ADVANCED PDE II - HOMEWORK 1

PIETER BLUE AND OANA POCOVNICU

Problem 1. Let $I \subset \mathbb{R}$ be an interval and let $E : I_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ and $B : I_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ be vector fields representing the electric and magnetic fields. The *Maxwell's equations* are given by:

$$\begin{aligned} \partial_t E &= \nabla \times B \\ \partial_t B &= -\nabla \times E \\ \nabla \cdot E &= 0 \\ \nabla \cdot B &= 0. \end{aligned}$$

Show that the components of these vector fields, E_i and B_i, $i = 1, 2, 3$, solve the linear wave equations:

$$\partial_t^2 E_i - \Delta E_i = 0, \qquad \partial_t^2 B_i - \Delta B_i = 0.$$

Conversely, let $E_0, B_0 : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $\nabla \cdot E_0 = \nabla \cdot B_0 = 0$. Show that the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 E - \Delta E = 0 \\ \partial_t^2 B - \Delta B = 0 \\ E\big|_{t=0} = E_0, \quad B\big|_{t=0} = B_0 \\ \partial_t E\big|_{t=0} = \nabla \times B_0, \qquad \partial_t B\big|_{t=0} = -\nabla \times E_0 \end{cases}$$

is a solution of Maxwell's equations.

Here, given a vector field $F = \langle F_1, F_2, F_3 \rangle$, we used the notations $\nabla \cdot F = \operatorname{div} F = \sum_{i=1}^3 \partial_{x_i} F_i$ and $\nabla \times F = \operatorname{curl} F = \langle \partial_{x_2} F_3 - \partial_{x_3} F_2,\ \partial_{x_3} F_1 - \partial_{x_1} F_3,\ \partial_{x_1} F_2 - \partial_{x_2} F_1 \rangle$.

Problem 2 (Alinhac's book on hyperbolic PDEs)**.** Let $u \in C^2(\mathbb{R} \times [0, T))$ be a solution of the Cauchy problem for the inviscid Burger's equation

$$\begin{cases} \partial_t u + u \partial_x u = 0 \\ u(x, 0) = h(x). \end{cases}$$

Let $x(t) = h(s)t + s$ be the projected characteristic starting from $(s, 0)$. Show that the function $q(t) := (\partial_x u)(x(t), t)$ satisfies $q' + q^2 = 0$. Compute q explicitly.

Show that if $h'(s) < 0$, then $q(t)$ becomes infinite as t approaches the value $t(s) = -\frac{1}{h'(s)} > 0$. Deduce from this that, given $h \in C^2(\mathbb{R})$, there exists a C^2 solution u of the above Cauchy problem exactly in the strip $\{0 \le t < \bar{T}\}$, where

$$\bar{T} := [\max(-h')]^{-1}.$$

The number $\bar{T}$ is called the *lifespan* of the smooth solution.

Problem 3 (Alinhac's book on hyperbolic PDEs)**.** Consider the Cauchy problem

$$\begin{cases} \partial_t u + u\partial_x u = u^2 \\ u(x,0) = u_0(x), \end{cases}$$

where $u_0 \in C_0^2(\mathbb{R})$ is compactly supported and not identically zero.
(a) Show the inequalities $\max(u_0 - u_0') > 0$, $\max(u_0 - u_0') \geq \max u_0$. Prove that if there exists x_0 such that

$$u_0(x_0) = \max u_0 > 0, \quad u_0''(x_0) < 0,$$

then $\max(u_0 - u_0') > \max u_0$.
(b) Use the method of characteristics to solve the Cauchy problem. Prove that a C^2 solution exists for

$$0 \leq t < \bar{T} := [\max(u_0 - u_0')]^{-1}.$$

(c) Let $u \in C^2(\mathbb{R} \times [0,T))$ be a solution of the above Cauchy problem. Denote by $(x(t),t)$ the projected characteristic starting from $(x_0, 0)$. Set $q(t) := (\partial_x u)(x(t),t)$. Establish the ODE satisfied by q and compute q explicitly. Deduce from this that $T \leq \bar{T}$ and hence $\bar{T}$ is the lifespan of the smooth solution.

Problem 4 (Evan's book)**.** Compute explicitly a shock solution of the inviscid Burger's equation

$$\partial_t u + u\partial_x u = 0, \quad t \geq 0,$$

with initial condition

$$u(x,0) = h(x) = \begin{cases} 1, & \text{if} \quad x < -1 \\ 0, & \text{if} \quad -1 < x < 0 \\ 2, & \text{if} \quad 0 < x < 1 \\ 0, & \text{if} \quad x > 1. \end{cases}$$

Problem 5 (Sogge's book)**.** Let $f \in C^3(\mathbb{R}^2)$, $g \in C^2(\mathbb{R}^2)$ supported on $\{x : |x| < R\}$. Use Poisson's formula to deduce that the solution $u \in C^2(\mathbb{R}^2)$ of the linear wave equation $\partial_t^2 u - \Delta u = 0$ with initial data $u(x,0) = f$, $\partial_t u(x,0) = g$, satisfies the decay estimate

$$|u(x,t)| \leq \frac{C}{\langle t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}}}.$$

Problem 6 (Bernstein's inequalities)**.** Prove the following Bernstein's inequalities:

(a) $\big\||\nabla|^s \mathbf{P}_N f\big\|_{L^p(\mathbb{R}^d)} \sim N^s \|\mathbf{P}_N f\|_{L^p(\mathbb{R}^d)}$,

(b) $\|\mathbf{P}_N f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|\mathbf{P}_N f\|_{L^p(\mathbb{R}^d)}$,

(c) $\|\mathbf{P}_{\leq N} f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|\mathbf{P}_{\leq N} f\|_{L^p(\mathbb{R}^d)}$,

for $1 \leq p \leq q \leq \infty$.

Hint: Use Young's inequality and the scaling property of the Fourier transform: $\mathcal{F}(f(\frac{\cdot}{N}))(\xi) = N^d \hat{f}(N\xi)$.

Problem 7 (Van der Corput lemma and consequences, Stein's book)**.** (a) Let $a, b \in \mathbb{R}$, $a < b$ and let $\phi : [a, b] \to \mathbb{R}$ be a smooth function such that $|\partial_x^k \phi(x)| \geq 1$ for all $x \in (a, b)$. Then

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-\frac{1}{k}}, \quad \forall \lambda \in \mathbb{R}$$

holds when
(i) $k = 1$ and ϕ' is monotonic or
(ii) $k \geq 2$.

The bound c_k is independent of ϕ.

Hint: Use integration by parts in (i) and induction on k in (ii). For (ii), it might be useful to consider $c \in [a, b]$ such that $|\partial_x^k \phi(c)| = \min_{x \in [a,b]} |\partial_x^k \phi(x)|$ and discuss the cases $\partial_x^k \phi(c) = 0$ and $\partial_x^k \phi(c) \neq 0$.

(b) Under the same assumptions on ϕ as in (a), show that

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right], \quad \forall \lambda \in \mathbb{R}$$

for all smooth functions $\psi : [a, b] \to \mathbb{C}$.

Hint: Set $F(x) := \int_a^x e^{i\lambda\phi(x)} dx$ for $x \in [a, b]$ and use integration by parts.

(c) (Asymptotics of Bessel functions) For $m \in \mathbb{Z}$, we define the Bessel function of order m by

$$J_m(r) := \int_0^{2\pi} e^{ir\sin\theta} e^{-im\theta} d\theta.$$

Use part (b) to deduce that

$$J_m(r) = O(r^{-\frac{1}{2}}) \quad \text{as} \quad r \to \infty.$$