ADVANCED PDE II - HOMEWORK 2

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Problem 1. In $\mathbb{R}^{1+n}$, the Fourier transform of a solution to the wave equation can be written as

$$\hat{u}(t,\vec{\xi}) = e^{i|\vec{\xi}|t} f_+(\vec{\xi}) + e^{-i|\vec{\xi}|t} f_-(\vec{\xi}). \tag{1}$$

Show that if f_+ and f_- are in $L^2(\mathrm{d}\xi)$, then $u(t,\vec{\xi})$ is a continuous function of t taking values in $L^2(\mathrm{d}\xi)$.

Problem 2.

(1) State the Hahn-Banach theorem.

(2) Let $n, s \in \mathbb{Z}^+$. Let $T > 0$. Let $F \in L^1([0,T]; H^s(\mathbb{R}^n))$ (although this doesn't matter). Let L be a differential operator and L^* be the formal adjoint, i.e. such that for all $\phi, \psi \in C_0^\infty((0,T)\times\mathbb{R}^n)$, $\int_0^T \int_{\mathbb{R}^n} \phi(L\psi)\mathrm{d}^n x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n} (L^*\phi)\psi \mathrm{d}^n x \mathrm{d}t$.

Suppose that for all $\psi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$

$$|\langle F, \psi\rangle| \le C \int_0^T \|(L^*\psi)(t,x)\|_{H_x^{-s-1}} \mathrm{d}t.$$

Show that there is $W \in (L^1([0,T]; H^{-s-1}))^*$ such that, for all $\psi \in C_0^\infty((-\infty,T)\times \mathbb{R}^n)$

$$W(L^*\psi) = \int_0^T \psi F \mathrm{d}^n x \mathrm{d}t.$$

Problem 3. Let $n \ge 1$. Consider the quasilinear wave equation

$$(G^{ij}\partial_i\partial_j + B^j\partial_j + A)u = F$$

with G, B, A, F satisfying condition $1(\infty,\Omega)$ holds and G is $1/100$ close to η. Suppose u is a solution. Further suppose $|F| < C|u|_1$ everywhere. Suppose $R > 1$.

Show that if there is an $R > 0$ such that $u(0,\vec{x})$ and $\partial_t u(0,\vec{x})$ both vanish for $|\vec{x}| > R$, then $u(t,\vec{x})$ vanishes for all $|\vec{x}| > R + 2t$.

Problem 4. Recall:

Definition 1.1. *Let U be a metric space and V be a complete metric space. Let $\Phi : U \times V \to V$. Φ is* ***uniformly continuous*** *in U if*

$$\forall x_1 \in U, \epsilon > 0 : \exists \delta > 0 : \forall x_2 \in U, y \in V : \quad \|x_2 - x_1\|_U < \delta \implies \quad \|\Phi(x_2,y) - \Phi(x_1,y)\|_V < \epsilon.$$

Φ is a ***uniform contraction mapping*** *in V if*

$$\exists r \in [0,1) : \forall x \in U; y_1, y_2 \in V : \qquad \|\Phi(x,y_2) - \Phi(x,y_1)\|_V \le r\|y_2 - y_1\|_V.$$

Theorem 1.2. *Let U be a metric space and V be a complete metric space.*

If $\Phi : U \times V \to V$ is uniformly continuous in U and a uniform contraction mapping in V, then there is a map $S : U \to V$ such that

(1) $\forall x \in U : \Phi(x, S(x)) = S(x)$;
(2) *If* $\Phi(x, y) = y$, *then* $y = S(x)$; *and*
(3) *The map* $S : U \to V$ *is continuous.*

Prove this theorem.

Problem 5. For $T > 0$, functions f, g on $\mathbb{R}^3$, and a function v on $[0, T] \times \mathbb{R}$, define $\Phi((f, g), v)$ to be the solution u of the initial value problem

$$-\partial_t^2 u + \sum_{i=1}^{3} \partial_i^2 u - u = v^3,$$
$$u(0, x) = f(x),$$
$$\partial_t u(0, x) = g(x).$$

Show that for any $(f, g) \in H^2 \times H^1$, there is a closed ball U in $H^2 \times H^1$, a $T > 0$, and a closed ball V in $C^0([0, T]; H^2) \cap C^1([0, T]; H^1)$, such that Φ is uniformly continuous in U and a uniform contraction mapping in V. [Hint: Prove an energy estimate]

State and prove a theorem about the well-posedness of $-\partial_t^2 u + \sum_{i=1}^{3} \partial_i^2 u - u = u^3$ in $\mathbb{R}^{1+3}$.

Problem 7. Let $L : \mathbb{R}^{(1+n)+1+(1+n)} \to \mathbb{R}$. If no argument is given, assume $L = L(x, u, \partial u) = L(x, u(x), \partial u(x))$, where $x \in \mathbb{R}^{1+n}$, $u : \mathbb{R}^{1+n} \to \mathbb{R}$ and ∂ denotes differentiation in $\mathbb{R}^{1+n}$. Use $\frac{\delta L}{\delta x^i}$ to denote the derivative of L with respect to its ith argument, use $\frac{\delta L}{\delta u}$ to denote its derivative with respect to its $((n + 1) + 1)$th argument, and $\frac{\delta L}{\delta \partial_i u}$ to denote its derivative with respect to its $((n + 1) + 1 + i)$th argument. Observe that the chain rule gives

$$\partial_i L = \frac{\delta L}{\delta x^i} + \frac{\delta L}{\delta u} \partial_i u + \sum_{j=0}^{n} \frac{\delta L}{\delta \partial_j u} \partial_j \partial_i u.$$

(Observe also that δ^i_j still denotes the Kronecker delta.) u is said to satisfy the Euler-Lagrange equations[1] if

$$\sum_{i=0}^{n} \partial_i \frac{\delta L}{\delta \partial_i u} - \frac{\delta L}{\delta u} = 0.$$

(1) Let

$$\mathcal{T}^i{}_j = \frac{\delta L}{\delta \partial_i u} \partial_j u - \delta^i_j L,$$
$$\mathcal{P}^i = \sum_{j=0}^{n} \mathcal{T}^i{}_j X^j.$$

[1] In APDE I, you should have seen that u solves the Euler-Lagrange equation, then it is critical point of $S = \int L \mathrm{d}^{1+n} x$. In elliptic problems, one typically looks for minimisers of S. Unfortunately, in hyperbolic problems, typically critical points of S are always saddle points, since S is unbounded above and below.

Show that if u satisfies the Euler-Lagrange equations, then

$$\sum_{i=0}^{n} \partial_i \mathcal{T}^i{}_j = -\frac{\delta L}{\delta x^j}.$$

(2) Find $\mathcal{T}^i{}_j$ if $L(x, u, \partial u) = \eta^{ij}(\partial_i u)(\partial_j u)$ for some constant η^{ij}.

Problem 7. Consider the inviscid Burgers' equation in $\mathbb{R}^{1+1}$,

$$\partial_t u + u \partial_x u = 0.$$

(1) Suppose $t > 0$ and u is a C^2 solution of this equation n $[0, t] \times \mathbb{R}$ and that uniformly in t, for $|x|$ sufficiently, $u(t, x) = 0$. Show

$$\|u(t, x)\|_{L^2_x} = \|u(0, x)\|_{L^2_x}.$$

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be in the Schwarz class. Consider $u_{-1}(t, x) = 0$ and, for $n \geq 0$, u_n defined by

$$\partial_t u_n + u_{n-1} \partial_x u_n = 0,$$
$$u_n(0, x) = f(x).$$

  (a) Show that there is a $T > 0$ such that for all $n \in \mathbb{N}$, $\sup_{t \in [0,T]} \|u\|_{H^3} < 2\|f\|_{H^3}$.
  (b) Show that the u_n converge in H^2 to a limit u.
  (c) Using various convergence properties, show that u is a C^1 function on $[0, T] \times \mathbb{R}$ and a solution of the inviscid Burgers' equation.