# Probabilistic global well-posedness of the energy-critical defocusing nonlinear wave equation below the energy space

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# Energy-critical nonlinear wave equations

**Energy-critical defocusing nonlinear wave equation:**

$$\text{(NLW)} \qquad \partial_t^2 u - \Delta u + |u|^{\frac{4}{d-2}} u = 0, \quad x \in \mathbb{R}^d, \quad d = 3, 4, 5, \quad u(t,x) \in \mathbb{R}$$

- Hamiltonian evolution corresponding to the conserved energy:

$$E(u(t), \partial_t u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 + \frac{d-2}{2d}|u|^{\frac{2d}{d-2}}\, dx$$

- Energy space: Recall Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$

$$\mathcal{E}(\mathbb{R}^d) = \left\{ (f,g) : E(f,g) := \int_{\mathbb{R}^d} \frac{1}{2} g^2 + \frac{1}{2}|\nabla f|^2 + \frac{d-2}{2d}|f|^{\frac{2d}{d-2}}\, dx < \infty \right\}$$
$$= \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) =: \dot{\mathcal{H}}^1(\mathbb{R}^d)$$

- Energy-critical equation: NLW is invariant under the scaling symmetry

$$u_\lambda(t,x) = \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x),$$

which leaves the $\dot{\mathcal{H}}^1(\mathbb{R}^d)$-norm invariant:

$$\|(u_\lambda(0), \partial_t u_\lambda(0))\|_{\dot{\mathcal{H}}^1} = \|(u(0), \partial_t u(0))\|_{\dot{\mathcal{H}}^1}$$

# On deterministic global well-posedness

**Defocusing nonlinear wave equation:**

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0$$

- Energy-critical: $p = 1 + \frac{4}{d-2}$ ($p = 5$ when $d = 3$; $p = 3$ when $d = 4$)
- Energy-subcritical if $p < 1 + \frac{4}{d-2}$: GWP in $\mathcal{H}^1$ via energy conservation
- Energy-supercritical if $p > 1 + \frac{4}{d-2}$: poorly understood

On the defocusing energy-critical NLW:

- **Small energy data theory**: Strauss (1968), Rauch (1981), Pecher (1984)
- **Global regularity** (smooth initial data lead to smooth solutions): Struwe (1994), Grillakis (1990, 1992), Shatah-Struwe (1993)
- **GWP in the energy space**: Shatah-Struwe (1994), Kapitanski (1994), Ginibre-Soffer-Velo (1992)
- **Scattering, global space-time bounds**: Bahouri-Shatah (1998), Bahouri-Gérard (1999), Nakanishi (1999), Tao (2006)
- **Ill-posedness below the energy space**: Christ-Colliander-Tao (2003)

# Randomization

**GOAL**: Prove almost sure global well-posedness of the enery-critical NLW for *rough* and *random* initial data below the energy space

On a compact manifold M:

- There exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(M)$ consisting of eigenfunctions of the Laplace-Beltrami operator
- Given $u_0(x) = \sum_{n=1}^{\infty} c_n e_n(x) \in H^s(M)$, one may define its randomization by:

$$u_0^\omega(x) := \sum_{n=1}^{\infty} g_n(\omega) c_n e_n(x),$$

  where $\{g_n\}_{n\in\mathbb{N}}$ are independent random variables

On $\mathbb{R}^d$:

- There is no countable basis of $L^2(\mathbb{R}^d)$ of eigenfunctions of the Laplacian:
  - work with eigenfunctions of the Laplacian with a confining potential such as harmonic oscillator $-\Delta + |x|^2$: Thomann (2009), Burq-Thomann-Tzvetkov (2010), Deng (2012), Poiret (2012)
  - work on $\mathbb{S}^d$ and transfer results to $\mathbb{R}^d$: De Suzzoni (2011, 2013, 2014)

## Wiener randomization:

- Naturally associated with the *Wiener decomposition*:

$$\mathbb{R}^d_\xi = \bigcup_{n\in\mathbb{Z}^d} Q_n, \qquad \text{where} \qquad Q_n := n + [-\tfrac{1}{2}, \tfrac{1}{2})^d$$

- Note that $\sum_{n\in\mathbb{Z}^d} \chi_{Q_n}(\xi) = 1$ and set $\chi_{Q_n}(D)u = \mathcal{F}^{-1}(\chi_{Q_n}\hat{u})$:

$$u = \sum_{n\in\mathbb{Z}^d} \chi_{Q_n}(D)u = \sum_{n\in\mathbb{Z}^d} \chi_{Q_0}(D-n)u$$

- Given $(u_0, u_1)$, we define Wiener randomization by:

$$(u_0^\omega, u_1^\omega) := \Big( \sum_{n\in\mathbb{Z}^d} g_{n,0}(\omega)\psi(D-n)u_0, \sum_{n\in\mathbb{Z}^d} g_{n,1}(\omega)\psi(D-n)u_1 \Big)$$

  - $\psi(\xi - n)$ = smoothed version of $\chi_{Q_n}(\xi) = \chi_{Q_0}(\xi - n)$
  - $\{g_{n,j}\}_{n\in\mathbb{Z}^d, j=0,1}$ = independent mean zero $\mathbb{C}$-valued random varibles with probability distributions $\mu_{n,j}$ satisfying:

$$\int e^{\gamma\cdot x} d\mu_{n,j}(x) \le e^{c|\gamma|^2}, \quad \text{for all } \gamma \in \mathbb{R}^2$$

## Basic properties of the Wiener randomization

- Naturally associated to function spaces from time-frequency analysis:

$$\text{Modulation space:} \quad \|u\|_{M^{p,q}_s(\mathbb{R}^d)} = \Big\| \langle n \rangle^s \|\psi(D-n)u\|_{L^p_x(\mathbb{R}^d)} \Big\|_{\ell^q_n(\mathbb{Z}^d)}$$

$$\text{Wiener amalgam space:} \quad \|u\|_{W^{p,q}_s(\mathbb{R}^d)} = \Big\| \langle n \rangle^s \|\psi(D-n)u\|_{\ell^p_n(\mathbb{Z}^d)} \Big\|_{L^q_x(\mathbb{R}^d)}$$

- *No* gain of differentiability from the randomization:

  $u \in H^s(\mathbb{R}^d) \setminus H^{s+\varepsilon}(\mathbb{R}^d) \quad \Longrightarrow \quad u^\omega \in H^s(\mathbb{R}^d) \setminus H^{s+\varepsilon}(\mathbb{R}^d)$ almost surely

- Improved integrability - Paley-Zygmund theorem:

  $u \in L^2(\mathbb{R}^d) \quad \Longrightarrow \quad u^\omega \in L^p(\mathbb{R}^d)$, $2 \leq p < \infty$, almost surely

- This yields *improved probabilistic Strichartz estimates*

  $\Longleftarrow$ one of the keys of our analysis

- Also employed by Lührman-Mendelson (2014), Zhang-Fang (2012)

# Main result

**Theorem 1:** P. 2014 ($d = 4, 5$), Oh-P. 2015 ($d = 3$)

For $d = 3, 4, 5$, let $s$ satisfy:

$$\tfrac{1}{2} < s < 1 \text{ when } d = 3, \quad 0 < s < 1 \text{ when } d = 4, \quad 0 \le s < 1 \text{ when } d = 5.$$

Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^d)$, let $(u_0^\omega, u_1^\omega)$ be its Wiener randomization. Then, the energy-critical defocusing NLW is almost surely globally well-posed (a.s. GWP).

More precisely, there exists $\widetilde{\Omega} \subset \Omega$ with $P(\widetilde{\Omega}) = 1$ such that, for any $\omega \in \widetilde{\Omega}$, there exists a unique global solution of NLW with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ in the class:

$$(u, \partial_t u) \in \big(S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)\big) + C\big(\mathbb{R}; \mathcal{H}^1(\mathbb{R}^d)\big) \subset C\big(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^d)\big).$$

$S(t)$ = propagator for the linear wave equation: $S(t)(f, g) := \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g$

- This is the first a.s. GWP result for *energy-critical* hyperbolic/dispersive PDEs (with initial data below the energy space)

# Remarks

- Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^d)$, define the induced probability measure on $\mathcal{H}^s(\mathbb{R}^d)$ by
$$\mu(A) = P\big((u_0^\omega, u_1^\omega) \in A\big)$$

$\Longrightarrow$ Theorem 1 states that there exists $\Sigma \subset \mathcal{H}^s$ with $\mu(\Sigma) = 1$ such that for any $(\phi_0, \phi_1) \in \Sigma$, NLW admits a unique global solution with $(u, \partial_t u)|_{t=0} = (\phi_0, \phi_1)$

- **Question 1:** Let $\Phi(t) : (\phi_0, \phi_1) \mapsto u(t)$ denote the solution map.
  Then, is it possible that $\Phi(t)(\Sigma)$ has small measure for $t \neq 0$?
- **Answer 1:** No, $\mu\big(\Phi(t)(\Sigma)\big) = 1$ for all $t \in \mathbb{R}$

# Remarks

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- **Question 1:** Let $\Phi(t) : (\phi_0, \phi_1) \mapsto u(t)$ denote the solution map. Then, is it possible that $\Phi(t)(\Sigma)$ has small measure for $t \neq 0$?
- **Answer 1:** No, $\mu\big(\Phi(t)(\Sigma)\big) = 1$ for all $t \in \mathbb{R}$

- **Question 2:** Is the solution map continuous?
- **Answer 2:** Yes, it is continuous *in probability* $(d = 3, 4)$
  $\Longleftarrow$ notion introduced by Burq-Tzvetkov (2014). For fixed $T, R > 0$:

$$\mu \otimes \mu\Big(\big((\phi_0,\phi_1), (\phi_0', \phi_1')\big) \in (\mathcal{H}^s)^2 : \big\|\Phi(t)(\phi_0, \phi_1) - \Phi(t)(\phi_0', \phi_1')\big\|_{L^\infty([0,T];\mathcal{H}^s)} > \delta \;\Big|\; (\phi_0, \phi_1), (\phi_0', \phi_1') \in B_R \text{ and } \|(\phi_0, \phi_1) - (\phi_0', \phi_1')\|_{\mathcal{H}^s} < \eta\Big) \leq g(\delta, \eta),$$

where $\lim_{\eta \to 0} g(\delta, \eta) = 0$ for each fixed $\delta > 0$

# Globalization arguments in the probabilistic setting

- **Invariant measure argument**: "use the invariance of Gibbs measures in the place of conservation laws":
  Bourgain (1994, 1996, 1997), Tzvetkov (2006), Burq-Tzvetkov (2007, 2008), Oh (2009), Nahmod-Oh-Rey-Bellet-Staffilani (2012), De Suzzoni (2013), Deng (2012), Richards (2012), Bourgain-Bulut (2013), ...
- **Probabilistic adaptations of deterministic globalization arguments**:
  - *Probabilistic high-low decomposition method* (subcritical):
    Colliander-Oh (2012), Lührmann-Mendelson (2014)
  - *Probabilistic a priori bound on the energy* (subcritical):
    Burq-Tzvetkov (2014), Lührmann-Mendelson (2015)
  - *Probabilistic compactness method*: Nahmod-Pavlović-Staffilani (2013), Burq-Thomann-Tzvetkov (2012)
- Theorem 1 considers the energy-critical NLW

  $\Longrightarrow$ We introduce a new globalization method: *probabilistic perturbation theory*

# Probabilistic perturbation theory

- Perturbation theory was used in the deterministic setting to obtain GWP:
  - energy-critical NLS: Colliander-Keel-Staffillani-Takaoka-Tao (2008), energy-critical NLW: Kenig-Merle (2008)
  - NLS with combined power nonlinearity: Tao-Vişan-Zhang (2007), Killip-Oh-P.-Vişan (2012)

- First instance in the *probabilistic* setting to obtain a.s. GWP
- Given randomized $(u_0^\omega, u_1^\omega)$, we consider:
  - linear part $z^\omega := S(t)(u_0^\omega, u_1^\omega)$: rough but random (better integrability)
  - nonlinear part $v^\omega := u - z^\omega$: "deterministic" but smoother
- Nonlinear part $v^\omega$ satisfies the *perturbed* energy-critical NLW:

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + F(v^\omega + z^\omega) = 0 \\ (v^\omega, \partial_t v^\omega)\big|_{t=0} = (0,0), \end{cases} \qquad F(u) := |u|^{\frac{4}{d-2}} u$$

- $F(v^\omega + z^\omega) = F(v^\omega) +$ error containing $z^\omega$

  *improved probabilistic Strichartz estimates* $\Longrightarrow$ the error is "small"

# Strategy of the proof

0. **Reduction to Almost a.s. GWP**:
Given $T$ and $\varepsilon > 0$, there exists $\Omega_{T,\varepsilon}$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$ such that for $\omega \in \Omega_{T,\varepsilon}$, there exists a unique solution $u^\omega$ to NLW on $[-T, T]$

"**Almost a.s. GWP implies a.s. GWP**":
   - For fixed $\varepsilon > 0$, let $T_j = 2^j$ and $\varepsilon_j = 2^{-j}\varepsilon$
   $\implies$ By almost a.s. GWP, construct $\Omega_j := \Omega_{T_j,\varepsilon_j}$
   - Then, let $\Omega_\varepsilon = \bigcap_{j=1}^\infty \Omega_j$
   $\implies$ NLW is globally well-posed on $\Omega_\varepsilon$ with $P(\Omega_\varepsilon^c) < \varepsilon$
   - Now, let $\widetilde{\Omega} = \bigcup_{\varepsilon>0} \Omega_\varepsilon$
   $\implies$ Then, NLW is globally well-posed on $\widetilde{\Omega}$ and $P(\widetilde{\Omega}^c) = \inf_{\varepsilon>0} \varepsilon = 0$

1. **Improved probabilistic Strichartz estimates**:
local-in-time, wide range of exponents

2. **"Good" *deterministic* local well-posedness theory** for the perturbed energy-critical NLW
    - In general, the local time of existence at a critical regularity depends on the profile of initial data
    - "good" means that the local time of existence depends only on the Sobolev norm of the initial data and on the size of the perturbation:
    - Ingredient 1: global solutions to the energy-critical defocusing NLW and their global space-time bounds
    - Ingredient 2: perturbation theory
3. **Probabilistic a priori energy estimate** - difficult in dimension $d = 3$
4. **Closing the argument**: energy estimate and probabilistic Strichartz estimates allow us to apply the "good" local well-posedness iteratively

# Step 1: Probabilistic Strichartz estimates

**Basic facts:**

- *Probabilistic fact:* For any $p \geq 2$ and any $\{c_n\}_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, we have

$$\Big\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) c_n \Big\|_{L^p(\Omega)} \leq C\sqrt{p} \|c_n\|_{\ell^2_n(\mathbb{Z}^d)}$$

- Bernstein's inequality: For a smooth projection $\mathbf{P}_N$ onto frequencies $\{|\xi| \sim N\}$,

$$\|\mathbf{P}_N f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|\mathbf{P}_N f\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq q \leq \infty$$

  The same estimate holds if $\mathbf{P}_N$ is a projection onto a region of volume $\sim N^d$

  $\Longrightarrow$ projecting onto a cube of size one in the Fourier space:

$$\|\psi(D - n)\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\psi(D - n)\phi\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq q \leq \infty$$

  Namely, no loss of regularity to go from the $L^p$-norm to the $L^q$-norm, $q \geq p$

## Proposition: Improved probabilistic Strichartz estimates

Let $I = [a,b] \subset \mathbb{R}$ be a compact time interval. If $(u_0, u_1) \in \dot{\mathcal{H}}^0(\mathbb{R}^d)$, then given $1 \leq q < \infty$ and $2 \leq r < \infty$, the following holds for all $\lambda > 0$:

$$P\big( \|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I;L_x^r)} > \lambda \big) \leq C \exp\left( -c \frac{\lambda^2}{|I|^{\frac{2}{q}} \|(u_0,u_1)\|_{\dot{\mathcal{H}}^0}^2} \right)$$

- Let $\lambda = K\|(u_0,u_1)\|_{\dot{\mathcal{H}}^0}$. Then, Proposition states that a Strichartz estimate:

  $$\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I;L_x^r)} \leq K\|(u_0,u_1)\|_{\dot{\mathcal{H}}^0}$$

  holds with a large probability ($\to 1$ as $|I| \to 0$ or $K \to \infty$)
- Let $\lambda = K|I|^\theta$ with $\theta < \frac{1}{q}$. Then, Proposition states that

  $$\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I;L_x^r)} \leq K|I|^\theta$$

  holds with a large probability ($\to 1$ as $|I| \to 0$ or $K \to \infty$)
- Local-in-time Strichartz estimates combined with a simple fixed point argument yield a.s. local well-posedness (LWP)

**Proof:** We only estimate $\cos(t|\nabla|)u_0^\omega$. Let $p \geq \max(q,r)$. Then, we have

$$
\begin{aligned}
\Big(\mathbb{E}\|\cos(t|\nabla|)u_0^\omega\|^p_{L^q_t(I;L^r_x)}\Big)^{\frac{1}{p}} &\leq \Big\|\|\cos(t|\nabla|)u_0^\omega\|_{L^p(\Omega)}\Big\|_{L^q_I L^r_x} \\
&\lesssim \sqrt{p}\Big\|\|\psi(D-n)\cos(t|\nabla|)u_0\|_{\ell^2_n}\Big\|_{L^q_I L^r_x} \\
&\overset{\text{Mink.}}{\lesssim} \sqrt{p}\Big\|\|\psi(D-n)\cos(t|\nabla|)u_0\|_{L^r_x}\Big\|_{L^q_I \ell^2_n} \\
&\overset{\text{Berns.}}{\lesssim} \sqrt{p}\Big\|\|\psi(D-n)\cos(t|\nabla|)u_0\|_{L^2_x}\Big\|_{L^q_I \ell^2_n} \\
&\sim \sqrt{p}|I|^{\frac{1}{q}}\|u_0\|_{L^2_x}
\end{aligned}
$$

Then, by Chebyshev's inequality we have

$$
P\Big(\|S(t)(u_0^\omega, u_1^\omega)\|_{L^q_t(I;L^r_x)} > \lambda\Big) < \left(\frac{C|I|^{\frac{1}{q}}p^{\frac{1}{2}}\left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}}\right)}{\lambda}\right)^p
$$

$\Longrightarrow$ Strichartz estimates follow by choosing an appropriate value of $p \geq \max(q,r)$

# Step 2: Deterministic local theory

- Given $I \subset \mathbb{R}$, let $X(I) = L_t^{\frac{d+2}{d-2}}(I; L_x^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)) = \mathcal{H}^1$-admissible Strichartz space

**Proposition:** Standard local well-posedness

Let $t_0 \in \mathbb{R}$ and $I \ni t_0$. Then, there exists $\delta > 0$ sufficiently small such that if

$$\|f\|_{X(I)} \le \delta^{\frac{d-2}{d+2}} \quad \text{and} \quad \|S(t-t_0)(v_0, v_1)\|_{X(I)} \le \delta,$$

the following perturbed NLW admits a unique solution $(v, \partial_t v) \in C(I; \dot{\mathcal{H}}^1(\mathbb{R}^d))$

$$\begin{cases} \partial_t^2 v - \Delta v + F(v+f) = 0 \\ (v, \partial_t v)\big|_{t=t_0} = (v_0, v_1). \end{cases}$$

Moreover, if $T < \infty$ is the maximal time of existence of the solution $v$, then

$$\|v\|_{X([t_0, T])} = \infty$$

- Length $|I|$ of time interval depends on the profile of $(v_0, v_1)$
- we will design a "good" LWP so that $|I|$ only depends on $\|(v_0, v_1)\|_{\mathcal{H}^1}$

# "Good" deterministic local theory

**Proposition: "Good" local well-posedness**

There exists sufficiently small $\tau = \tau\big(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^d)}, K, \gamma\big) > 0$ such that, if $f$ satisfies the condition

$$\|f\|_{X([t_0, t_0+\tau_*])} \leq K\tau_*^\gamma$$

for some $0 < \tau_* \leq \tau$, then the perturbed NLW

$$\begin{cases} \partial_t^2 v - \Delta v + F(v+f) = 0 \\ (v, \partial_t v)\big|_{t=t_0} = (v_0, v_1) \end{cases}$$

admits a unique solution $v$ in $C([t_0, t_0+\tau_*]; \mathcal{H}^1(\mathbb{R}^d))$. Moreover,

$$\|(v, \partial_t v)\|_{L_t^\infty([t_0,t_0+\tau_*];\dot{\mathcal{H}}^1)} + \|v\|_{L_t^q([t_0,t_0+\tau_*];L_x^r(\mathbb{R}^d))} \leq C\big(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^d)}\big),$$

for all wave admissible pairs $(q, r)$, where $C(\cdot)$ is a positive non-decreasing function.

- Time length depends only on the *size* of initial data (and the perturbation)

# Perturbation theory

**Proposition:** Perturbation theory

Let $v$ be a solution of the perturbed equation:

$$\partial_t^2 v - \Delta v + |v|^{\frac{4}{d-2}} v = e,$$

with initial data $(v, \partial_t v)\big|_{t=t_0} = (v_0, v_1) \in \mathcal{H}^1(\mathbb{R}^d)$, satisfying $\|v\|_{X(I)} \leq M$.

Given $(w_0, w_1) \in \mathcal{H}^1(\mathbb{R}^d)$, let $w$ be the solution of the energy-critical NLW with initial data $(w, \partial_t w)\big|_{t=t_0} = (w_0, w_1)$. Then, there exists $\varepsilon > 0$ sufficiently small such that if

$$\|(v_0 - w_0, v_1 - w_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^d)} \leq \varepsilon \quad \text{and} \quad \|e\|_{L_t^1(I; L_x^2(\mathbb{R}^d))} \leq \varepsilon,$$

then the following holds for all wave admissible pairs $(q, r)$:

$$\sup_{t \in I} \|(v(t) - w(t), \partial_t v(t) - \partial_t w(t))\|_{\dot{\mathcal{H}}^1} \leq C(M)\varepsilon$$

- This proposition follows from iterative application of local argument via deterministic Strichartz estimates

## Proof of the "good" LWP, $d = 4$:

- Strichartz space $X(I) = L_t^3\big(I; L_x^6(\mathbb{R}^4)\big)$, Nonlinearity $F(u) = |u|^2 u$
- Suffices to find $\tau$ such that $\|v\|_{L^3([t_0,t_0+\tau],L_x^6(\mathbb{R}^4))} \leq C\left(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^4)}\right)$
- **key fact:** global solution $w$ to the energy-critical defocusing NLW with initial data $(w, \partial_t w)_{|t=t_0} = (v_0, v_1) \in \mathcal{H}^1$ satisfies

$$\|w\|_{L^3(\mathbb{R},L_x^6(\mathbb{R}^4))} < C\left(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^4)}\right)$$

$\Longleftarrow$ concentration-compactness: Bahouri-Gérard (1999)

## Proof of the "good" LWP, $d = 4$:

- Strichartz space $X(I) = L^3_t\big(I; L^6_x(\mathbb{R}^4)\big)$, Nonlinearity $F(u) = |u|^2 u$
- Suffices to find $\tau$ such that $\|v\|_{L^3([t_0,t_0+\tau],L^6_x(\mathbb{R}^4))} \leq C\left(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^4)}\right)$
- **key fact:** global solution $w$ to the energy-critical defocusing NLW with initial data $(w, \partial_t w)_{|t=t_0} = (v_0, v_1) \in \mathcal{H}^1$ satisfies

$$\|w\|_{L^3(\mathbb{R},L^6_x(\mathbb{R}^4))} < C\left(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^4)}\right)$$

$\Longleftarrow$ concentration-compactness: Bahouri-Gérard (1999)

- Divide $[t_0, t_0 + \tau]$ into $J = J(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^4)}, \eta)$ sub-intervals $I_j = [t_j, t_{j+1}]$ such that $\|w\|_{L^3(I_j,L^6_x(\mathbb{R}^4))} \sim \eta \ll 1$
- On $I_0$: Verify the hypotheses of Perturbation theory via Duhamel formula and deterministic Strichartz estimates
- Perturbation theory on $I_0$ $\Longrightarrow$ $\sup_{t\in I_0} \big\|(v - w, \partial_t v - \partial_t w)\big\|_{\mathcal{H}^1} \leq C(4\eta)\varepsilon$
  $\Longrightarrow$ In particular, $\big\|\big(v(t_1) - w(t_1), \partial_t v(t_1) - \partial_t w(t_1)\big)\big\|_{\mathcal{H}^1} \leq C(4\eta)\varepsilon$
- Apply iteratively the perturbation proposition on the intervals $I_j$, $j = 1, \ldots, J = J(\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^d)}, \eta)$
- In the end, we obtain a condition on $\tau$ depending only on $\|(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{R}^d)}$, $K$, and $\gamma$

# Step 3: Probabilistic a priori energy estimate

In Step 2, we showed that the time of local existence depends only on the $\mathcal{H}^1$-norm $\Longrightarrow$ we need a *long-time* energy estimate with *large probability*

**Proposition:** Probabilistic energy bound

Let $d = 4$ or $5$, $0 < \varepsilon \ll 1$, $T > 0$. Then, there exists a set $\tilde{\Omega}_{T,\varepsilon} \subset \Omega$ with $P(\tilde{\Omega}^c_{T,\varepsilon}) < \varepsilon$ such that for all $t \in [0,T]$ and all $\omega \in \tilde{\Omega}_{T,\varepsilon}$:

$$\big\|(v^\omega, \partial_t v^\omega)\big\|_{L^\infty_t([0,T];\mathcal{H}^1(\mathbb{R}^d))} \leq C\left(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{R}^d)}\right)$$

- $(d=4)$ By taking a time derivative of $E\big(v^\omega(t)\big)$ and Gronwall's inequality:

$$\Big(E\big(v^\omega(t)\big)\Big)^{\frac{1}{2}} \leq C\big\|z^\omega\big\|^3_{L^3([0,T];L^6_x(\mathbb{R}^4))} e^{C\|z^\omega\|_{L^1_t([0,T];L^\infty_x(\mathbb{R}^4))}}$$

- By probabilistic Strichartz estimates, there exists $\tilde{\Omega}_{T,\varepsilon} \subset \Omega$ with $P(\tilde{\Omega}^c_{T,\varepsilon}) < \varepsilon$ such that for any $\omega \in \tilde{\Omega}_{T,\varepsilon}$:

$$\|z^\omega\|_{L^3([0,T],L^6_x)\cap L^1([0,T];L^\infty_x)} \leq KT^\theta\|u_0\|_{\mathcal{H}^s(\mathbb{R}^4)}$$

# Step 4: Closing the argument

**Goal:** Prove "almost" almost sure GWP:

Given $\varepsilon \ll 1$ and $T \gg 1$, there exists $\Omega_{T,\varepsilon} \subset \Omega$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$ such that for any $\omega \in \Omega_{T,\varepsilon}$ NLW admits a unique solution $u^\omega$ on $[0,T]$

- Probabilistic energy estimate (Step 3): for any $\omega \in \tilde{\Omega}_{T,\varepsilon}$,

$$\sup_{t\in[0,T]} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\big(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{R}^d)}\big)$$

- By probabilistic Strichartz estimates, there exists $\hat{\Omega}_{T,\varepsilon}$ with $P(\hat{\Omega}_{T,\varepsilon}^c) < \varepsilon$ such that

$$\|z^\omega\|_{X([k\tau,(k+1)\tau]\times\mathbb{R}^d)} \leq K\tau^\gamma, \quad k = 0, 1, \dots$$

- For any $\omega \in \Omega_{T,\varepsilon} := \tilde{\Omega}_{T,\varepsilon} \cap \hat{\Omega}_{T,\varepsilon}$, the hypotheses of the "good" LWP are satisfied on each $[k\tau,(k+1)\tau]$ with $\tau = \tau\big(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{R}^d)}, K, \gamma\big)$
- Apply iteratively the "good" local well-posedness for the perturbation $f = z^\omega$

# On $\mathbb{R}^3$

**Theorem 2:** Oh-P. 2015

The energy-critical defocusing quintic NLW on $\mathbb{R}^3$ is a.s. GWP with rough random data below the energy space.

**Main difficulty:** Probabilistic energy bound for nonlinear part $v^\omega$

$$\left| \frac{d}{dt} E\big(v^\omega(t)\big) \right| \leq \|\partial_t v^\omega\|_{L^2_x} \|F\big(v^\omega + z^\omega\big) - F\big(v^\omega\big)\|_{L^2_x}$$

and

$$\|z^\omega |v^\omega|^{\frac{4}{d-2}}\|_{L^2(\mathbb{R}^d)} \leq \|z^\omega\|_{L^\infty(\mathbb{R}^d)} \|v^\omega\|^{\frac{4}{d-2}}_{L^{\frac{8}{d-2}}(\mathbb{R}^d)}$$

But $E(v^\omega)$ only controls the $L^p(\mathbb{R}^d)$-norms of $v^\omega$ for $2 \leq p \leq \frac{2d}{d-2}$

$\Longrightarrow$ one needs $\frac{8}{d-2} \leq \frac{2d}{d-2} \quad \Longleftrightarrow \quad d \geq 4$

**New ingredients:**

- Integration by parts in time
- New probabilistic Strichartz estimate involving $L^\infty_t$:

$$P\big(\|S(t)(u_0^\omega, u_1^\omega)\|_{L^\infty_t([0,T];L^r_x(\mathbb{R}^3))} > \lambda\big) \lesssim T \exp\left( - c \frac{\lambda^2}{\max(1, T^2)\|(u_0, u_1)\|^2_{\mathcal{H}^\varepsilon(\mathbb{R}^3)}} \right)$$

# Periodic NLW

**Theorem 3:** Oh-P. 2015

The energy-critical defocusing NLW on $\mathbb{T}^d$, $d = 3, 4, 5$ is a.s. GWP with rough random data below the energy space.

- Suffices to prove almost a.s. GWP
- Reduce the problem on $\mathbb{T}^d$ to $\mathbb{R}^d$ by the "finite speed of propagation" of NLW

$$\begin{cases} \partial_t^2 \mathbf{u}^\omega - \Delta \mathbf{u}^\omega + |\mathbf{u}^\omega|^{\frac{4}{d-2}} \mathbf{u}^\omega = 0 \\ (\mathbf{u}^\omega, \partial_t \mathbf{u}^\omega)\big|_{t=0} = (\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T}), \end{cases} \qquad (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T}) = (\eta_T u^\omega_0, \eta_T u^\omega_1)$ with smooth cutoff $\eta_T(x) \equiv 1$ on $\langle T \rangle \cdot \mathbb{T}^d$

$$\Longrightarrow \quad \widehat{\mathbf{u}^\omega_{j,T}}(\xi) = \widehat{\eta_T u^\omega_j}(\xi) = \sum_{n \in \mathbb{Z}^d} \widehat{\eta_T}(\xi - n) g_{n,j}(\omega) \widehat{u}_j(n), \quad j = 0, 1$$

**Two issues:**

- Given $\xi \in \mathbb{R}^d$, we see infinitely many $g_n$'s

  $\Longrightarrow$ New probabilistic Strichartz estimates are needed

- Must justify the finite speed of propagation for rough solutions

# On nonlinear Schrödinger equations

*Conditional* a.s. GWP of the cubic NLS on $\mathbb{R}^d$, $d \geq 3$

**Theorem 4:** Bényi-Oh-P. 2015

The defocusing cubic NLS on $\mathbb{R}^d$, $d \geq 3$ is a.s. GWP with rough random data below the scaling critical regularity, provided that

1. probabilistic a priori energy bound for the nonlinear part
2. ($d \neq 4$) global space-time bound for solutions to deterministic cubic NLS

- Space-time bound holds in the energy-critical case $d = 4$ (Rickman-Vişan 2007)
  When $d \neq 4$, this question is widely open
- Condition 1 is in the spirit of the *conditional* GWP in $H^{s_{\text{crit}}}(\mathbb{R}^d)$ of the energy-supercritical NLW and NLS (Kenig-Merle 2010, Killip-Vişan 2010):
- For NLS, we need to use more intricate spaces than that for NLW:
  Fourier restriction norm method & $U^2$, $V^2$ spaces

Q: Probabilistic a priori energy estimate when $d = 4$?

**Open problem**: Prove almost sure scattering (linear asymptotic behavior) for NLW (and NLS) on $\mathbb{R}^d$ with rough random large data

- Small data: probabilistic *global-in-time* Strichartz estimates and a standard fixed point argument (Lührmann-Mendelson 2014)
- Standard way of proving scattering is via global space-time bounds
- Unfortunately, all our estimates are on finite time intervals $[0, T]$ and space-time bounds (on the nonlinear part) grow to $\infty$ as $T \to \infty$
- New ideas are needed