51}$$

In thermodynamic terms, the logarithm of the functional Z can be thought of as the grand canonical potential (generalized Helmholtz free energy). As noted by Miller and Robert [4, 5], it has an interesting variational property: the Euler equations for minimization of Z with respect to the function $\psi(\cdot)$ and the values of β and the numbers λ_i at given values of E and the numbers P_i are eqns (49) and (50). THIS STATEMENT MAY NOT BE QUITE CORRECT : NEEDS CHECKING

References

[1] A. J. Chorin, *Vorticity and turbulence*, Springer Applied Mathematical Sciences series no. 103

[2] R. H Kraichnan and D. Montgomery Rep. Prog. Phys. **43** 547-619 (1980) Two-dimensional turbulence.

[3] S. Kuksin, The Eulerian limit for 2D hydrodynamics, in preparation.

[4] J. Miller, Statistical mechanics of Euler equations in two dimensions. Phys Rev Lett **65** 2137-2140 (1990)

[5] R. Robert, A maximum-entropy principle for two-dimensional perfect fluid dynamics, J Stat Phys **65** 531-553 (1991)