ADVANCED PDE II - HOMEWORK 2

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For problems 2, 3, 4, you can choose between either solving problems 2 & 3 or solving problem 4.

Please submit your work by e-mail to pblue@ed.ac.uk and o.pocovnicu@hw.ac.uk by the **28th of March**. Please let us know as soon as possible if you find any typos or if something is unclear.

Problem 1 (O.P.)**.** Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$ and define the backward light cone with vertex at (t_0, x_0) to be $K(t_0, x_0) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : |x - x_0| < t_0 - t\}$. (If $t_0 > 0$, $K(t_0, x_0) \cap (\mathbb{R}_+ \times \mathbb{R}^3)$ coincides with $\mathcal{D}_+(B(x_0, t_0), 0, t_0)$ used in lectures.) Let u be a classical solution of the energy-critical NLW, $-\partial_t^2 u + \Delta u = u^5$, on $K(t_0, x_0)$. The purpose of this problem is to show the non-concentration of the potential energy of u on $K(t_0, x_0)$.

For questions 1-7 bellow, we will be assuming that $(t_0, x_0) = (0, \vec{0})$.

1) We have seen in Lecture 4 the following identity:

$$\partial_t \left(\frac{(\partial_t u)^2 + |\nabla u|^2}{2} + \frac{u^6}{6} \right) - \operatorname{div}(\partial_t u \nabla u) = 0. \tag{1}$$

By multiplying NLW by $\partial_{x_i} u$ and u respectively, show that the following two algebraic identities also hold:

$$\partial_t(\partial_t u \partial_{x_i} u) - \partial_{x_i} \left(\frac{(\partial_t u)^2 - |\nabla u|^2}{2} - \frac{u^6}{6} \right) - \operatorname{div}(\partial_{x_i} u \nabla u) = 0, \qquad i = 1, 2, 3, \tag{2}$$

and

$$\partial_t(u \partial_t u) - \operatorname{div}(u \nabla u) = (\partial_t u)^2 - |\nabla u|^2 - u^6. \tag{3}$$

2) Multiplying (1), (2), (3) by t, x, and 1 respectively and adding up the results, show the following scaling identity:

$$\partial_t(tQ) - \operatorname{div}(tP) + R = 0, \tag{4}$$

where

$$Q := \frac{(\partial_t u)^2 + |\nabla u|^2}{2} + \frac{u^6}{6} + \partial_t u \left(\frac{x}{t} \cdot \nabla u + \frac{u}{t} \right) \tag{0.1}$$

$$P := \frac{x}{t} \left(\frac{(\partial_t u)^2 - |\nabla u|^2}{2} - \frac{u^6}{6} \right) + \partial_t u \nabla u + \nabla u \left(\frac{x}{t} \cdot \nabla u + \frac{u}{t} \right) \tag{0.2}$$

$$R := \frac{u^6}{3}. \tag{0.3}$$

[This is called the scaling identity because the scaling transformation $u(t,x) \mapsto \lambda u(\lambda t, \lambda x)$ is generated by $t\partial_t u + x \cdot \nabla u + u$.]

3) Use some algebraic manipulations to show that

$$Q = \frac{u^6}{6} - \frac{1}{2}\mathrm{div}\left(u^2 \frac{x}{|x|^2}\right) + \frac{1}{2}\left((\partial_t u)^2 + \left|\nabla u + u\frac{x}{|x|^2}\right|^2 + 2\partial_t u \frac{x}{t} \cdot \left(\nabla u + u\frac{x}{|x|^2}\right)\right)$$

Then, prove that the following holds on $K(0,\vec{0})$:

$$Q \geq \frac{u^6}{6} - \frac{1}{2}\mathrm{div}\left(u^2 \frac{x}{|x|^2}\right). \tag{5}$$

4) Use some algebraic manipulations to show that on the boundary $\partial K(0,\vec{0})$ the following holds:

$$tQ + x \cdot P = \left(t\left|\partial_t u + \frac{x}{t} \cdot \nabla u\right|^2 + \frac{1}{2}\left(\partial_t + \frac{x}{t} \cdot \nabla\right)(u^2)\right)(-|x|, x).$$

In particular, setting $v(y) := u(-|y|, y)$, the above can be rewritten as

$$tQ + x \cdot P = -\frac{|x \cdot \nabla v|^2}{|x|} - \frac{x \cdot \nabla(v^2)}{2|x|} \tag{6}$$

on $\partial K(0,\vec{0})$.

5) Integrate the scaling identity (4) over the truncated cone $K_s^t := K(0,\vec{0}) \cap [s,t] \times \mathbb{R}^3$, for some $s < t < 0$, and use the divergence theorem together with (5) and (6) to show that

$$\begin{aligned} 0 \geq & t\int_{D(t)} Q dx - s\int_{D(s)} \left(\frac{u^6}{6} - \frac{1}{2}\mathrm{div}\left(u^2\frac{x}{|x|^2}\right)\right) dx \\ & - \frac{1}{\sqrt{2}}\int_{B(\vec{0},|s|)\setminus B(\vec{0},|t|)} \frac{|y \cdot \nabla v|^2}{|y|} + \frac{y \cdot \nabla(v^2)}{2|y|} dy, \end{aligned} \tag{7}$$

where $D(t) := K(0,\vec{0}) \cap \{t\} \times \mathbb{R}^3$, $D(s) := K(0,\vec{0}) \cap \{s\} \times \mathbb{R}^3$, and $B(\vec{0},r) := \{x \in \mathbb{R}^3 : |x| \leq r\}$ for any $r > 0$.

6) Show that

$$\frac{y \cdot \nabla(v^2)}{2|y|} = \mathrm{div}\left(\frac{v^2}{2|y|}y\right) - \frac{v^2}{|y|}.$$

Use this identity and apply the divergence theorem twice in the right-hand side of (7) to obtain

$$\begin{aligned} 0 \geq & t\int_{D(t)} Q dx - s\int_{D(s)} \frac{u^6}{6} dx - \frac{1}{\sqrt{2}}\int_{B(\vec{0},|s|)\setminus B(\vec{0},|t|)} \left(|y||\nabla v|^2 - \frac{v^2}{|y|}\right) dy \\ & - \frac{2+\sqrt{2}}{4}\int_{\partial B(\vec{0},|s|)} |v|^2 d\sigma. \end{aligned} \tag{8}$$

7) Using the trace theorem $H^1(B(\vec{0},1)) \hookrightarrow L^2(\partial B(\vec{0},1))$ and a change of variables, show that there exists $C_0 > 0$ such that for any $r > 0$ and any $f \in H^1(B(\vec{0},r))$ the following holds:

$$\int_{\partial B(\vec{0},r)} |f|^2 d\sigma \leq \frac{C_0}{r}\int_{B(\vec{0},r)} |f|^2 dx + C_0 r\int_{B(\vec{0},r)} |\nabla f|^2 dx. \tag{9}$$

8) Show that the energy flux across the lateral boundary M_s^t of K_s^t is given by

$$\text{Flux}(u; M_s^t) = \frac{1}{\sqrt{2}} \int_{B(\vec{0},|s|)\setminus B(\vec{0},|t|)} \frac{|\nabla v|^2}{2} + \frac{v^6}{6} dy.$$

Use this together with (9) to deduce from (8) that there exists $C > 0$ such that

$$0 \geq t \int_{D(t)} Q dx - s \left(\int_{D(s)} \frac{u^6}{6} dx - C\text{Flux}(u; M_s^0) - C\Big(\text{Flux}(u; M_s^0)\Big)^{\frac{1}{3}} \right). \tag{10}$$

9) Prove that for any $S < T < t_0$, there exists $C > 0$ such that the following holds:

$$\begin{aligned} \int_{B(x_0,t_0-S)} |u(S)|^6 dx \leq & C\frac{t_0 - T}{t_0 - S} \left(E_{B(x_0,t_0-T)}(u(T), \partial_t u(T)) + \Big(E_{B(x_0,t_0-T)}(u(T), \partial_t u(T))\Big)^{\frac{1}{3}} \right) \\ & + C \left(\text{Flux}(u; M_S^{t_0}(t_0, x_0)) + \Big(\text{Flux}(u; M_S^{t_0}(t_0, x_0))\Big)^{\frac{1}{3}} \right). \end{aligned} \tag{11}$$

Hint: For $(t_0, x_0) = (0, \vec{0})$ this follows directly from (10). For a general $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$, you'll need to reduce to the case $(t_0, x_0) = (0, \vec{0})$ by using the invariance of NLW under temporal and spacial translations, namely if $u(t, x)$ is a solution of NLW, so is $u_{(t_0,x_0)}(t, x) := u(t + t_0, x + x_0)$.

10) Deduce from (11) that

$$\lim_{S \to t_0^-} \int_{B(x_0,t_0-S)} |u(S)|^6 dx = 0.$$

Problem 2 (P.B.)**.** Let $n \geq 1$. Consider the quasilinear wave equation

$$(G^{ij} \partial_i \partial_j + B^j \partial_j + A)u = F$$

with G, B, A, F satisfying condition $1(\infty, \Omega)$ holds and G is $1/100$ close to η. Suppose u is a solution. Further suppose $|F| < C|u|_1$ everywhere. Suppose $R > 1$.

Show that if there is an $R > 0$ such that $u(0, \vec{x})$ and $\partial_t u(0, \vec{x})$ both vanish for $|\vec{x}| > R$, then $u(t, \vec{x})$ vanishes for all $|\vec{x}| > R + 2t$.

Problem 3 (P.B.)**.** Let $L : \mathbb{R}^{(1+n)+1+(1+n)} \to \mathbb{R}$. If no argument is given, assume $L = L(x, u, \partial u) = L(x, u(x), \partial u(x))$, where $x \in \mathbb{R}^{1+n}$, $u : \mathbb{R}^{1+n} \to \mathbb{R}$ and ∂ denotes differentiation in $\mathbb{R}^{1+n}$. Use $\frac{\delta L}{\delta x^i}$ to denote the derivative of L with respect to its ith argument, use $\frac{\delta L}{\delta u}$ to denote its derivative with respect to its $((n+1)+1)$th argument, and $\frac{\delta L}{\delta \partial_i u}$ to denote its derivative with respect to its $((n+1)+1+i)$th argument. Observe that the chain rule gives

$$\partial_i L = \frac{\delta L}{\delta x^i} + \frac{\delta L}{\delta u} \partial_i u + \sum_{j=0}^{n} \frac{\delta L}{\delta \partial_j u} \partial_j \partial_i u.$$

(Observe also that δ^i_j still denotes the Kronecker delta.) u is said to satisfy the Euler-Lagrange equations[1] if

$$\sum_{i=0}^{n} \partial_i \frac{\delta L}{\delta \partial_i u} - \frac{\delta L}{\delta u} = 0.$$

(1) Let

$$\mathcal{T}^i{}_j = \frac{\delta L}{\delta \partial_i u}\partial_j u - \delta^i_j L,$$
$$\mathcal{P}^i = \sum_{j=0}^{n} \mathcal{T}^i{}_j X^j.$$

Show that if u satisfies the Euler-Lagrange equations, then

$$\sum_{i=0}^{n} \partial_i \mathcal{T}^i{}_j = -\frac{\delta L}{\delta x^j}.$$

(2) Find $\mathcal{T}^i{}_j$ if $L(x, u, \partial u) = \eta^{ij}(\partial_i u)(\partial_j u)$ for some constant η^{ij}.

Problem 4 (P.B.)**.** Consider the inviscid Burgers' equation in $\mathbb{R}^{1+1}$,

$$\partial_t u + u\partial_x u = 0.$$

(1) Suppose $t > 0$ and u is a C^2 solution of this equation n $[0, t] \times \mathbb{R}$ and that uniformly in t, for $|x|$ sufficiently, $u(t, x) = 0$. Show

$$\|u(t, x)\|_{L^2_x} = \|u(0, x)\|_{L^2_x}.$$

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be in the Schwarz class. Consider $u_{-1}(t, x) = 0$ and, for $n \geq 0$, u_n defined by

$$\partial_t u_n + u_{n-1}\partial_x u_n = 0,$$
$$u_n(0, x) = f(x).$$

(a) Show that there is a $T > 0$ such that for all $n \in \mathcal{N}$, $\sup_{t\in[0,T]} \|u\|_{H^3} < 2\|f\|_{H^3}$.
(b) Show that the u_n converge in H^2 to a limit u.
(c) Using various convergence properties, show that u is a C^1 function on $[0, T] \times \mathbb{R}$ and a solution of the inviscid Burgers' equation.

[1] In APDE I, you should have seen that u solves the Euler-Lagrange equation, then it is critical point of $S = \int L \mathrm{d}^{1+n}x$. In elliptic problems, one typically looks for minimisers of S. Unfortunately, in hyperbolic problems, typically critical points of S are always saddle points, since S is unbounded above and below.