ADVANCED PDE II - HOMEWORK 1

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Problem 1. Let $I \subset \mathbb{R}$ be an interval and let $E : I_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ and $B : I_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ be vector fields representing the electric and magnetic fields. The *Maxwell's equations* are given by:

$$\partial_t E = \nabla \times B$$
$$\partial_t B = -\nabla \times E$$
$$\nabla \cdot E = 0$$
$$\nabla \cdot B = 0.$$

Show that the components of these vector fields, E_i and B_i , i = 1, 2, 3, solve the linear wave equations:

$$\partial_t^2 E_i - \Delta E_i = 0, \qquad \partial_t^2 B_i - \Delta B_i = 0.$$

Conversely, let $E_0, B_0 : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $\nabla \cdot E_0 = \nabla \cdot B_0 = 0$. Show that the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 E - \Delta E = 0\\ \partial_t^2 B - \Delta B = 0\\ E\big|_{t=0} = E_0, \quad B\big|_{t=0} = B_0\\ \partial_t E\big|_{t=0} = \nabla \times B_0, \quad \partial_t B\big|_{t=0} = -\nabla \times E_0 \end{cases}$$

is a solution of Maxwell's equations.

Here, given a vector field $F = \langle F_1, F_2, F_3 \rangle$, we used the notations $\nabla \cdot F = \operatorname{div} F = \sum_{i=1}^3 \partial_{x_i} F_i$ and $\nabla \times F = \operatorname{curl} F = \langle \partial_{x_2} F_3 - \partial_{x_3} F_2, \ \partial_{x_3} F_1 - \partial_{x_1} F_3, \ \partial_{x_1} F_2 - \partial_{x_2} F_1 \rangle$.

Problem 2 (Alinhac's book on hyperbolic PDEs). Let $u \in C^2(\mathbb{R} \times [0,T))$ be a solution of the Cauchy problem for the inviscid Burger's equation

$$\begin{cases} \partial_t u + u \partial_x u = 0\\ u(x,0) = h(x). \end{cases}$$

Let x(t) = h(s)t + s be the projected characteristic starting from (s, 0). Show that the function $q(t) := (\partial_x u)(x(t), t)$ satisfies $q' + q^2 = 0$. Compute q explicitly.

Show that if h'(s) < 0, then q(t) becomes infinite as t approaches the value $t(s) = -\frac{1}{h'(s)} > 0$. Deduce from this that, given $h \in C^2(\mathbb{R})$, there exists a C^2 solution u of the above Cauchy problem exactly in the strip $\{0 \le t < \overline{T}\}$, where

$$\bar{T} := [\max(-h')]^{-1}$$

The number \overline{T} is called the *lifespan* of the smooth solution.

Problem 3 (Alinhac's book on hyperbolic PDEs). Consider the Cauchy problem

$$\begin{cases} \partial_t u + u \partial_x u = u^2 \\ u(x,0) = u_0(x), \end{cases}$$

where $u_0 \in C_0^2(\mathbb{R})$ is compactly supported and not identically zero. (a) Show the inequalities $\max(u_0 - u'_0) > 0$, $\max(u_0 - u'_0) \ge \max u_0$. Prove that if there exists x_0 such that

$$u_0(x_0) = \max u_0 > 0, \quad u_0''(x_0) < 0,$$

then $\max(u_0 - u'_0) > \max u_0$.

(b) Use the method of characteristics to solve the Cauchy problem. Prove that a ${\cal C}^2$ solution exists for

$$0 \le t < \bar{T} := [\max(u_0 - u_0')]^{-1}$$

(c) Let $u \in C^2(\mathbb{R} \times [0,T))$ be a solution of the above Cauchy problem. Denote by (x(t),t) the projected characteristic starting from $(x_0,0)$. Set $q(t) := (\partial_x u)(x(t),t)$. Establish the ODE satisfied by q and compute q explicitly. Deduce from this that $T \leq \overline{T}$ and hence \overline{T} is the lifespan of the smooth solution.

Problem 4 (Evan's book). Compute explicitly the unique entropy solution of the inviscid Burger's equation

$$\partial_t u + u \partial_x u = 0, \quad t \ge 0,$$

with initial condition

$$u(x,0) = h(x) = \begin{cases} 1, & \text{if } x < -1\\ 0, & \text{if } -1 < x < 0\\ 2, & \text{if } 0 < x < 1\\ 0, & \text{if } x > 1. \end{cases}$$

Problem 5 (Sogge's book). Let $f \in C^3(\mathbb{R}^2)$, $g \in C^2(\mathbb{R}^2)$ supported on $\{x : |x| < R\}$. Use Poisson's formula to deduce that the solution $u \in C^2(\mathbb{R}^2)$ of the linear wave equation $\partial_t^2 u - \Delta u = 0$ with initial data u(x, 0) = f, $\partial_t u(x, 0) = g$, satisfies the decay estimate

$$|u(x,t)| \le \frac{C}{\langle t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}}}.$$

Problem 6 (Evan's book, example due to S. Kowalevski). Consider the *heat equation* $\partial_t u - \partial_x^2 u = 0$ on \mathbb{R} , with initial data $u(x, 0) = \frac{1}{1+x^2}$ (analytic near zero). Show that while all the partial derivatives on $\{t = 0\}$ are determined from the initial data, the associated Taylor series at the origin does not converge for any (t, x) with $t \neq 0$. In particular, one cannot apply the Cauchy-Kowalevski theorem to the heat equation.