ADVANCED PDE II - HOMEWORK 1

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Please submit your work by e-mail to pblue@ed.ac.uk and o.pocovnicu@hw.ac.uk by the 25th of February. Please let us know as soon as possible if you find any typos or if something is unclear.

Problem 1 (P.B.)**.** In $\mathbb{R}^{1+n}$, the Fourier transform of a solution to the linear wave equation can be written as

$$\hat{u}(t,\vec{\xi}) = e^{i|\vec{\xi}|t} f_+(\vec{\xi}) + e^{-i|\vec{\xi}|t} f_-(\vec{\xi}), \quad \text{for all} \quad (t,\vec{\xi}) \in \mathbb{R}^{1+n}.$$

Show that if f_+ and f_- are in $L^2(\mathrm{d}\xi)$, then $u(t,\vec{\xi})$ is a continuous function of t taking values in $L^2(\mathrm{d}\xi)$.

Problem 2 (O.P.)**.** Let $(u_0,u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. We consider the (energy-subcritical) cubic nonlinear wave equation on $\mathbb{R}^3$:

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3, \\ u(0) = u_0 \\ \partial_t u(0) = u_1. \end{cases} \tag{NLW}$$

Let

$$\Gamma u := S(t)(u_0,u_1) - \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} u^3(s) ds,$$

where $S(t)(u_0,u_1)$ denotes the solution of the linear wave equation with initial data (u_0,u_1) at $t=0$. We have seen in Lecture 4 that there exists $T = T(\|(u_0,u_1)\|_{\dot{H}^1\times L^2})$ such that Γ is a contraction on the ball

$$B_R := \{f \in L^\infty\big([0,T);\dot{H}^1(\mathbb{R}^3)\big) : \|f\|_{L^\infty([0,T);\dot{H}^1)} \le R\}$$

with $R := 2\|(u_0,u_1)\|_{\dot{H}^1\times L^2}$. By Banach's fixed point theorem and Duhamel's formula, this shows that (NLW) admits a unique solution in B_R.

a) Show that this solution belongs to the class

$(u,\partial_t u) \in C([0,T);\dot{H}^1(\mathbb{R}^3)) \times \mathbf{C}([0,T);L^2(\mathbb{R}^3))$. $\quad$ ($\mathbf{C^1}$ removed from $\mathbf{C^1([0,T);L^2}(\mathbb{R}^\mathbf{3}))$)

[As a first step, show that $\partial_t u \in L^\infty\big([0,T);L^2(\mathbb{R}^3)\big)$.]

b) Prove the uniqueness of the solution in the class $C([0,T);\dot{H}^1(\mathbb{R}^3)) \times \mathbf{C}([0,T);L^2(\mathbb{R}^3))$ **(**C^1 **removed from** $C^1([0,T);L^2(\mathbb{R}^3))$**).**

[From the Banach fixed point theorem, we only obtained the uniqueness in the ball B_R.]

c) (Stability with respect to initial data) Show that $(u_0,u_1) \mapsto u$ is locally Lipschitz continuous as a map from $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ to $C([0,T);\dot{H}^1(\mathbb{R}^3))$.

[Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. A mapping $f : X \to Y$ is called locally Lipschitz continuous if for every $R > 0$ there exists $K > 0$ such that for every $x, x' \in X$ with $\|x\|_X \leq R$ and $\|x'\|_X \leq R$ the following holds: $\|f(x) - f(x')\|_Y \leq K\|x - x'\|_X$.]

d) (Persistence of regularity) Let $s > 1$ and assume that $(u_0, u_1) \in \mathbf{H^s}(\mathbb{R}^\mathbf{3}) \times \mathbf{H^{s-1}}(\mathbb{R}^\mathbf{3})$ (**dots removed from** H^s **and** H^{s-1}). Prove that there exist $T' > 0$ and a unique solution of (NLW) in the class

$$(u, \partial_t u) \in C([0, T'); \mathbf{H^s}) \times \mathbf{C}([0, T'); \mathbf{H^{s-1}}).$$

(C^1 removed from $C^1([0, T'); H^{s-1}(\mathbb{R}^3))$ and dots also removed from H^s and H^{s-1}.)

<u>Hint</u>: You might want to use the fractional Leibniz rule (also known as the Kato-Ponce inequality), which states the following. Let $s > 0$, $1 < r < \infty$, $1 < p_1, p_2, q_1, q_2 \leq \infty$ such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then, there exists $C = C(s, n, r, p_1, p_2, q_1, q_2)$ such that

$$\|\langle\nabla\rangle^s(fg)\|_{L^r(\mathbb{R}^n)} \leq C\big(\|\langle\nabla\rangle^s f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{q_1}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|\langle\nabla\rangle^s g\|_{L^{q_1}(\mathbb{R}^n)}\big),$$

for any f and g such that all the norms on the right hand side of the inequality are finite. Here $\langle\nabla\rangle := \sqrt{1 + |\nabla|^2}$ is the Fourier multiplier defined by $\langle\nabla\rangle f := \mathcal{F}^{-1}(\sqrt{1 + |\xi|^2}\hat{f}(\xi))$.

e) Recall the definition of the energy of a solution u of (NLW):

$$E(u(t), \partial_t u(t)) := \int_{\mathbb{R}^3} \frac{(\partial_t u(t, x))^2 + |\nabla u(t, x)|^2}{2} + \frac{u^4(t, x)}{4} dx.$$

Let I be a time interval containing zero. Using similar arguments to those in Theorem 3.1 in Lecture 2 (Step 1), one can prove rigorously the conservation of the energy $E(u(t), \partial_t u(t)) = E(u_0, u_1)$ for all $t \in I$, for classical solutions of (NLW) $u \in C^3_{t,x}(I \times \mathbb{R}^3)$ such that $(u, \partial_t u) \in C(I; H^s(\mathbb{R}^3)) \times \mathbf{C}(I; H^{s-1}(\mathbb{R}^3))$ **(C^1 removed from** $C^1(I; H^{s-1}(\mathbb{R}^3))$ with s sufficiently large. [You do not need to prove this.]

Use this and some of the above points to prove rigorously the conservation of the energy for a local-in-time solution of (NLW) with initial data in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

[Recall that the 'proof of the energy conservation' that we saw in Lecture 4 only works for smooth solutions such that $u(t, \cdot)$ is compactly supported for all $t \in I$, since we advantageously applied the divergence theorem.]

Problem 3 (P.B.)**.** (1) State the Hahn-Banach theorem.

(2) Let $n, s \in \mathbb{Z}^+$. Let $T > 0$. Let $F \in L^1([0, T]; H^s(\mathbb{R}^n))$ (although this doesn't matter). Let L be a **linear** differential operator and L^* be the formal adjoint, i.e. such that for all $\phi, \psi \in C_0^\infty((0, T) \times \mathbb{R}^n)$, $\int_0^T \int_{\mathbb{R}^n} \phi(L\psi)\mathrm{d}^n x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n} (L^*\phi)\psi\mathrm{d}^n x\mathrm{d}t$.

Suppose that for all $\psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$

$$|\langle F, \psi\rangle| \leq C \int_0^T \|(L^*\psi)(t, x)\|_{H_x^{-s-1}}\mathrm{d}t.$$

Show that there is $W \in (L^1([0, T]; H^{-s-1}))^*$ such that, for all $\psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$

$$W(L^*\psi) = \int_0^T \psi F\mathrm{d}^n x\mathrm{d}t.$$