Well-posedness of Energy-Critical Nonlinear Schrödinger Equations

Kelvin Ka Wing Cheung

Supervisors: Tadahiro Oh and Oana Pocovnicu

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1 Introduction

The Cauchy problem for the defocusing energy-critical nonlinear Schrödinger equation (EC-NLS) is

$$\begin{cases} i\partial_t u + \Delta u &= |u|^{\frac{4}{d-2}} u \\ u(t_0, x) &= u_0(x) \end{cases} \tag{EC-NLS}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. In general, seminlinear Schrödinger equations are used to model a number of physical phenomena; for example in Bose-Einstein condensates (see for instance [3]). Our interest lies in well-posedness of the equation, namely, we wish to answer questions of the following sort:

(i) whether the solution exists, and if so, whether it exists for globally (i.e for all times $t \in \mathbb{R}$) or just locally (i.e. only a short time interval I);

(ii) whether the said solution is unique;

(iii) whether the solution depends continuously on the initial data, i.e. whether the solution map $u_0 \mapsto u$ is continuous (with respective to some well-defined topologies).

For this problem, local existence was proved by Cazenave and Weissler in 1989 [5] using the classical Strichartz estimates and other elementary considerations. As it turns out, the question of uniqueness and continuous dependence on initial data follow almost immediately from the existence of the solution. The heart of the matter lies in the question of global existence. Indeed, the process of upgrading local existence to global existence is highly non-trivial and requires a number of sophisticated mathematical tools from harmonic analysis. To give a brief historical account of the problem, in 1999, Bourgain [4] proved that the Cauchy problem (EC-NLS) is globally well-posed for radially symmetric data in the homogeneuous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ in dimension $d = 3$ using the idea of *induction on energy.* Tao [18] built upon this idea and extended Bourgain's result to dimension $d \geq 5$ in 2005. The radial assumption was finally removed in 2008 by Colliander, Keel, Staffilani, Takaoka, and Tao [7] for dimension $d = 3$, which further advanced the induction on energy argument. This global well-posedness result was also extended to dimension $d = 4$ by Ryckman and Vişan [16] and to $d \geq 5$ by Vişan [20].

This project is a joint work with the fellow MIGSAA student Tolomeo, who is responsible for part 2 of this report. Our main goal is to understand and expose the proof of global well-posedness for $d = 3$. In this report (i.e. part 1), we shall first give a brief review of basic Strichartz theory and some properties of the general NLS. We then give a proof of the local well-posedness of (EC-NLS). Once this has been accomplished, we shall discuss two important analytic results. The first of which is a stability theorem which essentially states that the (EC-NLS) is stable under perturbation. The second result is a *linear profile decomposition* which, heuristically speaking, describes all the defects of compactness of the Strichartz inequality. These tools are essential to start the induction on energy argument in part 2, namely to construct a minimum blow up solution[1], which will eventually serve as a minimal counterexample in the inductive argument.

2 Preliminaries and Notations

Some general notations: We shall always use the letter d to denote the dimension of the ambient Euclidean space $\mathbb{R}^d$. For real quantities A and B, we use the notation $A \lesssim B$ to mean $A \leq CB$ for some constant $C \in (0,\infty)$ which may depend on the dimension d. We also write $A \sim B$ to mean $A \lesssim B$ and $B \gtrsim A$, where $\gtrsim$ has the obvious meaning. Any extra dependencies on the implicit constant will be written as subscripts on $\lesssim$, $\gtrsim$ and $\sim$. So for example, $A \lesssim_{p,q} B$ means $A \leq CB$ for some constant $C \in (0,\infty)$ that may depend on the parameters d, p and q.

Test functions and distributions: We write $C^\infty(\mathbb{R}^d)$ for the space of all smooth functions $f : \mathbb{R}^d \to \mathbb{C}$. Certain subspaces of $C^\infty(\mathbb{R}^d)$ form spaces of test functions:

- $C_c^\infty\left(\mathbb{R}^d\right) = \left\{f \in C^\infty(\mathbb{R}^d) : \text{supp}(f) \text{ is compact}\right\}$;
- $\mathcal{S}(\mathbb{R}^d) = \left\{f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left|x^\alpha D^\beta f(x)\right| < \infty \text{ for all multi-indices } \alpha, \beta\right\}$.

C^∞, C_c^∞, $\mathcal{S}$ instead of $C^\infty(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$. We shall do the same for other function spaces as well.

The dual space of $\mathcal{S}$ is the space of tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$, which consists of all continuous linear functionals of $\mathcal{S}$ (with respect to the topology of the Schwartz space). More concretely, a linear functional of $\mathcal{S}$ is in $\mathcal{S}'$ if there exists $N \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$|\langle T, \varphi\rangle| \leq C \sum_{|\alpha|,|\beta|\leq N} \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta f(x)\right|,$$

for all $\varphi \in \mathcal{S}$, where $\langle T, \varphi\rangle$ denotes the action of T on φ. Note that for $p \in [1,\infty]$, any $f \in L^p = L^p(\mathbb{R}^d)$ (with Lebesgue measure) forms a tempered distribution through the action

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^d} f\varphi.$$

We caution that this is slightly different from the L^2 inner product, given by

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^d} f\bar{g}.$$

[1] the precise meaning of a minimum blow up solution in this context will be explained in part 2

for $f, g \in L^2$, so in particular, $\langle f, g\rangle = \langle f, \bar{g}\rangle_{L^2}$.

Fourier transform: We use the following convention of Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Recall that $\mathcal{F}$ is invertible on $\mathcal{S}$ and L^2, with the inverse Fourier transform given by

$$\mathcal{F}^{-1}f(x) = \check{f}(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

In particular, $\mathcal{F}$ defines a unitary operator on L^2. This is implied by Parseval's Theorem:

$$\langle f, g\rangle_{L^2} = \left\langle \hat{f}, \hat{g}\right\rangle_{L^2} \quad \text{and} \quad \int_{\mathbb{R}^d} \hat{f} g = \int_{\mathbb{R}^d} f\hat{g}.$$

Parseval's Theorem also motivates the definition of Fourier transform for tempered distributions: for $v \in \mathcal{S}'$, we define $\hat{v}$ to be the tempered distribution given

$$\langle \hat{v}, \varphi\rangle = \langle v, \hat{\varphi}\rangle.$$

Sobolev Spaces: The primary spaces we will be working on are (fractional) Sobolev spaces. Recall that the derivative of a tempered distribution is defined as follows: for $v \in \mathcal{S}'$, define $\partial^\alpha v$ (for $\alpha \in \mathbb{N}_0$) by

$$\langle \partial^\alpha v, \varphi\rangle = (-1)^{|\alpha|} \langle v, \partial^\alpha \varphi\rangle.$$

It is easy to show that

$$\mathcal{F}\partial^\alpha v = (i\xi)^\alpha \partial^\alpha \hat{v} \quad \text{and} \quad \partial^\alpha \hat{v} = \mathcal{F}((-ix)^\alpha v).$$

Now let $f \in L^2$ and $k \in \mathbb{N}$, and consider

$$\left\|\nabla^k f\right\|_{L^2} = \left(\sum_{j=1}^d \int_{\mathbb{R}^d} \left|\partial_{x_j}^k f\right|^2\right)^{\frac{1}{2}}.$$

Applying Plancherel's Theorem, and the property of differentiating Fourier transform, we obtain $\left\|\nabla^k f\right\|_{L^2} = \left\|\mathcal{F}^{-1}\left(|\xi|^k \hat{f}\right)\right\|_{L^2}$. Motivated by this, we define for $s \in \mathbb{R}$ the *fractional differentiation* operators $|\nabla|^s$ and $|\langle\nabla\rangle^s|$ by

$$|\nabla^s| f := \mathcal{F}^{-1}\left(|\xi|^s \hat{f}\right) \quad \text{and} \quad \langle\nabla\rangle^s f = \mathcal{F}^{-1}\left(\left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{f}\right).$$

We note that for $\varphi \in \mathcal{S}$, $\langle \nabla \rangle^s \varphi$ is well-defined for all $s \in \mathbb{R}$, and $|\nabla|^s \varphi$ is well-defined for $s > -d$ (because the function $|\xi|^s$ is locally integrable for such s). On the other hand, for $s \leq -d$, $|\nabla|^s \varphi$ is only defined for certain Schwartz functions. Since we will never use operators of such low order, we shall ignore this technicality.

For $p \in (1, \infty)$, these operators give rise to the Sobolev norms

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} = \|\langle \nabla \rangle^s f\|_{L^p(\mathbb{R}^d)} \quad \text{and } \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} = \||\nabla|^s f\|_{L^p(\mathbb{R}^d)} .$$

These are the non-homogeneous and homogeneous Sobolev norms respectively. We shall then define the non-homogeneous space $W^{s,p}(\mathbb{R}^d)$ and homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^d)$ to be the completion of $\mathcal{S}(\mathbb{R}^d)$ with their respective norms. When $p = 2$ we shall write

$$H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d) \quad \text{and } \dot{H}^s(\mathbb{R}^d) = \dot{W}^{s,2}(\mathbb{R}^d).$$

Note that H^s and $\dot{H}^s$ are Hilbert spaces with inner products

$$\langle f, g \rangle_{H^s} = \int_X \langle \xi \rangle^{2s} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \quad \text{and} \quad \langle f, g \rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} |\xi|^{2s} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

In general, these Sobolev spaces are spaces of tempered distributions. However, we have the following characterisation:

$$\begin{aligned} W^{s,p} &= \dot{W}^{s,p} \cap L^p \qquad && \text{for } s > 0; \\ W^{s,p} &= \dot{W}^{s,p} + L^p \qquad && \text{for } s < 0. \end{aligned}$$

Of course, for $s = 0$ we simply have $W^{0,p} = \dot{W}^{0,p} = L^p$. Note that for $s = k \in \mathbb{N}_0$, the inhomogeneous spaces $W^{k,p}$ coincide with the classical Sobolev spaces, in the sense that the $W^{k,p}$ norm is equivalent to the classical Sobolev norm:

$$\|f\|_{W^{k,p}} \sim_{k,p} \sum_{j=0}^{d} \left\|\partial_{x_j} f\right\|_{L^p} .$$

Mixed Spaces: For $p, q \in [1, \infty]$, $T \subseteq \mathbb{R}^{d_1}$ and $X \subseteq \mathbb{R}^{d_2}$, we define the mixed Lebesgue space[2] $L^p_t L^q_x(T \times X)$ to be the space of all complex valued function f on $T \times X$ such that

$$\|f\|_{L^p_t L^q_x(T\times X)} := \left(\int_{t\in T} \left(\int_{x \in X} |f(t, x)|^q \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} < \infty.$$

As one might expect, these spaces enjoy many nice properties of the usual Lebesgue spaces. For example, $L^p_t L^q_x(T \times X)$ is a Banach space with the above norm; and the (continuous) dual space is (isometrically isomorphic to) $L^{p'}_t L^{q'}_x(T \times X)$ where p' and q' are the Hölder

[2] also known as Strichartz spaces in literature

conjugates of p and q respectively, and a linear functional g acts on $f \in L_t^p L_x^q(T \times X)$ in the expected way:

$$\langle f, g \rangle = \int_{T \times X} gf.$$

We also record the Minkowski integral inequality: for $1 \leq p < q \leq \infty$, we have

$$\|f\|_{L_t^p L_x^q(T \times X)} \leq \|f\|_{L_t^q L_x^p(X \times T)}. \tag{2.1}$$

Note that when $p = q = 2$ we simply have the above space is simply $L_{t,x}^2(T \times X)$, and is of course a Hilbert space. Usually, T is the "time" domain $\mathbb{R}$ while X is the "spatial" domain $\mathbb{R}^d$. In this case we shall, unless otherwise stated, write $L_t^p L_x^q$ instead of the more cumbersome notation $L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)$. The subscripts of dummy variables t and x, while not necessary, are nonetheless helpful for clarity.

Similarly, we may also define other "mixed spaces". For example, $L_t^p W_x^{s,q}(\mathbb{R} \times \mathbb{R}^d)$ denotes (for $s > 0$, $p \in (0, \infty)$) the space of all measurable functions $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ such that the $f(t, \cdot) \in W^{s,p}$ for all $t \in \mathbb{R}$ and $\tilde{f}$ given by $\tilde{f}(t) = \|f(t, \cdot)\|_{W_x^{s,q}(\mathbb{R}^d)}$ is an $L^p(\mathbb{R})$ function.

3 The Energy-Critical Schrödinger Equation

The Cauchy problem for the general nonlinear Schrödinger Equation given by

$$\begin{cases} i\partial_t u + \Delta u &= F(u) \\ u(t_0, \cdot) &= u_0 \end{cases} \tag{NLS}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. We use the letters t and x to denote the time ($\mathbb{R}$) and spatial ($\mathbb{R}^d$) variables respectively. The Laplacian is on spatial variable only, i.e. $\Delta = \sum_{i=1}^d \partial_{x_i}^2$. The nonlinearity F is some function $F : \mathbb{C} \to \mathbb{C}$. We will be concerned with the particular nonlinearity $F(u) = \mu|u|^p u$ where $\mu \in \{-1, 1\}$. The cases $\mu = 1$ and $\mu = -1$ are called *defocusing* and *focusing* respectively. For our final goal of global well-posedness (in part 2), we will eventually restrict our attention to the defocusing case only. But most of our results here in part 1 will still hold for the focusing case.

At a formal level, the PDE can be turned into an integral equation using Fourier transform in a standard way. Indeed, suppose that u satisfies (NLS). By taking Fourier transform in x, we have

$$\begin{cases} \partial_t \hat{u}(t,\xi) + i|\xi|^2 \hat{u}(t,\xi) &= -i\widehat{F(u)}(t,\xi) \\ \widehat{u}(t_0,\xi) &= \widehat{u}_0(\xi) \end{cases}$$

For each $\xi \in \mathbb{R}^d$, this is an ODE in t, and can be easily solved by multiplying by the integrating factor $e^{i|\xi|^2 t}$, giving

$$\widehat{u} e^{i|\xi|^2 t} - \widehat{u_0} e^{i|\xi|^2 t_0} = -i \int_{t_0}^t \widehat{F(u)}(s,\xi) e^{i|\xi|^2 s} \, ds.$$

Rearranging and reverting the Fourier transform, we obtain the *Duhamel formula*:

$$u = S(t-t_0)u_0 - i \int_{t_0}^t S(t-s) F(s,\xi) \, ds, \tag{3.1}$$

where $S(t)$ is the Schrödinger propagator, given by

$$S(t)f(x) = e^{it\Delta} f(x) = \mathcal{F}^{-1}(e^{-i|\cdot|^2 t} \hat{f})(x).$$

Note that all steps in the above computation are justified provided $u_0(x)$ and $u(x)$ are sufficiently nice functions so that there is no problem reverting the Fourier transform. In particular, for the linear equation with $F \equiv 0$, and say with Schwartz initial data u_0, the equation is solved by $S(t-t_0)u_0$.

Proposition 3.1 (Basic Properties of Schrödinger Propagator)**.** *Let $I \subseteq \mathbb{R}$ be an interval that is not necessarily finite. Then*

(i) $\overline{S(t)f} = S(-t)\overline{f}$;

(ii) $S(t_1)S(t_2) = S(t_1 + t_2)$;

(iii) *For each* $t \in \mathbb{R}$, $S(t)$ *is a unitary operator on* H^s *and* $\dot{H}^s$ *for any* $s \in \mathbb{R}$..

(iv) *If* $f \in \dot{H}^s$, *then* $S(t)f \in C_t\dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d)$;

(v) *If* $F \in C_t\dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d)$, *then* $\displaystyle\int_{t_0}^t S(t-t')F(t')\,dt' \in C_t\dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d)$.

Here, $C_t\dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d)$ *consists of all space-time functions* u *such that each* $u(t,\cdot) \in \dot{H}^s$ *and each* $u(\cdot, x)$ *is continuous.*

We can in fact compute $S(t)f$ explicitly, at least for $f \in \mathcal{S}$. Indeed, by the properties of Fourier transform,

$$\begin{aligned} S(t)f(x) &= \mathcal{F}^{-1}(e^{-i|\cdot|^2 t}\hat{f})(x) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}^{-1}(e^{-i|\cdot|^2 t}) * f(x). \end{aligned}$$

Note that the function $e^{-i|\cdot|^2 t}$ is not even in L^1, so the above inverse Fourier transform is taken in the sense of a tempered distribution. To compute $\mathcal{F}^{-1}(e^{-i|\cdot|^2 t})$, we exploit the well-known fact that

$$\mathcal{F}e^{-z|\cdot|^2}(\xi) = \left(\frac{1}{2z}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

for $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) > 0$. In particular, we have for $\epsilon > 0$,

$$\mathcal{F}^{-1}\left(e^{-(\epsilon+it)|\cdot|^2}\right)(x) = \left(\frac{1}{2(\epsilon+it)}\right)^{\frac{d}{2}} e^{-\frac{|x|^2}{4(\epsilon+it)}} \to \left(\frac{1}{2it}\right)^{\frac{d}{2}} e^{-\frac{|x|^2}{4it}}.$$

as $\epsilon \to 0$. We thus yield the explicit formula

$$S(t)f(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{\frac{i|\cdot|^2}{4t}} * f(x). \tag{3.2}$$

By appying Young's inequality for convolution, we immediately get the following particular dispersive estimate

$$\|S(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{\frac{d}{2}}}\|f\|_{L^1_x} \tag{3.3}$$

for all $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^∞, we may extend $S(t)$ to a bounded linear operator from L^∞ to L^1. By interpolation, we have the following extension:

Proposition 3.2 (Dispersive Estimate)**.** *Let $p \in [2, \infty]$. For any $f \in L^{p'}$,*

$$\|S(t)f\|_{L^p} \lesssim_p |t|^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)} \|f\|_{L^{p'}}$$

Proof. By the unitarity of $S(t)$ in L^2, we have $\|S(t)f\|_{L^2} = \|f\|_{L^2}$. The result then follows from an application of Riesz-Thorin interpolation Theorem. □

Theorem 3.3 (Strichartz Estimates [17])**.** *A pair $(q, r) \in [2, \infty]^2$ is said to be (Schrödinger) admissible if $(q, r, d) \neq (2, \infty, 2)$ and*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{3.4}$$

Suppose that (q, r), $(\tilde{q}, \tilde{r})$ are admissible pairs, and that $I \subseteq \mathbb{R}$ is an interval containing t_0 that is not necessarily finite, then the following estimates hold:

(i) *The homogeneous Strichartz estimate:*

$$\|S(t)f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{q,r} \|f\|_{L_x^2(\mathbb{R}\times\mathbb{R}^d)}.$$

(ii) *The dual homogeneous Strichartz estimate:*

$$\left\|\int_{\mathbb{R}} S(-t')F(t')\,dt'\right\|_{L_x^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{q,r} \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.$$

(iii) *The nonhomogeneous Strichartz estimate:*

$$\left\|\int_{t_0}^{t} S(t-t')F(t')\,dt'\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{q,r,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

We now return to (NLS) with nonlinearity $F(u) = \mu|u|^p u$ where $\mu \in \{-1, 1\}$. This problem has a natural scaling symmetry. Indeed, for simplicity, let us assume u is a solution to (NLS) in the classical sense. Then

$$u^\lambda(t, x) = \lambda^{-\frac{2}{p}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is also a solution to (NLS) with initial data $u_0^\lambda(x) := u_0(\lambda x)$.

Now suppose that the a initial data u_0 belongs in the homogeneous Sobolev space $\dot{H}^s$ for some $s \in \mathbb{R}$. A simple calculation shows that

$$\left\|u^\lambda\right\|_{\dot{H}^s} = \lambda^{\frac{d}{2}-\frac{2}{p}-s} \|u\|_{\dot{H}^s}.$$

It follows that we have scale invariance if $s = s_{\text{crit}} := \frac{d}{2} - \frac{2}{p}$. We thus call the initial value problem (NLS) is (scaling) *critical* if the initial data belongs in $\dot{H}^{s_{\text{crit}}}$.

The *energy* is the quantity

$$E(u(t)) := \frac{1}{2} \|u(t,\cdot)\|_{\dot{H}^1}^2 + \mu \frac{d-2}{2d} \|u(t,\cdot)\|_{L^{\frac{2d}{d-2}}}^{\frac{d-2}{2d}} .$$

By differentiating in time, one can show that the energy is *conserved*, meaning that it is invariant in time:

$$E(u(t)) = E(u(0))$$

for any $t \in \mathbb{R}$. If $p = \frac{2d}{d-2}$, we have $s_{\text{crit}} = 1$. By a change of variable and the preceding discussion, this value of p renders the energy invariant under scaling. The *energy-critical NLS* is then given by

$$\begin{cases} i\partial_t u + \Delta u &= \mu |u|^{\frac{4}{d-2}} u \\ u|_{t=t_0} &= u_0 \end{cases} \tag{EC-NLS}$$

where the initial data $u_0 \in \dot{H}^1$.

For ease of exposition, we shall restrict attention to dimension $d = 3$. Many of the proofs here can in fact be adapted to higher dimension d, but doing so will introduce strange fractions in terms of d, which, in our opinion, obscure the underlying ideas of the proofs. Note that for $d = 3$, we would have $p = 6$, and the (EC-NLS) becomes the quintic NLS. We now define precisely what we mean by a solution.

Definition 3.4 (Solution). *Let $I \subseteq \mathbb{R}$ be a (not necessarily finite) interval. A function $u : I \times \mathbb{R}^3 \to \mathbb{C}$ is a (strong) solution to (EC-NLS) if $u \in C_t \dot{H}_x^1 \cap L_{t,x}^{10}(K \times \mathbb{R}^3)$ for each compact $K \subseteq I$ and satisfies satisfies the Duhamel formula*

$$u(t,x) = S(t-t_0)u_0 - i\mu \int_{t_0}^{t} S(t-s) \left(|u|^4 u\right)(s,x)\, ds. \tag{3.5}$$

The interval I is called the lifespan of u. If u cannot be extended to a larger interval than I, then I is called the maximum lifespan. We say u is a global solution if its maximum lifespan is $\mathbb{R}$, otherwise u is called a local solution.

We will explain the reason for the presence of the strange looking space $L_{t,x}^{10}(K \times \mathbb{R})$ in due course. Note that a solution is a fixed point of the operator $\Gamma = \Gamma_{u_0}$, where

$$\Gamma(u) = \text{RHS}(3.5).$$

Thus the contraction mapping theorem is useful tool to prove the existence and uniqueness of a solution, at least locally. We will need to show that Γ is a contraction on some complete subspace of $C_t\dot{H}^1_x \cap L^{10}_{t,x}(I\times\mathbb{R}^3)$. In fact, we shall prove this on a closed ball of the space $\dot{S}(I)$ defined by the norm

$$\|u\|_{\dot{S}(I)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L^q_t\dot{W}^{1,r}_x(I\times\mathbb{R}^3)} = \max\Big(\|u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}, \|u\|_{L^2_t L^6_x(I\times\mathbb{R}^3)}\Big).$$

The second equality infers that the above norm is determined by the "endpoint" spaces, and follows from Hölder interpolation. Indeed, for any admissible (q,r), one has

$$\begin{aligned}\|u\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} &\le \|u\|^{1-\theta(q)}_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}\|u\|^{\theta(q)}_{L^2_t L^6_x(I\times\mathbb{R}^3)} \le \max\Big(\|u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}, \|u\|_{L^2_t L^6_x(I\times\mathbb{R}^3)}\Big)\\ &\le \|u\|_{\dot{S}(I)},\end{aligned}$$

where the first inequality needs some further justification. Here, $0\le\theta(q)\le 1$ needs to satisfy

$$\frac{1}{q} = \frac{\theta(q)}{\infty} + \frac{1-\theta(q)}{2} \quad\text{and}\quad \frac{1}{r} = \frac{\theta(q)}{2} + \frac{1-\theta(q)}{6}$$

for the interpolation to work. Let $\theta(q)$ be defined by the first expression above. Then the second equality follows immediately from the admissibility relation:

$$\frac{1}{r} = \frac{1}{2} - \frac{2}{3q} = \frac{1}{2} - \frac{1-\theta(q)}{3} = \frac{\theta}{2} - \frac{1-\theta(q)}{6}.$$

By Sobolev inequality (with the numbers $\frac{1}{3} = \frac{13}{30} - \frac{1}{10}$), and that the pair $\left(10, \frac{30}{13}\right)$ is admissible, one has the embedding

$$\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x(I\times\mathbb{R}^3)} \le \|u\|_{\dot{S}(I)}. \tag{3.6}$$

Moreover, the admissible pair $(2,\infty)$ gives

$$\|u\|_{L^\infty_t\dot{H}^1(I\times\mathbb{R}^3)} \le \|u\|_{\dot{S}(I)}. \tag{3.7}$$

The inequalities (3.6) and (3.7) show that the space $\dot{S}(I)$ embed (continuously) into $L^\infty_t\dot{H}^1_x\cap L^{10}_{t,x}(I\times\mathbb{R}^3)$. If Γ is a contraction on a closed ball of $\dot{S}(I)$, then the unique fixed point u recovered from contraction mapping theorem is indeed a solution in the sense of Definition 3.4 (noting that u is automatically continuous in time in light of Proposition 3.1 (iv) and (v)).

Theorem 3.5 (Local Wellposedness[5]). *Let I be a (not necessarily finite) interval containing t_0. There exists $\eta_1 > 0$, such that if the initial data $u_0\in\dot{H}^1(\mathbb{R}^d)$ and I satisfy*

$$\|S(t-t_0)u_0\|_{\dot{S}(I)} \le \eta$$

for some $0<\eta\le\eta_1$, then a unique solution u to (EC-NLS) exists in the closed ball

$$\overline{B_{2\eta}} := \Big\{u\in\dot{S}(I) : \|u\|_{\dot{S}(I)} \le 2\eta\Big\}.$$

Proof. For this proof, all space-time norms are defined on $I \times \mathbb{R}^d$. We shall first try to get some control over $\|\Gamma(u) - \Gamma(v)\|_{\dot{S}(I)}$ for $u, v \in \dot{S}(I)$. First note the following crude estimate

$$\left||u|^4 u - |v|^4 v\right| \lesssim |u - v|(|u|^4 + |v|^4),$$

(see Lemma A.13). We thus obtain the following bounds:

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\|_{\dot{S}(I)} &\lesssim \left\||u|^4 u - |v|^4 v\right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}} && \text{(Strichartz)} \\
&\lesssim \left\|\left(|u|^4 + |v|^4\right)|u - v|\right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}} \\
&\leq \left(\|u\|_{L_{t,x}^{10}}^4 + \|v\|_{L_{t,x}^{10}}^4\right)\|u - v\|_{L_t^{10} L_x^{\frac{30}{13}}} && \text{(Hölder)} \\
&\lesssim \left(\|u\|_{\dot{S}(I)}^4 + \|v\|_{\dot{S}(I)}^4\right)\|u - v\|_{\dot{S}(I)}. && \text{(Sobolev)}
\end{aligned}$$

To make Γ into a contraction, we want the factor on $\|u - v\|_{\dot{S}(I)}$ to be less than 1, so we need some control over the $\dot{S}(I)$ norms on u and v. Thus we shall consider u and v in some closed ball $\overline{B_{2\eta}}$ in the $\dot{S}(I)$ norm where η is ranged over $0 < \eta < \eta_1$ for some suitably chosen η_1. To find such R, observe that by Strichartz we have

$$\|\Gamma(u)\|_{\dot{S}(I)} \leq \|S(t - t_0)u_0\|_{\dot{S}(I)} + C\left\||u|^4 u\right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}} \qquad \text{(Strichartz)}$$

for some constant $C > 0$. The second term can be further refined by distributing the gradient over each factor in $|u|^4 u = u^3 \bar{u}^2$:

$$\begin{aligned}
\left\||u|^4 u\right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}} &\sim \left\|u^4 \nabla u\right\|_{L_t^2 L_x^{\frac{6}{5}}} && \text{(Product Rule)} \\
&\leq \|u\|_{L_{t,x}^{10}}^4 \|\nabla u\|_{L_t^{10} L_x^{\frac{30}{10}}} && \text{(Hölder)} \\
&\lesssim \|u\|_{\dot{S}(I)}^5. && \text{(Sobolev)}
\end{aligned}$$

Summarising, we have the two inequalities

$$\|\Gamma(u) - \Gamma(v)\|_{\dot{S}(I)} \leq C_1\left(\|u\|_{\dot{S}(I)}^4 + \|v\|_{\dot{S}(I)}^4\right)\|u - v\|_{\dot{S}(I)};$$
$$\|\Gamma(u)\|_{\dot{S}(I)} \leq \|S(t - t_0)u_0\|_{\dot{S}(I)} + C_2\|u\|_{\dot{S}(I)}^5.$$

We now assume $\|S(t - t_0)u_0\|_{\dot{S}(I)} < \eta$. Looking at the second inequality, we have

$$\|\Gamma(u)\|_{\dot{S}(I)} \leq \eta + C_2(2\eta)^5$$

for $u \in \overline{B_{2\eta}}$. The right hand side is $\leq 2\eta$ provided $C_2(2\eta)^5 < \eta$. If we further ensure that $2(2\eta)^4 C_1 \leq \frac{1}{2}$, then the first inequality implies that Γ is a contraction from $\overline{B_{2\eta}}$ to itself. Both of these conditions are satisfied if $\eta \leq \eta_1$ for some sufficiently small η_1 that depends on the harmless constants C_1 and C_2. Contraction mapping theorem then implies that Γ has a unique fixed point u in $\overline{B_{2\eta}}$. $\square$

Remark 3.6. *The above local wellposedness result has the following implications:*

(i) ***Short time existence:*** *By dominated convergence theorem. given any* $u_0 \in \dot{H}^1_x$*, one can always choose a sufficiently small interval* I *containing* t_0 *such that the condition* $\|S(t)u_0\|_{\dot{S}(I)} \leq \eta$ *holds. The above theorem then implies that one can always obtain a short time solution given any initial data* $u_0 \in \dot{H}^1_x$*.*

(ii) ***Small data global wellposedness:*** *By (homogeneous) Strichartz estimate, one has* $\|S(t)u_0\|_{\dot{S}(\mathbb{R})} \leq \|u_0\|_{\dot{H}^1_x}$*. Therefore, if* $\|u_0\|_{\dot{H}^1_x} \leq \eta$*, we can invoke the local wellposedness result to get global existence of a solution* u*.*

4 Perturbation Theory

Consider the following perturbed energy-critical NLS:

$$\begin{cases} i\partial_t \tilde{u} + \Delta \tilde{u} &= \mu |\tilde{u}|^4 u + e \\ \tilde{u}(t_0, x) &= \tilde{u}_0(x) \end{cases} \tag{P-EC-NLS}$$

where $e : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a small function in some sense. Suppose that $\tilde{u}$ is a solution to (P-EC-NLS) with initial data $\tilde{u}_0$, where the precise definition of a solution is as in Definition 3.4 with the Duhamel formula in (3.5) replaced by

$$u(t,x) = S(t-t_0)u_0 - i\mu \int_{t_0}^{t} S(t-s)\left(|u|^4 u + e\right)(s,x)\, ds. \tag{4.1}$$

For initial $u_0 \in \dot{H}^1_x$ close to $\tilde{u}_0$, we would like to show that there is a unique solution u to (EC-NLS) that stays close to $\tilde{u}$. More precisely, one can show the following stability result.

Theorem 4.1 (Energy-critical Stability Result [7])**.** *Suppose that $\tilde{u} : I \to \mathbb{R}^3$ is a solution to (P-EC-NLS) where $I \subseteq \mathbb{R}$ be a compact interval and contains t_0. Assume that we have the following bounds:*

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)} \leq E \tag{4.2}$$

$$\|\tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq L \tag{4.3}$$

for some constant $E, L > 0$. Then there exists a small $\epsilon_1 = \epsilon_1(E,L) > 0$ such that if a function $u_0 \in \dot{H}^1$ and the error e satisfy the bounds

$$\|\tilde{u}(t_0) - u_0\|_{\dot{H}^1_x(\mathbb{R}^3)} < \epsilon \tag{4.4}$$

$$\|e\|_{L^2_t \dot{W}^{\frac{6}{5}}_x(I\times\mathbb{R}^3)} < \epsilon \tag{4.5}$$

for some $0 < \epsilon < \epsilon_1$, then there is a unique solution $u : I \times \mathbb{R}^3 \to \mathbb{C}$ to (EC-NLS) with initial data $u(t_0, \cdot) = u_0$ such that

$$\|u - \tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq C(E,L)\epsilon \tag{4.6}$$

$$\|u - \tilde{u}\|_{\dot{S}(I)} \leq C(E,L)\epsilon \tag{4.7}$$

$$\|u\|_{\dot{S}(I)} \leq C(E,L) \tag{4.8}$$

for some constant $C(E,L) > 0$.

Proof. Without loss of generality, we may assume $t_0 = \inf I$. By the local wellposedness result (or more precisely, Remark 3.6 (ii)), there is a solution $u : I_0 \times \mathbb{R}^3$ to (EC-NLS) with initial data $u(t_0) = u_0$. For the time being, we shall make the following additional assumptions:

(i) The lifespan of u is at least as long as the lifespan of $\tilde{u}$. Thus we may assume $I_0 = I$;

(ii) $\|\tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \leq \epsilon$.

Let $v = u - \tilde{u}$, and define for each $t \in I$ the function

$$A(t) := \left\| |u|^4 u - |\tilde{u}|^4 \tilde{u} \right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}([t_0,t]\times\mathbb{R}^3)}.$$

Note that A is a continuous in t. We shall use a common trick called *continuity argument* to show that $A(t)$ can be made uniformly small in t by possibly decreasing the size of $\bar{\epsilon}$. The first step is to attempt to bound $A(t)$ by itself in some "non-trivial way". We first note that by the Duhamel's formula,

$$|\nabla| v = S(t-t_0)|\nabla| v - \mu i \int_{t_0}^{t} S(t-t')|\nabla| \left(|u|^4 u - |\tilde{u}|^4 \tilde{u} - e \right) dt'.$$

Therefore,

$$\begin{aligned} \|v\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}([t_0,t]\times\mathbb{R}^3)} &\lesssim \|v(t_0)\|_{\dot{H}_x^1} + \left\| |u|^4 u - |\tilde{u}|^4 \tilde{u} \right\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}([t_0,t]\times\mathbb{R}^3)} + \|e\|_{L_t^2 \dot{W}_x^{1,\frac{6}{5}}([t_0,t]\times\mathbb{R}^3)} \\ &\lesssim \epsilon + A(t), \end{aligned} \tag{$\dagger$}$$

where the first inequality is obtained by Duhamel's formula and estimating each term by Strichartz, and the last inequality is due to (4.4). On the other hand, since

$$\begin{aligned} \left| \nabla \left(|u|^4 u - |\tilde{u}|^4 \tilde{u} \right) \right| &\lesssim |u|^4 |\nabla u| + |\tilde{u}|^4 |\nabla \tilde{u}| \\ &= |v + \tilde{u}|^4 |\nabla (v + \tilde{u})| + |\tilde{u}|^4 |\nabla \tilde{u}|. \end{aligned}$$

Hence we can estimate using Hölder, Sobolev, ($\dagger$) and assumption (ii), we have (after suppressing the cumbersome $[t, t_0] \times \mathbb{R}^3$ in the notation)

$$\begin{aligned} A(t) &\lesssim \|v + \tilde{u}\|_{L_{t,x}^{10}}^4 \|v + \tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}} + \|\tilde{u}\|_{L_{t,x}^{10}}^4 \|\tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}} \\ &\lesssim \|v + \tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}}^5 + \|\tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}}^5 \\ &\lesssim \left(\|v\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}} + \|\tilde{u}\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}} \right)^5 \\ &\leq (A(t) + 2\epsilon)^5 \end{aligned}$$

We now claim that for sufficiently small ϵ, we in fact have

$$A(t) \leq \epsilon \tag{4.9}$$

for all $t \in I$. Indeed, for a fixed ϵ, let $t_\epsilon = \sup\{t \in I : A(t) \leq \epsilon\}$. We note that $A(t_0) = 0$, hence $t_\epsilon \geq 0$. Now suppose that $t_\epsilon < \sup I$. Then by the continuity of A, we must have $A(t_\epsilon) = \epsilon$. The above bound on $A(t)$ then implies

$$\epsilon \leq C(3\epsilon)^5 \quad \Longleftrightarrow \quad \epsilon^{-4} \leq 3^5 C \tag{4.10}$$

for every n. Clearly this cannot hold for every $\epsilon > 0$. Hence there exists $\epsilon_0 > 0$ such that $t_{\epsilon_0} = \sup I$. In particular, (4.10) cannot hold for any $0 < \epsilon \leq \epsilon_0$ and hence $t_\epsilon = \sup I$ whenever $0 < \epsilon \leq \epsilon_0$. We deduce that $A(t) \leq \epsilon$ whenever $\epsilon \in (0, \epsilon_0)$.

We are now ready to prove (4.6) to (4.8) under said assumptions. (4.6) now follows easily from Sobolev and (†):

$$\|v\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|v\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x (I\times\mathbb{R}^3)} \lesssim A(t^*) + \epsilon \lesssim \epsilon.$$

For (4.7), we use Duhamel and Strichartz in the same manner as in (†), followed by the smallness of $A(t^*)$:

$$\|v\|_{\dot{S}(I)} \lesssim \|v(t_0)\|_{\dot{H}^1_x} + A(t^*) + \|e\|_{L^2_t \dot{W}^{1,\frac{6}{5}}_x ([t_0,t]\times\mathbb{R}^3)} \lesssim \epsilon.$$

For (4.8), we expand u with Duhamel and estimate each term the usual way (as in the proof of local wellposedness) to get

$$\begin{aligned}
\|u\|_{\dot{S}(I)} &\lesssim \|u(t_0)\|_{\dot{H}^1_x} + \|u\|^5_{L^2_t \dot{W}^{1,\frac{6}{5}}_x (I\times\mathbb{R}^3)} \\
&\lesssim \|\tilde{u}(t_0)\|_{\dot{H}^1_x} + \|v(t_0)\|_{\dot{H}^1_x} + \left(\|\tilde{u}\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x (I\times\mathbb{R}^3)} + \|v\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x (I\times\mathbb{R}^3)} \right)^5 \\
&\lesssim E + \epsilon + (2\epsilon)^5 \\
&\lesssim E
\end{aligned}$$

where we used (4.2), (4.4), and reduced the size of $\epsilon_0 = \epsilon_0(E)$ if necessary so that $\epsilon + (2\epsilon)^5 \leq E$ for all $\epsilon \leq \epsilon_0$.

Let us now summarise what we have proved so far. By having the extra assumptions (i) and (ii), we have shown that provided $0 < \epsilon \leq \epsilon_0(E)$, one has

$$\|u - \tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq C(I)\epsilon \tag{4.11}$$

$$\|u - \tilde{u}\|_{\dot{S}(I)} \leq C(I)\epsilon \tag{4.12}$$

$$\|u\|_{\dot{S}(I)} \leq C(I)E \tag{4.13}$$

$$\big\| |u|^4 u - |\tilde{u}|^4 \tilde{u} \big\|_{L^2_t \dot{W}^{1,\frac{6}{5}}_x (I\times\mathbb{R}^3)} \leq \epsilon. \tag{4.14}$$

We now proceed to remove assumption (ii). The idea is to partition the interval I so that on each piece, $\tilde{u}$ has its $L^{10}_t \dot{W}^{1,\frac{30}{13}}_x$ norm controlled by ϵ. This will allow us to apply what we proved so far on each piece and then assemble everything back together. In order to do this, we need to check that at least $\tilde{u} \in \dot{S}(I)$.

To do so, we first partition I into N_0 subintervals J_k so that

$$\|\tilde{u}\|_{L^{10}_{t,x}(J_k\times\mathbb{R}^3)} \leq \eta$$

for each $k \leq N_0$, where $\eta > 0$ is some small quantity to be chosen later. This is possible because of (4.3), which in particular implies that the number of intervals we need is at most $\left(\frac{L}{\eta}+1\right)^{10}$. For a fixed k, we use Duhamel again to estimate

$$\begin{aligned}\|\tilde{u}\|_{\dot{S}(J_k)} &\lesssim \|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)} + \|\tilde{u}\|_{\dot{S}(J_k)} \|\tilde{u}\|^4_{L^{10}_{t,x}(J_k\times\mathbb{R}^3)} + \|e\|_{L^2_t L^{\frac{6}{5}}_x(I\times\mathbb{R}^3)} \\ &\lesssim E + \eta^4 \|\tilde{u}\|_{\dot{S}(J_k)} + \epsilon,\end{aligned}$$

where we invoked (4.2) and (4.5) to get the last inequality. By a continuity argument, we may choose η sufficiently small to get that $\|\tilde{u}\|_{\dot{S}(J_k)} \lesssim E + \epsilon$. Summing over the k's, we obtain

$$\|\tilde{u}\|_{\dot{S}(I)} \leq C(E, L).$$

Now that we have control over $\|\tilde{u}\|_{\dot{S}(J_k)}$, we can partition I into $N_1 = N_1(E, L)$ intervals $I_k = [t_k, t_{k+1}]$ so that

$$\|\tilde{u}\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x(I_k\times\mathbb{R}^3)} \leq \epsilon$$

for each k. If we can verify that the condition (4.4) holds for each interval, namely

$$\|u(t_k) - \tilde{u}(t_k)\|_{\dot{H}^1_x} < \epsilon \tag{††}$$

for each k, then we have (4.11) to (4.14) for each interval I_k, but noting that the constants in those inequalities depend on k. One can then obtain (4.6), (4.7) and (4.8) by summing over k on (4.11), (4.12) and (4.13) respectively (Note that the dependency on E and L on the constants will come from the number of intervals N_1 which depend on E and L). We shall verify (††) inductively. The case $k = 0$ holds by assumption. Assume that (††) holds for some $k \geq 0$. Then by Duhamel and Strichartz,

$$\begin{aligned}\|v(t_{k+1})\|_{\dot{H}^1_x} &\lesssim \|v(t_0)\|_{\dot{H}^1_x} + \left\||u|^4 u - |\tilde{u}|^4\tilde{u}\right\|_{L^2_t \dot{W}^{1,\frac{6}{5}}_x([t_0,t_{k+1}]\times\mathbb{R}^3)} + \|e\|_{L^2_t \dot{W}^{1,\frac{6}{5}}_x(I\times\mathbb{R}^3)} \\ &\leq \epsilon + \sum_{j=0}^{k} \left\||u|^4 u - |\tilde{u}|^4\tilde{u}\right\|_{L^2_t \dot{W}^{1,\frac{6}{5}}_x([t_j,t_{j+1}]\times\mathbb{R}^3)} + \epsilon \\ &\leq (k+3)\epsilon.\end{aligned}$$

Namely, $\|v(t_{k+1})\|_{\dot{H}^1_x} \leq C(k)\epsilon := \epsilon_k$ for some constant $C(k) > 0$. By reducing the size of $\epsilon_1 = \epsilon_1(k)$ if necessary, we can make $\epsilon \leq \epsilon_1$ arbitrarily small, and hence ensure $\epsilon_k < \epsilon_0(E)$

which allows us to continue the induction. Note that the final version of ϵ_1 will depend on L and E since ϵ_1 depends on $N_1(L,E)$. This proves the theorem assuming (i).

Finally, to remove the assumption (i), let I_0 be the maximum lifespan of u and $T := \sup I_0$. Suppose for a contradiction that $T < \sup I := \tilde{T}$. What we have proved so far holds in the interval I_0. In particular, we have $\|u\|_{\dot{S}^1(I_0)} \le C(E,L)$. Now, the Duhamel formula starting from some $t_0' < T$ reads

$$u(t) = S(t-t_0')u(t') - i\mu \int_{t_0'}^{t} S(t-s)|u|^4 u(s)\,ds.$$

Hence

$$\left\|S(t-t_0')u(t_0')\right\|_{\dot{S}^1([t',T))} \le \|u(t)\|_{\dot{S}^1\left([t_0',T)\right)} + \|u\|_{\dot{S}^1\left([t_0',T)\right)} \|u\|^4_{L^{10}_{t,x}([t_0',T)\times\mathbb{R}^3)}.$$

The RHS can be made to be less than $\frac{\eta_1}{2}$ by choosing t_0' sufficiently close to T, where η_1 is the local existence threshold in Theorem 3.5. Since the LHS is a continuous function in T, there exists $\delta > 0$ such that

$$\left\|S(t-t_0')u(t_0')\right\|_{\dot{S}^1([t',T+\delta))} \le \eta_1.$$

But then by local well-posedness, there is a solution on $[t_0',T)$ with initial data $u|_{t=t_0'} = u(t_0')$, and hence the solution can be extended beyond T, contradicting the maximality of I_0. □

Remark 4.2.

(i) ***Unconditional uniqueness:*** *Note that in the local well-posedness result, we only proved uniqueness in a closed ball. One can use the above stability result to upgrade this to unconditional uniqueness (in the sense of Definition 3.4) by simply taking $e = 0$ and $u_0 = \tilde{u}_0$. Though one can also prove this without the stability result using a continuity argument.*

(ii) ***Continuous dependence on initial data:*** *The continuous dependence on initial data is implied by the stability result corresponding to the case when $e = 0$. Namely, if we have initial datum u_0 and $\tilde{u}_0$ sufficiently close to each other in the sense of (4.4), then the corresponding solutions u and $\tilde{u}$ remain close in $L^{10}_{t,x} \cap L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)$ by the virtue of (4.6) and (4.7).*

5 Concentration compactness

In this section, we take a digression from well-posedness and prove a *linear profile decomposition* for the Schrödinger propagator. As we mentioned in the introduction, such a decomposition is essentially a statement that captures all defects of compactness in the Strichartz inequality. Unfortunately, its statement and proof are rather long and involved. As such, we have decided to first study the profile decomposition for the easier L^2 based Gagliardo-Nirenberg inequality, which states that for a function $f \in H^1(\mathbb{R}^d)$,

$$\|f\|_{L^p} \lesssim_p \|f\|_{\dot{H}^1}^{\theta} \|f\|_{L^2}^{1-\theta}, \tag{5.1}$$

where

$$1 < p < 2, \qquad \theta = \frac{d(p-2)}{2p}$$

(see also Lemma A.1 in the Appendix). Note that the RHS of the inequality is bounded above by $\|f\|_{H^1}$. This implies the continuity of the embedding $H^1 \subseteq L^p$, or equivalently, the continuity of the identity operator $\mathrm{Id} : H^1 \to L^p$. To go one step further, can we show that this operator is compact?

To motivate why one might want compactness, let us consider the problem of showing the existence of an extremiser in the inequality (5.1). That is, we want to show that the the best constant of inequality (5.1), given by

$$S := \sup \left\{ J(f) := \frac{\|f\|_{L^p}}{\|f\|_{\dot{H}^1}^{\theta} \|f\|_{L^2}^{1-\theta}} : f \in H^1 \right\} \tag{5.2}$$

is attained at some $f \in H^1$. This is easy to prove if compactness holds. Indeed, let f_n be a sequence in H^1 such that $J(f_n) \to S$ as $n \to \infty$. By first rescaling the values of f_n, we may assume each $\|f_n\|_{\dot{H}^1} = 1$. If we also apply an appropriate $\dot{H}^1$-preserving scaling, we may assume that each $\|f_n\|_{L^2} = 1$. By compactness, we may pass to a subsequence so that f_n converges in L^p to some $f \in L^p$. Since $\|f_n\|_{\dot{H}^1} = 1 = \|f_n\|_{L^2}$, we may (by Banach Alaoglu's Theorem) pass to another subsequence so that f_n converges weakly in L^2 and $\dot{H}^1$ to f. Note that we also have $\|f_n\|_{L^2} \to \|f\|_{L^2}$ and $\|f_n\|_{H^1} \to \|f\|_{H^1}$, and hence f_n in fact converges strongly in the same spaces. Uniqueness of limits then imply that f_n converges to f in L^p, L^2 and $\dot{H}^1$ sense, and so f is our extremiser.

Unfortunately, compactness does not hold in this setting[3]. This can be seen by considering the unitary group of translations in H^1. Indeed, for any non-zero $\phi \in H^1$, consider the "travelling profile"

$$f_n := \phi(\cdot - x_n)$$

[3]Though one can still show that the space of radial functions in H^1 can be embedded compactly into L^p. One can then use the same argument as in the preceding paragraph to show the existence of an extremiser. But we shall not pursue this further here.

where $x_n \in \mathbb{R}^d$ converges to ∞. Clearly, $\|f_n\|_{H^1} = \|\phi\|_{H^1} = C > 0$. On the other hand, it is easy to see that $f_n \rightharpoonup 0$ in L^p (the half arrow denotes weak convergence here). And hence f_n has no convergent subsequence in L^p.

More generally, one can construct examples of sequences in H^1 with no convergent subsequences by considering a "superposition of travelling profiles". Suppose that $\phi^1, ..., \phi^J \in H^1$ are non-zero. Consider

$$f_n := \sum_{j=1}^{J} \phi^j(\cdot - x_n^j)$$

where each $x_n^j \to \infty$ as $n \to \infty$. Again, this implies that the above sequence converges weakly to 0 in L^p. This time, the norms $\|f_n\|_{H^1}$ need not stay constant. In fact, the travelling profiles might end up cancelling each other as $n \to \infty$. To prevent this from happening, one can impose the asymptotic orthogonality condition that $|x_n^j - x_n^k| \to \infty$ for each $j \neq k$ which ensures that the travelling profiles stay "far" from each other. In fact, we have the asymptotic decoupling

$$\|f_n\|_{H^1}^2 = \sum_{j=1}^{J} \left\|\phi^j\right\|_{H^1}^2 + o_n(1)$$

as $n \to \infty$. Hence $\|f_n\|_{H^1}$ is once again away from 0 and so f_n has no convergent subsequence in L^p.

The concentration compactness phenomenon, in this setting, tells us that these are essentially the only ways in which compactness fails. Loosely speaking, a profile decomposition tells us that if $\{f_n\}_{n=1}^\infty$ is a bounded sequence in H^1 that fails to have any convergent subsequence in L^q, then f_n at least has a subsequence that "converges" to a superposition of concentrating or travelling profiles of the form above.

Theorem 5.1 (Profile Decomposition for Gagliardo-Nirenberg Inequality [9])**.**
Let $d \geq 3$. Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in H^1. Then after possibly passing to a subsequence of $\{f_n\}_{n=1}^\infty$, there exists $J^* \in \mathbb{N}_0 \cup \{\infty\}$ such that for each finite $0 \leq J \leq J^*$ and each $n \in \mathbb{N}$, we have the decomposition

$$f_n = \sum_{j=1}^{J} \phi^j(\cdot - x_n^j) + r_n^J \tag{5.3}$$

where the profiles $\phi^j \in H^1$ are non-zero, the remainders $\left\{r_n^J\right\}_{n=1}^\infty \subseteq H^1$, and the translation parameters $\{x_n^j\}_{n=1}^\infty \subseteq \mathbb{R}^d$, such that for any $p \in (2, 2^*)$ where $2^* = \frac{2d}{d-2}$,

(i) $\lim\limits_{J \to J^*} \limsup\limits_{n \to \infty} \left\|r_n^J\right\|_{L^p} = 0$*;*

(ii) $\|f_n\|_{H^1}^2 = \sum_{J=1}^{J} \left\|\phi^J\right\|_{H^1}^2 + \left\|r_n^J\right\|_{H^1}^2 + o_n(1)$ *as* $n \to \infty$*;*

(iii) $\lim_{J\to J^*}\limsup_{n\to\infty}\left(\|f_n\|_{L^p}^p - \sum_{j=1}^J \left\|\phi^j\right\|_{L^p}^p\right) = 0;$

(iv) $r_n^J(\cdot + x_n^J) \rightharpoonup 0$ *in* H^1 *as* $n \to \infty$*;*

(v) $\lim_{n\to\infty} |x_n^k - x_n^j| = \infty$ *whenever* $j \neq k$*.*

Here, o_n *is the usual little-o notation, that is* $o_n(1)$ *is a quantity that converges to* 0 *as* $n \to \infty$*. Also, if* J^* *is finite, the limit* $\lim_{J\to J^*} a(J)$ *is simply taken to mean* $a(J^*)$*.*

Proof. We first fix some notations. For a bounded sequence $v = \{v_n\}_{n=1}^\infty \in H^1$, let

$$\mathcal{W}(v) = \left\{w \in H^1 :\ \text{up to a subsequence, } v_n(\cdot + x_n) \rightharpoonup w \text{ for some } \{x_n\}_{n=1}^\infty \subseteq \mathbb{R}^d\right\}$$

be the set of profiles of v. This set is non-empty since every bounded sequence in a normed space has a weakly convergent subsequence. We also let

$$\eta(v) = \sup_{w\in\mathcal{W}(v)} \|w\|_{H^1}.$$

Note that this quantity is finite since $\eta(v) \leq \limsup_{n\to\infty} \|v_n\|_{H^1}$.

We first recursively extract a large "bubble of concentration" ϕ^J from the remainder term and then check at each step J that Properties (ii) to (v) hold, with Property (iii) replaced with

$$\lim_{n\to\infty}\left(\|f_n\|_{L^p}^p - \sum_{j=1}^J \left\|\phi^j\right\|_{L^p}^p - \left\|r_n^J\right\|_{L^p}^p\right) = 0. \tag{iii*}$$

for any $p \in (2, 2^*)$. This of course implies (iii) once we have established (i), for which we shall verify last.

To start, we set $r_n^0 := f_n$. Assume that we have completed step K for some $K \geq 0$ so that Properties (ii) to (v) hold for each $J \leq K$, and wish to find ϕ^{K+1}, $\left\{r_n^{K+1}\right\}_{n=1}^\infty$ and $\{x_n^{K+1}\}_{n=1}^\infty$. If $\eta(r) = 0$, we terminate the iteration as before and set $J^* = K + 1$. Otherwise we have $\eta(r^K) > 0$ and we find some $\phi^{K+1} \in \mathcal{W}(r)$ such that $\left\|\phi^{K+1}\right\|_{H^1} > \frac{1}{2}\eta(r)$. By definition, this means that after passing to a subsequence, there exists $\{x_n^{K+1}\}_{n=1}^\infty \subseteq \mathbb{R}^d$ such that

$$r_n^K(\cdot + x_n^{K+1}) \rightharpoonup \phi^{K+1} \text{ as } n \to \infty.$$

Let $r_n^{K+1} := f_n - \sum_{j=1}^{K+1} \phi^j(\cdot - x_n^j)$. Note that by construction in the previous step, we have

$$r_n^{K+1} = r_n^K - \phi^{K+1}(\cdot - x_n^{K+1}).$$

In particular, this means (iv) holds at the $(K + 1)$-th level.

$$r_n^{K+1}(\cdot + x_n^{K+1}) \rightharpoonup 0 \quad \text{as } n \to \infty. \tag{5.4}$$

Turning to (ii), we have, by the translation invariance of the H^1 norm,

$$\begin{aligned}\left\|r_n^{K+1}\right\|_{H^1}^2 &= \left\langle r_n^K(\cdot + x_n^{K+1}) - \phi^{K+1}, r_n^K(\cdot + x_n^{K+1}) - \phi^{K+1}\right\rangle_{H^1} \\ &= \left\|r_n^K\right\|_{H^1}^2 + 2\mathrm{Re}\left\langle r_n^K(\cdot + x_n^{K+1}), \phi^K\right\rangle_{H^1} + \left\|\phi^K\right\|_{H^1}^2 \\ &= \left\|r_n^K\right\|_{H^1}^2 + \left\|\phi^K\right\|_{H^1}^2 + o_n(1)\end{aligned}$$

as $n \to \infty$. Hence by induction,

$$\begin{aligned}\|f\|^2 &= \sum_{j=1}^{K} \left\|\phi^j\right\|_{H^1}^2 + \left\|r_n^K\right\|_{H^1}^2 + o_n(1) \\ &= \sum_{j=1}^{K+1} \left\|\phi^j\right\|_{H^1}^2 + \left\|r_n^{K+1}\right\|_{H^1}^2 + o_n(1),\end{aligned}$$

proving (ii). To prove (iii*), we first restrict our attention to the cube $[-R,R]^d \subseteq \mathbb{R}^d$ for some $R > 0$. By the Rellich Kondrachov Theorem, we may pass to a subsequence so that $w_K := \{r^K(\cdot + x_n^{K+1})_n\}_{n=1}^\infty$ is (strongly) convergent in $L^2\left([-R,R]^d\right)$. By a diagonal argument, we may pass to a subsequence so that w_K converges in the whole $L^2(\mathbb{R}^d)$. Passing to a subsequence once more, we have that w_K converges almost everywhere to some $L^2(\mathbb{R}^d)$ function. Note that this function coincides with ϕ^{K+1}. Hence we may apply the lemma of Brézis-Lieb (Lemma A.8) to obtain for $p \in (2, 2^*)$,

$$\begin{aligned}\left\|r_n^K\right\|_{L^p}^p &= \left\|r_n^K(\cdot + x_n^{K+1}) - \phi^{K+1}\right\|_{L^p}^p + \left\|\phi_n^{K+1}\right\|_{L^p}^p + o_n(1) \\ &= \left\|r_n^{K+1}\right\|_{L^p}^p + \left\|\phi_n^{K+1}\right\|_{L^p}^p + o_n(1).\end{aligned}$$

By induction, we thus have

$$\begin{aligned}\|f_n\|_{L^p}^p &= \sum_{j=1}^{K} \left\|\phi_n^j\right\|_{L^p}^p + \left\|r_n^K\right\|_{L^p}^p + o_n(1) \\ &= \sum_{j=1}^{K+1} \left\|\phi_n^j\right\|_{L^p}^p + \left\|r_n^{K+1}\right\|_{L^p}^p + o_n(1).\end{aligned}$$

This proves (iii*). We now check (v). By induction, we just need to show that for any $J \le K$, $|x_n^J - x_n^{K+1}| \to \infty$ as $n \to \infty$. Suppose not, and let J be the largest $J \le K$ so that $|x_n^J - x_n^{K+1}| \not\to \infty$. Then $x_n^J - x_n^{K+1}$ must have a bounded subsequence in n, and hence by passing to a subsequence, $\lim_{n\to\infty} x_n^J - x_n^{K+1} = x_0$ for some $x_0 \in \mathbb{R}$ and some $J \le K$. Decomposing f_n at levels J and $K+1$, we find

$$\sum_{j=1}^{J} \phi^j(\cdot - x_n^j) + r_n^J = \sum_{j=1}^{K+1} \phi^j(\cdot - x_n^j) + r_n^{K+1},$$

and hence

$$r_n^J(\cdot + x_n^{K+1}) = \sum_{j=J+1}^{K+1} \phi^j(\cdot - x_n^j + x_n^{K+1}) + r_n^{K+1}(\cdot + x_n^{K+1}),$$

We see LHS converges weakly to 0 in H^1 by writing $r_n^J(\cdot+x_n^{K+1}) = r_n^J(\cdot+x_n^J+(x_n^{K+1}-x_n^J))$. On RHS, we see that every term except $\phi^{K+1}(\cdot)$ converges weakly to 0. But this implies that $\phi^{K+1} \equiv 0$, which is a contradiction. This proves (v).

We finally turn our attention to (v). We first note that by construction, $\eta(r^J)$ vanishes as $J \to J^*$. Indeed, if J^* is finite, then we have $\eta(r^{J^*}) = 0$. For $J^* = \infty$, (iii) implies that the series $\sum_{j=1}^{\infty} \|\phi^j\|_{H^1}$ converges, and in particular, $\|\phi^j\|_{H^1} \to 0$ as $j \to \infty$. By construction, we have

$$\eta(r^j) < 2\|\phi^j\|_{H^1} \to 0 \quad \text{as } j \to \infty. \tag{5.5}$$

This fact will be needed later to control r_n^J on low frequencies. Indeed, we shall consider the frequency cutoff

$$\widehat{Q_R f}(\xi) := \mathbb{1}_{|\xi|\le R}(\xi)\hat{f}(\xi).$$

Note that $Q_R f = K_R * f$ where $\widehat{K_R f} = \mathbb{1}_{[-R,R]}$. We shall split

$$\|r^J\|_{L^p} \le \|Q_R r_n^J\|_{L^p} + \|(\mathrm{id} - Q_R) r_n^J\|_{L^p}.$$

We estimate the second term as follows. Let s be the real number satisfying $\frac{s}{d} = \frac{1}{2} - \frac{1}{p}$. Note that $p < 2^*$ implies that $s < 1$. An application of Sobolev embedding followed by Plancherel's Theorem gives

$$\begin{aligned}
\|(\mathrm{id} - Q_R) r_n^J\|_{L^p} &\lesssim_p \|(\mathrm{id} - Q_R) r_n^J\|_{\dot{H}^s} \\
&= \left(\int_{|\xi|>R} |\xi|^{2s} |\widehat{r_n^J}(\xi)|^2 \, d\xi \right)^{1/2} \\
&\le \left(\int_{|\xi|>R} \frac{|\xi|^2}{R^{2-2s}} |\widehat{r_n^J}(\xi)|^2 \, d\xi \right)^{1/2} \\
&\le R^{s-1} \|r_n^J\|_{H^1},
\end{aligned}$$

where the penultimate inequality follows from multiplying the integrand by $\left(\frac{|\xi|}{R}\right)^{2-2s}$, which is greater than 1. For the first term, observe that by interpolation followered by Plancherel's Theorem,

$$\begin{aligned}
\|Q_R r_n^J\|_{L^p} &\le \|Q_R r_n^J\|_{L^2}^{2/p} \|Q_R r_n^J\|_{L^\infty}^{1-2/p} \le \|r_n^J\|_{L^2}^{2/p} \|Q_R r_n^J\|_{L^\infty}^{1-2/p} \\
&\le \|r_n^J\|_{H^1}^{2/p} \|Q_R r_n^J\|_{L^\infty}^{1-2/p}
\end{aligned}$$

Now, we may rewrite $\limsup_{n\to\infty} \left\|Q_R r_n^J\right\|_{L^\infty}$ as $\sup\left(\limsup_{n\to\infty} \left|Q_R r_n^J\,(x_n)\,\right|\right)$, where the supremum is taken over all $\mathbb{R}^n$ sequences $\{x_n\}_{n=1}^{\infty}$. We have the estimate

$$\begin{aligned}
\limsup_{n\to\infty} \left|Q_R r_n^J\right|(x_n) &= \limsup_{n\to\infty} \left|K_R * r_n^J(x_n)\right| \\
&= \limsup_{n\to\infty} \left|\int K_R(-x) r_n^J(x+x_n)\,dx\right| \\
&\le \sup\left\{\left|\int K_R(-x) w(x)\,dx\right| : w \in \mathcal{W}(r^J)\right\} \\
&\le \|K_R\|_{L^2}\, \eta\left(r^J\right),
\end{aligned}$$

where the last inequality follows from Cauchy Schwartz. Putting everything together,

$$\limsup_{n\to\infty} \left\|r_n^J\right\|_{L^p} \lesssim \limsup_{n\to\infty} \left(R^{s-1}\left\|r_n^J\right\|_{H^1} + \left\|r_n^J\right\|_{H^1}^{2/p}\left(\|K_R\|_{L^2}\,\eta\left(r^J\right)\right)^{1-2/p}\right).$$

The first term on RHS can be made arbitrarily small by choosing R large. For the second term, we recall that $\eta(r^J) \to 0$ as $J \to J^*$. Hence choosing J large will also ensure that the second term is small. It then follows that

$$\limsup_{n\to\infty} \left\|r_n^J\right\|_{L^p} \to 0 \text{ as } J \to J^*. \qquad \square$$

As a small application, one can use the profile decomposition as a replacement for compactness to prove the existence of an extremiser.

Corollary 5.2. *The Gagliardo-Nirenberg inequality (5.1) has an extremiser.*

Proof. As in our argument above, let $\{f_n\}_{n=1}^{\infty} \subseteq \dot{H}^1(\mathbb{R}^d)$ be an optimising sequence, for which we may assume $\|f\|_{\dot{H}^1} = \|f_n\|_{L^2} = 1$ for each n. We now decompose a subsequence of f_n as in Theorem 5.3. Let S be the best constant of the inequality as in (5.2). By property (iv) we obtain

$$S^p = \lim_{n\to\infty} \|f_n\|_{L^p}^p = \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{L^p}^p \le S^p \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{\dot{H}^1}^{p\theta} \left\|\phi^j\right\|_{L^2}^{p(1-\theta)} \le S^p \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{H^1}^p,$$

Now property (iii) and our choice of f_n implies that $\sum_{j=1}^{J^*} \left\|\phi^j\right\|_{H^1}^2 \le 1$. Moreover, since $p > 2$, we in fact have

$$S^p \le S^p \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{H^1}^p \le S^p \sum_{j=1}^{J^*} \left\|\phi^j\right\|_{H^1}^2 \le S^p,$$

which can only mean $J^* = 1$ and that $\left\|\phi^1\right\|_{\dot{H}^1} = 1$. Our decomposition then reads $f_n = \phi^1(\cdot - x_n^1) + r_n^1$. Property (ii) then implies that $f_n(\cdot + x_n^1) \rightharpoonup \phi^1$ in H^1. This weak convergence can be upgraded to strong convergence as $\|f_n\|_{H^1} = \left\|\phi^1\right\|_{\dot{H}^1} = 1$. Since translation leaves the L^2, H^1 and L^p norms invariant, we may use ϕ^1 as our extremiser. $\square$

One can also form a profile decomposition for the Sobolev inequality (Theorem A.3):

$$\|f\|_{L^{2^*}} \lesssim \|f\|_{\dot{H}^1}\,. \tag{5.6}$$

For convenience later on, let us fix the notations

$$\tau_y f(x) = f(x-y) \tag{5.7}$$

$$\delta_\lambda f(x) = \lambda^{-\frac{d-2}{2}} f\left(\lambda^{-1}x\right) \tag{5.8}$$

for spatial translation and scaling parameters $y \in \mathbb{R}^d$, $\lambda \in (0,\infty)$, and a function $f : \mathbb{R}^d \to \mathbb{C}$. Note that $\tau_y^{-1} = \tau_{-y}$ and $\delta_\lambda^{-1} = \delta_{\lambda^{-1}}$.

The identity operator $\mathrm{Id} : \dot{H}^1 \to L^{2^*}$ is not compact. We have seen how the translation group $\{\tau_y\}_{y\in\mathbb{R}^d}$ causes a lack of compactness in Gagliardo-Nirenberg inequality. For the Sobolev inequality, the translation group is once again a culprit. Since we are now working on the critical space $\dot{H}^1$, we also have the unitary group of $\dot{H}^1$-preserving scaling to worry about. Indeed, it is not difficult to see that if we have a sequence of scaling parameters $\{\lambda_n\}_{n=1}^\infty$ that converges to ∞, then $\delta_{\lambda_n} f \rightharpoonup 0$ in L^{2^*}. If $f \in \dot{H}^1$ is non-zero, then $\delta_{\lambda_n} f$ has no convergent subsequence in L^{2^*} since $\left\|\delta_{\lambda_n} f\right\|_{\dot{H}^1} = \|f\|_{\dot{H}^1} > 0$.

The profile decomposition for Sobolev inequality is given as follows:

Theorem 5.3 (Profile Decomposition for Sobolev Inequality [10][13])**.**
Let $d \geq 3$. Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $\dot{H}^1$. Then after possibly passing to a subsequence of $\{f_n\}_{n=1}^\infty$, there exists $J^* \in \mathbb{N}_0 \cup \{\infty\}$ such that for each finite $0 \leq J \leq J^*$ and each $n \in \mathbb{N}$, we have the decomposition

$$f_n = \left(\sum_{j=1}^J \tau_{x_n^j}\delta_{\lambda_n^j}\phi^j\right) + r_n^J \tag{5.9}$$

where the profiles $\phi^j \in \dot{H}^1$ are non-zero, the remainders $\left\{r_n^J\right\}_{n=1}^\infty \subseteq \dot{H}^1$, and the translation and scaling parameters $\left\{\left(x_n^j, \lambda_n^j\right)\right\}_{n=1}^\infty \subseteq \mathbb{R}^d \times (0,\infty)$ satisfy

(i) $\lim\limits_{J\to J^*} \limsup\limits_{n\to\infty} \left\|r_n^J\right\|_{L^{2^*}} = 0$;

(ii) $\|f_n\|_{\dot{H}^1}^2 = \sum_{j=1}^J \left\|\phi^j\right\|_{\dot{H}^1}^2 + \left\|r_n^J\right\|_{\dot{H}^1}^2 + o_n(1)$ *as* $n \to \infty$;

(iii) $\lim\limits_{J\to J^*} \limsup\limits_{n\to\infty} \left(\|f_n\|_{L^{2^*}}^{2^*} - \sum_{j=1}^J \left\|\phi^j\right\|_{L^{2^*}}^{2^*}\right) = 0$;

(iv) $\delta_{\lambda_n^j}^{-1}\tau_{x_n^j}^{-1} r_n^J \rightharpoonup 0$ *in* $\dot{H}^1$ *as* $n \to \infty$;

(v) $\lim\limits_{n\to\infty} \left(\dfrac{|x_n^k - x_n^j|^2}{\lambda_n^j\lambda_n^k} + \dfrac{\lambda_n^k}{\lambda_n^j} + \dfrac{\lambda_n^j}{\lambda_n^k}\right) = \infty$ *whenever* $j \neq k$.

We shall omit the proof for this result since it is very similar to the proof of the profile decomposition for Schrödinger propagator, which we shall present shortly.

We finally move onto our last profile decomposition, and indeed, the one we need for the induction on energy argument. We shall once again restrict our attention to dimension $d = 3$ (the proof for higher dimensions is almost the same, but this restriction will let us avoid nasty fractions in terms of d). The inequality for which we shall study is

$$\|S(t)f\|_{L^{10}_{t,x}} \lesssim \|S(t)f\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x} \lesssim \|f\|_{\dot{H}^1_x}, \tag{5.10}$$

which follows from Sobolev and Strichartz inequalities. This gives rise to the continuous operator $T : \dot{H}^1_x \to L^{10}_{t,x}$ defined by $Tf = S(t)f$. In addition to spatial translation and $\dot{H}^1$-preserving scaling, the unitary group of time translation is also causing compactness to fail in this setting. By time translation, we mean the action of symmetry group $\{S(t')\}_{t'\in\mathbb{R}}$ on $\dot{H}^1_x$. Indeed, it is easy to see that if $t_n \to \infty$, then $S(t - t_n)f$ converges weakly to 0 in $L^{10}_{t,x}$ for any $f \in \dot{H}^1$. In fact, one can show something slightly stronger:

Lemma 5.4. *Suppose that $\{(x_n, t_n)\}_{n=1}^{\infty} \subseteq \mathbb{R}^d \times \mathbb{R}$ satisfy $|x_n| \to \infty$ or $|t_n| \to \infty$. Then for any $f \in \dot{H}^1$, $S(t_n)\tau_{x_n} f \rightharpoonup 0$ in $\dot{H}^1$ as $n \to \infty$.*

Proof. By a density argument, it suffices to assume $f \in C_c^\infty(\mathbb{R}^3)$, and to prove $\left|\langle S(t_n)\tau_{x_n} f, \varphi\rangle_{\dot{H}^1}\right| \to 0$ as $n \to \infty$ for every $\varphi \in C_c^\infty(\mathbb{R}^3)$. Assume $|t_n| \to \infty$. Then

$$\begin{aligned}\left|\langle S(t_n)\tau_{x_n} f, \varphi\rangle_{\dot{H}^1}\right| &\le \left\|S(t_n)\tau_{x_n} f\right\|_{L^6_x} \left\||\nabla|^2\varphi\right\|_{L^{\frac{6}{5}}_x} \\ &\lesssim \frac{1}{|t_n|}\|f\|_{L^{\frac{6}{5}}_x} \left\||\nabla|^2\varphi\right\|_{L^{\frac{6}{5}}_x} \\ &\to 0\end{aligned}$$

as $n \to \infty$, where we used by Hölder and dispersive estimate above. Now assume that $|t_n| \not\to \infty$ but $|x_n| \to \infty$. Take a subsequence $\{(x_{n_k}, t_{n_k})\}_{k=1}^{\infty}$ so that

$$\limsup_{n\to\infty} \left|\langle S(t_n)\tau_{x_n} f, \varphi\rangle_{\dot{H}^1}\right| = \lim_{k\to\infty} \left|\left\langle S(t_{n_k})\tau_{x_{n_k}} f, \varphi\right\rangle_{\dot{H}^1}\right|.$$

By passing to a subsequence once more (which we still denote by $\{(x_{n_k}, t_{n_k})\}_{k=1}^{\infty}$), we may assume $t_{n_k} \to t_\infty \in \mathbb{R}$ as $k \to \infty$. Then

$$\begin{aligned}\left|\left\langle S(t_{n_k})\tau_{x_{n_k}} f, \varphi\right\rangle_{\dot{H}^1}\right| &= \left|\left\langle \tau_{x_{n_k}} f, S(-t_{n_k})\varphi\right\rangle_{\dot{H}^1}\right| \\ &\le \left|\left\langle \tau_{x_{n_k}} f, S(-t_\infty)\varphi\right\rangle_{\dot{H}^1}\right| + \|f\|_{\dot{H}^1_x} \|S(-t_{n_k}) - S(-t_\infty)\|_{\mathrm{op}} \|\varphi\|_{\dot{H}^1_x}\end{aligned}$$

where the last inequality results from triangle and Cauchy Schwartz inequalities. The first quantity on the RHS converges to 0 because $|x_n| \to \infty$, while the second quantity also converges to 0 because $S(-t_{n_k}) \to S(-t_\infty)$ as operators on $\dot{H}^1$. ☐

The profile decomposition for Schrödinger propagator reads as follows, and its proof will occupy the rest of this section.

Theorem 5.5 (Profile decomposition for Schrödinger propagator[14][19]).
Suppose that $\{f_n\}_{n=1}^{\infty}$ *is a bounded sequence in* $\dot{H}^1$*. Then after possibly passing to a subsequence of* $\{f_n\}_{n=1}^{\infty}$*, there exists* $J^* \in \mathbb{N}_0 \cup \{\infty\}$ *such that for each finite* $0 \le J \le J^*$ *and each* $n \in \mathbb{N}$*, we have the decomposition*

$$f_n = \left(\sum_{j=1}^{J} \tau_{x_n^j} \delta_{\lambda_n^j} S(t_n^j) \phi^j \right) + r_n^J \tag{5.11}$$

where the profiles $\phi^j \in \dot{H}^1$ *are non-zero, the remainders* $\{r_n^J\}_{n=1}^{\infty} \subseteq \dot{H}^1$*, and the symmetry parameters* $\{\left(\lambda_n^j, t_n^j, x_n^j\right)\}_{n=1}^{\infty} \subseteq (0,\infty) \times \mathbb{R} \times \mathbb{R}^d$ *satisfy*

(i) $\lim_{J \to J^*} \limsup_{n\to\infty} \left\| S(t) r_n^J \right\|_{L_{t,x}^{10}} = 0$;

(ii) $\|f_n\|_{\dot{H}^1}^2 = \sum_{j=1}^{J} \left\|\phi^j\right\|_{\dot{H}^1}^2 + \left\|r_n^J\right\|_{\dot{H}^1}^2 + o_n(1)$ *as* $n \to \infty$;

(iii) $\|f_n\|_{L_x^6}^6 - \sum_{j=1}^{J} \left\|S(t_n^j)\phi^j\right\|_{L_x^6}^6 + \left\|r_n^J\right\|_{L_x^6}^6 + o_n(1)$ *as* $n \to \infty$;

(iv) $\limsup_{J\to J^*} \lim_{n\to\infty} \left(\|S(t) f_n\|_{L_x^{10}}^{10} - \sum_{j=1}^{J} \left\|S(t)\phi^j\right\|_{L_x^{10}}^{10} \right) = 0$;

(v) $S(-t_n^J)\delta_{\lambda_n^J}^{-1} \tau_{x_n^J}^{-1} r_n^J \rightharpoonup 0$ *in* $\dot{H}^1$ *as* $n \to \infty$;

(vi) $\lim_{n\to\infty} \left(\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} + \frac{|x_n^k - x_n^j|^2}{\lambda_n^j \lambda_n^k} + \frac{|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \right) = \infty$ *whenever* $j \ne k$;

(vii) *for each fixed* j*, either* $t_n^j \equiv 0$*, or* $\lim_{n\to\infty} t_n^j \in \{\infty, -\infty\}$.

Remark 5.6. *The asymptotic decouplings in* $\dot{H}_x^1$ *and* L_x^6 *topologies in (ii) and (iii) an be combined to give energy decoupling:*

$$\lim_{n\to\infty} \left(E(f_n) - \sum_{j=1}^{J} E(S(t_n^j)\phi^j) - E(r_n^J) \right) = 0$$

We first prove a refinement of our inequality (5.10).

Lemma 5.7 (Refined Strichartz Inequality)**.** *For $f \in \dot{H}^1(\mathbb{R}^3)$,*

$$\|S(t)f\|_{L^{10}_{t,x}} \lesssim \|f\|_{\dot{H}^1}^{\frac{1}{5}} \sup_{N \in 2^{\mathbb{Z}}} \|S(t)P_N f\|_{L^{10}_{t,x}}^{\frac{4}{5}}.$$

Proof. We shall first write down the following long chain on inequalities and only justify the steps afterwards. Here, we write f_N to mean the Littlewood-Paley projection $P_N f$. All summations on capital letters are over the dyadic numbers $2^{\mathbb{Z}}$.

$$\begin{aligned}
\|S(t)f\|_{L^{10}_{t,x}}^{10} &\overset{\text{(i)}}{\sim} \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |S(t)f_N|^2 \right)^{1/2} \right\|_{L^{10}_{t,x}}^{10} \\
&= \int_{\mathbb{R}\times\mathbb{R}^d} \prod_{i=1}^{5} \left(\sum_{N_i \in 2^{\mathbb{Z}}} |S(t)f_{N_i}|^2 \right) \\
&\overset{\text{(ii)}}{\lesssim} \sum_{N_1 \le N_2 \le N_3 \le N_4 \le N_5} \int_{\mathbb{R}\times\mathbb{R}^d} \prod_{i=1}^{5} |S(t)f_{N_i}|^2 \\
&\overset{\text{(iii)}}{\le} \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}} \right)^8 \sum_{N_1 \le \ldots \le N_5} \|S(t)f_{N_1}\|_{L^{q_1}_t L^{r_1}_x} \|S(t)f_{N_5}\|_{L^{q_2}_t L^{r_2}_x} \\
&\overset{\text{(iv)}}{\lesssim} \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}} \right)^8 \sum_{N_1 \le \ldots \le N_5} N_1^{c_1} N_5^{c_2} \|S(t)f_{N_1}\|_{L^{q_1}_t L^{p_1}_x} \|S(t)f_{N_5}\|_{L^{q_2}_t L^{p_2}_x} \\
&\overset{\text{(v)}}{\lesssim} \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}} \right)^8 \sum_{N_1 \le \ldots \le N_5} N_1^{c_1} N_5^{c_2} \|f_{N_1}\|_{L^2_x} \|f_{N_5}\|_{L^2_x} \\
&\lesssim \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}} \right)^8 \sum_{N_1 \le N_5} N_1^{c_1-1} N_5^{c_2-1} \|f_{N_1}\|_{\dot{H}^1} \|f_{N_5}\|_{\dot{H}^1} \sum_{N_1 \le N_2, N_3, N_4 \le N_5} 1 \\
&\lesssim \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}} \right)^8 \sum_{N_1 \le N_5} N_1^{c_1-1} N_5^{c_2-1} \left(\log_2 \left(\frac{N_5}{N_1} \right) \right)^3 \|f_{N_1}\|_{\dot{H}^1} \|f_{N_5}\|_{\dot{H}^1}.
\end{aligned}$$

(i) See the square function estimate in Lemma A.6.

(ii) We break the summation as follows:

$$\sum_{N_1} \cdots \sum_{N_5} \le \sum_{\substack{\text{all permutations} \\ \text{of } (N_1,\ldots,N_5)}} \left(\sum_{N_1} \sum_{\substack{N_2 \\ N_2 \ge N_1}} \cdots \sum_{\substack{N_5 \\ N_5 \ge N_4}} \right) = 5! \sum_{N_1 \le N_2 \le N_3 \le N_4 \le N_5}$$

where the second equality is possible because the summand $\prod_{i=1}^{5}|S(t)f_{N_i}|^2$ is symmetric in each argument.

(iii) We break the integral with Hölder, where the parameters $1 \le q_1, q_2, r_1, r_2 \le \infty$ are chosen later. Of course they satisfy

$$\frac{8}{10} + \frac{1}{q_1} + \frac{1}{q_2} = 1 = \frac{8}{10} + \frac{1}{r_1} + \frac{1}{r_2}. \tag{5.12}$$

(iv) We use Bernstein estimate (see Lemma A.5) first to gain a factor of $N_1^{c_1}$ and $N_5^{c_2}$, where c_1 and c_2 are chosen later. This means we want

$$\frac{3}{p_1} - \frac{3}{r_1} = c_1 \qquad \frac{3}{p_2} - \frac{3}{r_2} = c_2 \tag{5.13}$$

(v) We want to use Strichartz here, so we want the pairs (q_1, p_1) and (q_2, p_2) to be admissible:

$$\frac{2}{q_1} + \frac{3}{p_1} = \frac{3}{2} = \frac{2}{q_2} + \frac{3}{p_2} \tag{5.14}$$

For reasons that will be clear later, we also want the exponents $c_1 - 1 = -(c_2 - 1)$. It turns out that one can choose parameters

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}, \quad p_1 = \frac{30}{11}, \quad p_2 = 2, \quad q_1 = 5, \quad q_2 = \infty, \quad r_1 = 5, \quad r_2 = \infty$$

(these numbers are not unique nor important, but one should of course ensure that these parameters satisfy the constraints). Thus the LHS is controlled by

$$\left(\sup_{L\in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}}\right)^8 \sum_{N_1 \le N_5} \left(\frac{N_1}{N_5}\right)^{\frac{1}{2}} \left(\log_2\left(\frac{N_5}{N_1}\right)\right)^3 \|f_{N_1}\|_{\dot{H}^1} \|f_{N_5}\|_{\dot{H}^1}.$$

Observe that $\left(\frac{N_1}{N_5}\right)^{\frac{1}{2}} \left(\log_2\left(\frac{N_5}{N_1}\right)\right)^3 \lesssim_\epsilon \left(\frac{N_1}{N_5}\right)^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. Applying Cauchy Schwartz on the sum over N_5, we have

$$\sum_{N_5} \sum_{\substack{N_1 \\ N_1 \le N_5}} \left(\frac{N_1}{N_5}\right)^{\frac{1}{2}-\epsilon} \|f_{N_1}\|_{\dot{H}^1} \|f_{N_5}\|_{\dot{H}^1} \le \left\| \|f_{N_5}\|_{\dot{H}^1} \right\|_{\ell^2_{N_5}(2^{\mathbb{Z}})} \left\| \sum_{\substack{N_1 \\ N_1 \le N_5}} \left(\frac{N_1}{N_5}\right)^{1-2\epsilon} \|f_{N_5}\|_{\dot{H}^1} \right\|_{\ell^2_{N_5}(2^{\mathbb{Z}})}$$

To close the argument, we want to show that the integral operator T defined by

$$T(a_{N_5}) = \sum_{\substack{N_1 \\ N_1 \le N_5}} \left(\frac{N_1}{N_5}\right)^{1-2\epsilon} a_{N_5}$$

is bounded from $\ell^2(2^{\mathbb{Z}}) \to \ell^2(2^{\mathbb{Z}})$. This can be seen to be the case by Schur's test (see Lemma A.7). Indeed, the integral kernel here is $K(N_1, N_5) = \mathbb{1}_{N_1 \leq N_5} \left(\frac{N_1}{N_5}\right)^{1-2\epsilon}$, and the use of Schur's test is justified by the uniform bounds

$$\sum_{N_1} K(N_1, N_5) = \sum_{\substack{N_1 \\ N_1 \leq N_5}} \left(\frac{N_1}{N_5}\right)^{1-2\epsilon} = \sum_{m=0}^{\infty} 2^{m(2\epsilon-1)} \lesssim 1 \quad \forall N_5 \in 2^{\mathbb{Z}},$$

$$\sum_{N_5} K(N_1, N_5) = \sum_{\substack{N_5 \\ N_5 \geq N_1}} \left(\frac{N_1}{N_5}\right)^{1-2\epsilon} = \sum_{m=0}^{\infty} 2^{m(2\epsilon-1)} \lesssim 1 \quad \forall N_1 \in 2^{\mathbb{Z}}$$

provided $\epsilon = \frac{1}{1000}$, say. Thus

$$\begin{aligned}\|S(t)f\|_{L^{10}_{t,x}}^{10} &\lesssim \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}}\right)^8 \left\|\|f_N\|_{\dot{H}^1}\right\|^2_{\ell^2_N(2^{\mathbb{Z}})} \\ &\sim \left(\sup_{L \in 2^{\mathbb{Z}}} \|S(t)f_L\|_{L^{10}_{t,x}}\right)^8 \|f\|_{\dot{H}^1}{}^2\end{aligned}$$

by the square function estimate. □

Before we continue on with the gory details of the proof, let us first record the following easily verified relations between the symmetries:

$$\tau_y \delta_\lambda f = \delta_\lambda \tau_{\frac{y}{\lambda}} f \qquad \text{and} \qquad \delta_\lambda \tau_y f = \tau_{\lambda y} \delta_\lambda f \tag{5.15}$$

$$S(t')\delta_\lambda f = \delta_\lambda S\left(\lambda^{-2} t'\right) f \qquad \text{and} \qquad \delta_\lambda S(t') f = S\left(\lambda^2 t'\right) \delta_\lambda f \tag{5.16}$$

$$\tau_y S(t') f = S(t') \tau_y \tag{5.17}$$

As in the proof of the profile decomposition for Gagliardo-Nirenberg inequality, the main idea of the proof of Theorem 5.5 will be to recursively extract a large bubble of concentration from f_n. The asymptotic decoupling in $\dot{H}^1$ is nothing more than a consequence of weak convergence and the fact that $\dot{H}^1$ is a Hilbert space, and the former is built into our construction of ϕ^J and r_n^J. That the remainder r_n^J vanishes asymptotically in $L^{10}_{t,x}$ topology is a bit more subtle, and requires the help of Lemma 5.7. For decoupling in L^6 and $L^{10}_{t,x}$ topologies, we will once again appeal to Brezis-Lieb lemma; the tricky part will be to prove the almost sure convergence required to apply this lemma.

Proposition 5.8 (Inverse Strichartz Inequality)**.** *Suppose that $\{f_n\}_{n=1}^\infty$ is a bounded sequence in $\dot{H}^1$ satisfying*

$$\lim_{n\to\infty} \|f_n\|_{\dot{H}^1} = A \quad \text{ and } \quad \lim_{n\to\infty} \|S(t)f_n\|_{L^{10}_{t,x}} = \epsilon$$

for some $A \geq 0$ and $\epsilon > 0$. Then up to a subsequence of $\{f_n\}_{n=1}^\infty$, there exist $\phi \in \dot{H}^1$ and sequences $\{x_n\}_{n=1}^\infty \subseteq \mathbb{R}^d$, $\{t_n\}_{n=1}^\infty \subseteq \mathbb{R}$, and $\{\lambda_n\}_{n=1}^\infty \subseteq (0, \infty)$, such that

(i) $S(-t_n)\delta_{\lambda_n}^{-1}\tau_{x_n}^{-1}f_n \rightharpoonup \phi$ *in* $\dot{H}^1$*;*

(ii) $\|f_n\|_{\dot{H}^1}^2 = \left\|f_n - \tau_{x_n}\delta_{\lambda_n}S(t_n)\phi\right\|_{\dot{H}^1}^2 + \|\phi\|_{\dot{H}^1}^2 + o_n(1)$ *as* $n \to \infty$*;*

(iii) $\|\phi\|_{\dot{H}^1}^2 \gtrsim A^2\left(\frac{\epsilon}{A}\right)^{\frac{15}{4}}$;

(iv) $\|S(t)f_n\|_{L_{t,x}^{10}}^{10} = \left\|S(t)f_n - S(t+\lambda_n^2 t_n)\tau_{x_n}\delta_{\lambda_n}\phi\right\|_{L_{t,x}^{10}}^{10} + \|S(t)\phi\|_{L_{t,x}^{10}}^{10} + o_n(1)$ *as* $n \to \infty$*;*

(v) $\|S(t)\phi\|_{L_{t,x}^{10}}^{10} \gtrsim \epsilon^{10}\left(\frac{\epsilon}{A}\right)^{\frac{35}{4}}$;

(vi) $\|f_n\|_{L_x^6}^6 = \left\|f_n - \tau_{x_n}\delta_{\lambda_n}S(t_n)\phi\right\|_{L_x^6}^6 + \|S(t_n)\phi\|_{L_x^6}^6 + o_n(1)$ *as* $n \to \infty$.

Proof. We first note that (i) and (ii) actually hold for *any* choice of $\{\lambda_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$. Indeed, let $g_n := S(-t_n)\delta_{\lambda_n}^{-1}\tau_{x_n}^{-1}f_n$. Then $\|g_n\|_{\dot{H}^1} = \|f_n\|_{\dot{H}^1}$ and so $\{g_n\}_{n=1}^{\infty}$ is bounded in $\dot{H}^1$, Hence there is some $\phi \in \dot{H}^1$ such that up to subsequence, $g_n \rightharpoonup \phi$. Property (ii) then follows from the identity

$$\left\|f_n - \tau_{x_n}\delta_{\lambda_n}S\left(t_n\right)\phi\right\|_{\dot{H}^1} = \|g_n - \phi\|_{\dot{H}^1}$$

and the Hilbert space fact

$$g_n \rightharpoonup \phi \implies \|g_n\|_{\dot{H}^1}^2 - \|g_n - \phi\|_{\dot{H}^1}^2 \to \|\phi\|_{H^1}^2 \quad \text{as } n \to \infty.$$

The proof then boils down to choosing x_n, t_n and λ_n appropriately so that the remaining properties hold.
Using Lemma 5.7, we have the estimate

$$\|S(t)f_n\|_{L_{t,x}^{10}} \lesssim \|f_n\|_{\dot{H}^1}^{\frac{1}{5}} \sup_{N\in 2^{\mathbb{Z}}} \|S(t)P_N f_n\|_{L^{10}}^{\frac{4}{5}},$$

for each $n \in \mathbb{N}$. By choosing $N_n \in 2^{\mathbb{Z}}$ so that

$$\|S(t)P_{N_n}f_n\|_{L_{t,x}^{10}} > \frac{1}{2}\sup_{N\in 2^{\mathbb{Z}}} \|S(t)P_N f_n\|_{L_{t,x}^{10}},$$

we have the estimate

$$\begin{aligned}\epsilon^{\frac{5}{4}}A^{-\frac{1}{4}} &= \lim_{n\to\infty} \|S(t)f_n\|_{L_{t,x}^{10}}^{\frac{5}{4}} \|f_n\|_{\dot{H}^1}^{-\frac{1}{4}} \lesssim \lim_{n\to\infty} \|S(t)P_{N_n}f_n\|_{L_{t,x}^{10}} \\ &\le \lim_{n\to\infty} \|S(t)P_{N_n}f_n\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{3}} \|S(t)P_{N_n}f_n\|_{L_{t,x}^{\infty}}^{\frac{2}{3}}\end{aligned}$$

where we used Hölder interpolation on the last inequality. Using Strichartz estimate (the pair $\left(\frac{10}{3}, \frac{10}{3}\right)$ are admissible), followed by Bernstein's Lemma, we have

$$\|S(t)P_{N_n}f_n\|_{L^{\frac{10}{3}}_{t,x}} \lesssim \|P_{N_n}f_n\|_{L^2} \sim \frac{1}{N_n}\|P_{N_n}|\nabla|f_n\|_{L^2} \leq \frac{1}{N_n}\|f_n\|_{\dot{H}^1} \lesssim \frac{A}{N_n}.$$

It follows that the above inequality reduces to

$$\epsilon^{\frac{5}{4}}A^{-\frac{1}{4}} \lesssim \lim_{n\to\infty}\left(\frac{A}{N_n}\right)^{\frac{1}{3}}\|S(t)P_{N_n}f_n\|^{\frac{2}{3}}_{L^\infty_{t,x}}.$$

Set $\lambda_n := N_n^{-1}$, and choose for each $n \in \mathbb{N}$ some $x_n \in \mathbb{R}^d$ and $t_n \in \mathbb{R}$ to approximate the $L^\infty_{t,x}$ norm, in the sense that

$$\lim_{n\to\infty} N_n^{-\frac{1}{2}}\left|S(-\lambda_n^2 t_n)P_{N_n}f_n(x_n)\right| \gtrsim A\left(\frac{\epsilon}{A}\right)^{\frac{15}{8}}.$$

Having chosen our parameters, we may pass to a subsequence if necessary so that $g_n = S(-t_n)\,\delta_{\lambda_n}^{-1}\tau_{x_n}^{-1}f_n = \delta_{\lambda_n}^{-1}\tau_{x_n}^{-1}S\left(-\lambda_n^2 t_n\right)f_n$ is weakly convergent to some $\phi \in \dot{H}^1$. As said at the start, (i) and (ii) are immediate. To see (iii), we let K_N be the convolution kernel of $P_N f$. Explicitly, $K_N = \mathcal{F}^{-1}\left[\psi\left(\frac{\cdot}{N}\right)\right]$ where ψ is as defined in the definition of P_N. Then

$$\begin{aligned}
|\langle K_1, \phi\rangle_{L^2}| &= \lim_{n\to\infty}|\langle K_1, g_n\rangle| \\
&= \lim_{n\to\infty}\left|\int N_n^{-\frac{1}{2}}S(-\lambda_n^2 t_n)f_n\left(\frac{x}{N_n}+x_n\right)\overline{K_1}(x)\,dx\right| \\
&= \lim_{n\to\infty}\left|N_n^{-\frac{1}{2}}\int S(-\lambda_n^2 t_n)f_n(x)\,N_n^3\overline{K_1}(N_n(x-x_n))\,dx\right| \\
&= \lim_{n\to\infty}\left|N_n^{-\frac{1}{2}}\int S(-\lambda_n^2 t_n)f_n(x)\,\overline{K_{N_n}}(x-x_n)\,dx\right| \\
&= \lim_{n\to\infty} N_n^{-\frac{1}{2}}|P_{N_n}S(-\lambda_n^2 t_n)f_n(x_n)|.
\end{aligned}$$

On the other hand, an application of Hölder followed by Sobolev inequality gives

$$|\langle \phi, K_1\rangle_{L^2}| \leq \|\phi\|_{L^{2^*}}\|K_1\|_{L^{(2^*)'}} \lesssim \|\phi\|_{\dot{H}^1}.$$

Putting everything together then gives $\|\phi\|_{\dot{H}^1} \gtrsim A\left(\frac{\epsilon}{A}\right)^{\frac{15}{8}}$, proving (iii). The above also implies that

$$\begin{aligned}
A\left(\frac{\epsilon}{A}\right)^{\frac{15}{8}} &\lesssim |\langle K_1, \phi\rangle_{L^2}| = |\langle S(t)K_1, S(t)\phi\rangle_{L^2}| \leq \|S(t)\phi\|_{L^{10}_x}\|S(t)\psi\|_{L^{\frac{10}{9}}_x} \\
&\leq |t|^{\frac{6}{5}}\|S(t)\phi\|_{L^{10}_x}\|K_1\|_{L^{10}_x} \lesssim \|S(t)\phi\|_{L^{10}_x}
\end{aligned}$$

provided $|t| \leq 1$, where we used dispersive estimate in the penultimate inequality. Integrating over $0 \leq t \leq 1$ gives

$$A\left(\frac{\epsilon}{A}\right)^{\frac{15}{8}} \lesssim \|S(t)\phi\|_{L^{10}_{t,x}([0,1]\times\mathbb{R}^3)} \leq \|S(t)\phi\|_{L^{10}_{t,x}},$$

which is equivalent to (v).

We now turn to (vi). Passing to a subsequence, we have that $t_n \to t_\infty$ for some $t_\infty \in [-\infty, \infty]$. Suppose that $t_\infty \in \{\infty, -\infty\}$. By dispersive estimate, we have for any Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^3)$,

$$\|S(t_n)\psi\|_{L^6_x} \lesssim \frac{1}{|t_n|}\|\psi\|_{L_x^{\frac{6}{5}}} \to 0 \quad \text{as } n \to \infty.$$

Moreover,

$$\begin{aligned}\|S(t_n)\phi\|_{L^6_x} &\leq \|S(t_n)\phi - S(t_n)\psi\|_{L^6_x} + \|S(t_n)\psi\|_{L^6_x}\\ &\lesssim \|\phi - \psi\|_{\dot{H}^1} + \|S(t_n)\psi\|_{L^6_x}.\end{aligned}$$

By a density argument, we deduce that $\|S(t_n)\phi\|_{L^6_x} \to 0$ as $n \to \infty$. One can then deduce property (vi). For the case when $t_\infty \in \mathbb{R}$, we claim that $\delta_{\lambda_n}^{-1}\tau_{x_n}^{-1} f_n \rightharpoonup S(t_\infty)\phi$ in $\dot{H}^1$. Indeed, testing $h_n := \delta_{\lambda_n}^{-1}\tau_{x_n}^{-1} f_n$ against $\varphi \in \dot{H}^1$,

$$\begin{aligned}\langle h_n, \varphi\rangle_{\dot{H}^1} &= \langle S(-t_n)h_n, S(-t_n)\varphi\rangle_{\dot{H}^1}\\ &= \langle S(-t_n)h_n, S(-t_\infty)\varphi\rangle_{\dot{H}^1} + \langle S(-t_n)h_n, S(-t_n)\varphi - S(-t_\infty)\varphi\rangle_{\dot{H}^1}.\end{aligned}$$

The second term vanishes as $n \to \infty$ by an application of Cauchy-Schwartz and the fact that $S(-t_n) \to S(-t_\infty)$ in operator norm. By weak convergence, the first term converges to

$$\langle \phi, S(-t_\infty)\varphi\rangle_{\dot{H}^1} = \langle S(t_\infty)\phi, \varphi\rangle_{\dot{H}^1}.$$

This proves our claim. Now by Rellich Kondrachov Theorem, we may infer that for any compact set K, a subsequence of h_n converges in $L^2(K)$. A diagonal argument then shows that $h_n \to G$ for some $G \in L^2(\mathbb{R}^3)$. Passing to a subsequence once more, we get that $h_n \to G$ almost everywhere. But since h_n is weakly convergent to $S(t_\infty)\phi$ in $\dot{H}^1$, we may conclude that $G = S(t_\infty)\phi$ almost everywhere. Finally, applying Brezis-Lieb Lemma shows that

$$\|h_n\|^6_{L^6_x} = \|h_n - S(t_\infty)\phi\|^6_{L^6_x} + \|S(t_\infty)\phi\|^6_{L^6_x} + o_n(1).$$

To see that this implies (vi), we note that $\|S(t_\infty)\phi\|_{L^6_x} = \lim_{n\to\infty}\|S(t_n)\phi\|_{L^6_x}$ and $\|h_n\|_{L^6_x} = \|f_n\|_{L^6_x}$. Moreover, we also have

$$\|h_n - S(t_\infty)\phi\|_{L^6_x} = \|h_n - S(t_n)\phi\|_{L^6_x} + o_n(1)$$

We now turn to (iv). We again want to apply Brezis-Lieb Lemma to $S(t)g_n$ and $S(t)\phi$. We thus want to show that

(†) $\{S(t)g_n\}_{n=1}^{\infty}$ is a bounded sequence in $L^{10}_{t,x}$ that converges pointwise to $S(t)\phi$ for a.e. $(t,x) \in \mathbb{R}\times\mathbb{R}^d$.

Once we prove (†), the conclusion of Brezis-Lieb Lemma then tells us that

$$\|S(t)g_n\|^{10}_{L^{10}_{t,x}} - \|S(t)g_n - S(t)\phi\|^{10}_{L^{10}_{t,x}} - \|S(t)\phi\|^{10}_{L^{10}_{t,x}} \to 0$$

as $n \to \infty$. By applying the change of variable $(t,x) \mapsto \left(\frac{t}{\lambda_n^2} - t_n, \frac{x-x_n}{\lambda_n}\right)$, we get that

$$\|S(t)g_n\|^{10}_{L^{10}_{t,x}} = \|S(t)f_n\|^{10}_{L^{10}_{t,x}}$$

$$\|S(t)g_n - S(t)\phi\|^{10}_{L^{10}_{t,x}} = \left\|S(t)f_n - S(t+\lambda_n^2 t_n)\tau_{x_n}\delta_{\lambda_n}\phi\right\|^{10}_{L^{10}_{t,x}},$$

which then yield (iv).

To prove the claim (†), we notice that the boundedness of $S(t)g_n$ in $L^{10}_{t,x}$ follows from the uniform bound by Sobolev Inequality:

$$\|S(t)g_n\|_{L^{10}_{t,x}} \lesssim \|g_n\|_{\dot{H}^1} = \|f_n\|_{\dot{H}^1} = A < \infty.$$

For almost everywhere convergence, we need to work a bit harder. We first restrict our attention to the closed cube $Q_k \subseteq \mathbb{R}^d$ of length k centred at the origin. Let $\chi_k \in C_c^{\infty}$ so that $\chi_k(x) = 1$ on Q_k and $\chi_k = 0$ outside $2Q_k$. We use the corollary of Ries-Komorogorov Theorem to show that $\mathcal{F} := \{\chi_k S(t)g_n\}_{n=1}^{\infty}$ is relatively compact in $L^2_{t,x}$. Thus we need to show that $\mathcal{F}$ is bounded in $L^2(\mathbb{R}^3)$ and for all $\epsilon > 0$, there exists $R > 0$ such that for all $f \in \mathcal{F}$,

$$\int_{|(t,x)|>R} |\chi_k S(t)g_n(x)|^2 \, d(x,y) + \int_{|(\tau,\xi)|>R} |\mathcal{F}_{t,x}(\chi_k S(t)g_n)(\tau,\xi)|^2 \, d(\tau,\xi) < \epsilon.$$

That $\mathcal{F}$ is bounded follows from the chain of inequality

$$\|\chi_k S(t)g_n\|_{L^2_{t,x}} \lesssim \|S(t)g_n\|_{L^{10}_{t,x}} \lesssim \|g_n\|_{\dot{H}^1_x} \lesssim A,$$

where we first used Hölder and ditched the cutoff χ_k in the implicit constant, followed by (5.10). Next, the L^2 norm of the tail,

$$\int_{|(t,x)|>R} |\chi_k S(t)g_n(x)|^2 \, dx$$

vanishes for large R, since $\chi_k S(t)g_n$ is supported on Q_k. It remains to estimate the tail of the Fourier side. Observe that

$$|(\tau,\xi)| > R \implies \tau^2 + \xi^2 > R^2$$

$$\implies \tau^2 > \frac{R^2}{2} \quad \text{or} \quad \xi^2 > \frac{R^2}{2}$$
$$\implies |\tau| + |\xi|^2 \gtrsim R \quad \text{(provided } R > 1).$$

Therefore,

$$\begin{aligned}
&\int_{|(\tau,\xi)|>R} |\mathcal{F}_{t,x}(\chi_k S(t) g_n)(\tau,\xi)|^2 \, d(\tau,\xi) \\
&\lesssim R^{-\frac{1}{2}} \int_{\mathbb{R}\times\mathbb{R}^d} \left| \sqrt{|\tau| + |\xi|^2} \mathcal{F}_{t,x}(\chi_k S(t) g_n)(\tau,\xi) \right|^2 d(\tau,\xi) \\
&= R^{-\frac{1}{2}} \left(\left\| |\nabla_t^{\frac{1}{2}}| \chi_k S(t) g_n \right\|_{L^2_{t,x}}^2 + \| |\nabla_x| \chi_k S(t) g_n \|_{L^2_{t,x}}^2 \right).
\end{aligned}$$

This is small for large R, provided of course we show that the two norms above are finite. To see this, we use product rule to estimate

$$\begin{aligned}
\| |\nabla_x| \chi_k S(t) g_n \|_{L^2_{t,x}} &\lesssim \|\chi_k\|_{L^2_t L^\infty_x} \| |\nabla_x| S(t) g_n \|_{L^\infty_t L^2_x} + \| |\nabla_x| \chi_k \|_{L^{\frac{d+2}{2}}_{t,x}} \|S(t) g_n\|_{L^{\frac{2(d+2)}{d-2}}_{t,x}} \\
&\lesssim \| |\nabla_x| \chi_k \|_{L^2_t L^\infty_x} \|g_n\|_{\dot{H}^1_x} + \| |\nabla_x| \chi_k \|_{L^{\frac{d+2}{2}}_{t,x}} \|g_n\|_{\dot{H}^1_x} < \infty
\end{aligned}$$

where we used Strichartz and (5.10) on the second inequality. The other term is similar, but we need to use fractional Leibniz rule to estimate

$$\left\| |\nabla_t|^{\frac{1}{2}} \chi_k S(t) g_n \right\|_{L^2_{t,x}} \lesssim \|\chi_k\|_{L^2_t L^\infty_x} \left\| |\nabla_t|^{\frac{1}{2}} S(t) g_n \right\|_{L^\infty_t L^2_x} + \left\| |\nabla_t|^{\frac{1}{2}} \chi_k \right\|_{L^{\frac{5}{2}}_{t,x}} \|S(t) g_n\|_{L^{10}_{t,x}}.$$

By testing against functions in $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, we see that $|\nabla_t|^{\frac{1}{2}} S(t) = |\nabla_x| S(t)$ as tempered distributions. It follows by the same argument for the x derivative that $\left\| |\nabla_t|^{\frac{1}{2}} \chi_k S(t) g_n \right\|_{L^2_{t,x}}$ is finite.

All this work has shown that the conditions of Riesz-Kolmorogorov Theorem are satisfied. Thus $\{\chi_k S(t) g_n\}_{n=1}^\infty$ is relatively compact in $L^2_{t,x}$ for each $k \in \mathbb{N}$, and hence relatively compact in $L^2_{t,x}(Q_k)$. Since $\{\chi_k S(t) g_n\}_{n=1}^\infty = \{S(t) g_n\}_{n=1}^\infty$ in Q_k, we may infer using a diagonal argument that $\{S(t) g_n\}_{n=1}^\infty$ is relatively compact in $L^2_{t,x}$. Going one step further, we may pass to a subsequence so that $\{S(t) g_n\}_{n=1}^\infty$ is pointwise convergent a.e. to some $G \in L^2_{t,x}$. It remains to show that $G(t,x) = S(t)\phi(x)$ for a.e. (t,x).

We notice that since g_n converges weakly to ϕ in $\dot{H}^1$, g_n also converges weakly to L^2. We now claim that $S(t) g_n$ converges to $S(t)\phi$ as a tempered distribution in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$. This will allow us to infer that $G = S(t)\phi$ a.e. Indeed, testing against a $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\langle S(t)(g_n - \phi(t,\cdot)), \varphi \rangle_{t,x} = \left\langle g_n - \phi, \int_{\mathbb{R}} S(-t) \varphi(t,\cdot) \, dt \right\rangle_x$$

$$= \left\langle g_n - \phi, \overline{\int_{\mathbb{R}} S(-t)\varphi(t,\cdot)\,dt} \right\rangle_{L^2_x}.$$

The right hand side tends to 0 as $n \to \infty$ once we show that $\int_{\mathbb{R}} S(-t)\varphi(t,\cdot)\,dt$ belongs in L^2_x. But this is true because

$$\left\| \int_{\mathbb{R}} S(-t)\varphi(t,\cdot)\,dt \right\|_{L^2_x} \leq \int_{\mathbb{R}} \|\varphi(t,\cdot)\|_{L^2_x}\,dt = \|\varphi\|_{L^1_t L^2_x} < \infty$$

where the first inequality follows from Minkowski integral inequality and the unitarity of $S(-t)$ in L^2. This completes the proof of (†). ☐

We finally proceed prove the profile decomposition for Schrödinger propagator.

Proof of Theorem 5.5. We shall recursively apply Proposition 5.8 so that Properties (ii) and (v) are built into our construction. The rest of the properties will be verified separately. We first set up the following notations:

$$\begin{aligned} g_n^j &:= \tau_{x_n^j} \delta_{\lambda_n^j} S\left(t_n^j\right); \\ h_n^j &:= \tau_{x_n^j} \delta_{\lambda_n^j}. \end{aligned}$$

Note that $g_n^j = h_n^j S(t_n^j)$ and

$$\left(g_n^j\right)^{-1} = S(-t_n^j)\left(h_n^j\right)^{-1} = S\left(-t_n^j\right)\delta^{-1}_{\lambda_n^j}\tau^{-1}_{x_n^j} = \delta^{-1}_{\lambda_n^j}\tau^{-1}_{x_n^j} S\left(-(\lambda_n^j)^2 t_n^j\right) f_n.$$

To start the construction, we let $r_n^0 := f_n$. Suppose that we have constructed the decomposition up to the level $J \geq 0$ so that properties (ii) and (v) are satisfied. Since $\{f_n\}_{n=1}^{\infty}$ is bounded in $\dot{H}^1$, we may pass to a subsequence so that the limits

$$\lim_{n\to\infty} \left\|r_n^J\right\|_{\dot{H}^1} := A_J \quad \text{and} \quad \lim_{n\to\infty} \left\|S(t) r_n^J\right\|_{L^{10}_{t,x}} := \epsilon_J$$

exist. If $\epsilon_J = 0$, we may stop there and set $J^* = J$. Otherwise we apply Proposition 5.8 on r_n^J to get a profile $\phi^{J+1} \in \dot{H}^1$, and a sequence $\left\{(\lambda_n^{J+1}, t_n^{J+1}, x_n^{J+1})\right\}_{n=1}^{\infty} \subseteq (0,\infty) \times \mathbb{R} \times \mathbb{R}^d$ satisfying the conditions of Proposition 5.8. Namely, ϕ^{J+1} is the $\dot{H}^1$ weak limit of $\left(g_n^{J+1}\right)^{-1} r_n^J$. We now set

$$r_n^{J+1} = r_n^J - g_n^{J+1}\phi^{J+1}.$$

Then $\left(g_n^{J+1}\right)^{-1} r_n^{J+1}$ converges weakly to 0 in $\dot{H}^1$ as $n \to \infty$, that is, (v) is proved for the $J+1$-th step. Nest, (ii) from Proposition 5.8 implies that

$$\begin{aligned} \left\|r_n^J\right\|^2_{\dot{H}^1} &= \left\|r_n^J - g_n^{J+1}\phi^{J+1}\right\|^2_{\dot{H}^1} + \left\|\phi^{J+1}\right\|^2_{\dot{H}^1} + o(1) \\ &= \left\|r_n^{J+1}\right\|^2_{\dot{H}^1} + \left\|\phi^{J+1}\right\|^2_{\dot{H}^1} + o(1) \end{aligned} \tag{†}$$

as $n \to \infty$. Thus according to our inductive hypothesis (for (ii)), we have

$$\begin{aligned}\|f_n\|_{\dot{H}^1}^2 &= \sum_{j=1}^{J} \left\|\phi^j\right\|_{\dot{H}^1}^2 + \left\|r_n^J\right\|_{\dot{H}^1}^2 + o(1) \\ &= \sum_{j=1}^{J+1} \left\|\phi^j\right\|_{\dot{H}^1}^2 + \left\|r_n^{J+1}\right\|_{\dot{H}^1}^2 + o(1)\end{aligned}$$

as $n \to \infty$. Similarly, one obtains

$$\begin{aligned}\|f_n\|_{L_x^6}^6 &= \sum_{j=1}^{J+1} \left\|S(t_n^j)\phi^j\right\|_{L_x^6}^6 + \left\|r_n^{J+1}\right\|_{L_x^6}^6 + o_n(1); \\ \|S(t)f_n\|_{L_x^{10}}^{10} &= \sum_{j=1}^{J+1} \left\|S(t)\phi^j\right\|_{L_x^{10}}^{10} + \left\|S(t)r_n^{J+1}\right\|_{L_x^{10}}^{10} + o_n(1),\end{aligned}$$

from the decoupling statements (vi) and (iv) from Proposition 5.7 respectively. This proves (iii), and also (iv) once we have proved (i). We now turn to (iii). Looking back at our construction, if J^* is finite, then the construction was stopped at the point J^*, which can only happen if $\epsilon_{J^*} = 0$, and so (iii) holds. Now assume that $J^* = \infty$. At the J-th step, we have

$$\epsilon_{J+1}^{10} = \lim_{n\to\infty} \left\|S(t)r_n^{J+1}\right\|_{L_{t,x}^{10}}^{10} \le \epsilon_J^{10} \left(1 - C_d \left(\frac{\epsilon_J}{A_J}\right)^{\frac{21}{4}}\right)$$

for some constant C depending only on d Also,

$$A_{J+1}{}^2 = \lim_{n\to\infty} \left\|r_n^{J+1}\right\|_{\dot{H}^1}^2 \le A_J{}^2 \left(1 - K_d \left(\frac{\epsilon_J}{A_J}\right)^{\frac{15}{4}}\right) \le A_J^2,$$

where the second inequality follows from (†) and (iii) from Proposition 5.8. This implies that $\epsilon_J \to 0$ as $J \to \infty$. Indeed, if $\epsilon_J \to \epsilon > 0$, then $\epsilon_J/A_J \ge \epsilon_J/A_1$ since A_J is decreasing in J. Consequently, the ratio A_{J+1}/A_J is uniformly bounded above by a constant less than 1; more specifically,

$$\frac{A_{J+1}}{A_J} \le \left(1 - K_d \left(\frac{\epsilon}{A_1}\right)^{\frac{15}{4}}\right)^{\frac{1}{2}} < 1,$$

which is a contradiction. This completes the proof of (v). It remains to prove (i), and we shall do so by contradiction. Suppose that there is a minimal counterexample (j,k) for (i) with $j < k$, in the sense that (i) fails for (j,k) but (i) holds for (j,l) whenever $j < l < k$. Then we may pass to a subsequence so that

$$\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} \to \bar{\lambda}; \quad \frac{x_n^k - x_n^j}{\sqrt{\lambda_n^j \lambda_n^k}} \to \bar{x}; \quad \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j \lambda_n^k} \to \bar{t} \qquad (\dagger\dagger)$$

as $n \to \infty$, for some $(\bar{\lambda}, \bar{x}, \bar{t}) \in (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}$. By construction ϕ^k is the $\dot{H}^1$ weak limit of $\left(g_n^k\right)^{-1} r_n^{k-1}$. Note that ϕ^k is non-trivial. Our goal is to show that (††) implies that the $\dot{H}^1$ weak limit of

$$\left(g_n^k\right)^{-1} r_n^{k-1} = \left(g_n^k\right)^{-1} r_n^j - \sum_{l=j+1}^{k-1} \left(g_n^k\right)^{-1} g_n^l \phi^l \tag{† † †}$$

is in fact 0, hence obtaining a contradiction.

Observe that for a sequence $\{u_n\}_{n=1}^{\infty} \in \dot{H}^1$,

$$\begin{aligned}
\left(g_n^k\right)^{-1} u_n &= \left(g_n^k\right)^{-1} g_n^j \left[\left(g_n^j\right)^{-1} u_n\right] \\
&= S\left(-t_n^k\right) \left(h_n^k\right)^{-1} h_n^j S\left(t_n^j\right) \left[\left(g_n^j\right)^{-1} u_n\right] \\
&= \left(h_n^k\right)^{-1} h_n^j S\left(t_n^j - t_n^k \left(\frac{\lambda_n^k}{\lambda_n^j}\right)^2\right) \left[\left(g_n^j\right)^{-1} u_n\right] \\
&= \tau_{\frac{x_n^k - x_n^j}{\lambda_n^j}} \delta_{\frac{\lambda_n^k}{\lambda_n^j}} S\left(t_n^j - t_n^k \left(\frac{\lambda_n^k}{\lambda_n^j}\right)^2\right) \left[\left(g_n^j\right)^{-1} u_n\right].
\end{aligned}$$

Now, the operator acting on $\left(g_n^j\right)^{-1} u_n$ converges in $\mathcal{B}(\dot{H}^1)$. This follows from that

$$\begin{aligned}
t_n^j - t_n^k \left(\frac{\lambda_n^k}{\lambda_n^j}\right)^2 &= \frac{t_n^j \left(\lambda_n^j\right)^2 - t_n^k \left(\lambda_n^k\right)^2}{\lambda_n^j \lambda_n^k} \cdot \frac{\lambda_n^k}{\lambda_n^j} \to \frac{\bar{t}}{\bar{\lambda}} \\
\frac{x_n^k - x_n^j}{\lambda_n^j} &= \frac{x_n^k - x_n^j}{\sqrt{\lambda_n^k \lambda_n^j}} \sqrt{\frac{\lambda_n^k}{\lambda_n^j}} \to \bar{x} \bar{\lambda}^{-\frac{1}{2}}
\end{aligned}$$

as $n \to \infty$. Thus by Lemma A.11, proving $\left(g_n^j\right)^{-1} u_n \rightharpoonup 0$ implies $\left(g_n^k\right)^{-1} u_n \rightharpoonup 0$. For $u_n = r_n^j$, we clearly have $\left(g_n^j\right)^{-1} r_n^j \rightharpoonup 0$ as $n \to \infty$ by construction. Hence the first term on († † †) converges weakly to 0. For the second term, we will need to show this for $u_n = g_n^l \phi^l$ for each $j < l < k$, that is, to show

$$I_n := \left(g_n^j\right)^{-1} g_n^l \phi^l \rightharpoonup 0 \quad \text{as } n \to \infty.$$

In the same spirit as above, we may unpack I_n as

$$I_n = \tau_{\frac{x_n^j - x_n^l}{\lambda_n^l}} \delta_{\frac{\lambda_n^j}{\lambda_n^l}} S\left(t_n^l - t_n^j \left(\frac{\lambda_n^j}{\lambda_n^l}\right)^2\right) \phi^l. \tag{5.18}$$

By a density argument, it suffices to show this for the case when $\phi^l \in C_c^\infty(\mathbb{R}^d)$. Recall that (i) holds (j,l). Thus we shall split into three cases.

Case 1: $\dfrac{\lambda_n^j}{\lambda_n^l} + \dfrac{\lambda_n^l}{\lambda_n^j} \to \infty$ as $n \to \infty$.

Let $\psi \in C_c^\infty(\mathbb{R}^d)$. By Cauchy-Schwartz inequality,

$$\langle I_n, \psi\rangle_{\dot{H}^1} \leq \min\left\{ \|I_n\|_{\dot{H}_x^2} \|\psi\|_{L_x^2}, \|I_n\|_{L_x^2} \|\psi\|_{\dot{H}_x^2} \right\}.$$

By a change of variables,

$$\|I_n\|_{\dot{H}_x^2} = \frac{\lambda_n^j}{\lambda_n^l} \left\| \phi^l \right\|_{\dot{H}_x^2} \quad \text{and} \quad \|I_n\|_{L_x^2} = \frac{\lambda_n^l}{\lambda_n^j} \left\| \phi^l \right\|_{L_x^2}.$$

It follows that $\langle I_n, \psi\rangle_{\dot{H}^1} \to 0$ as $n \to \infty$. By Lemma A.12, we have $I_n \rightharpoonup 0$ in $\dot{H}^1$.

Case 2: $\dfrac{\lambda_n^j}{\lambda_n^l} \to \bar{\lambda}^1 \in (0,\infty)$ and $\dfrac{|t_n^l \left(\lambda_n^l\right)^2 - t_n^j \left(\lambda_n^j\right)^2 |}{\lambda_n^l \lambda_n^j} \to \infty$.

We claim that

$$J_n := \tau_{\frac{x_n^j - x_n^l}{\lambda_n^l}} \delta_{\lambda_1} S\left(t_n^l - t_n^j \left(\frac{\lambda_n^j}{\lambda_n^l}\right)^2 \right) \phi^l$$

is weakly convergent to 0. Indeed, since

$$t_n^l - t_n^j \left(\frac{\lambda_n^j}{\lambda_n^l}\right)^2 = \frac{|t_n^l \left(\lambda_n^l\right)^2 - t_n^j \left(\lambda_n^j\right)^2 |}{\lambda_n^l \lambda_n^j} \cdot \frac{\lambda_n^j}{\lambda_n^l} \to \infty.$$

In view of (5.18), Lemma 5.4 then implies that $I_n \rightharpoonup 0$.

Case 3: $\dfrac{\lambda_n^j}{\lambda_n^l} \to \bar{\lambda}^1 \in (0,\infty)$, $\dfrac{|t_n^j \left(\lambda_n^j\right)^2 - t_n^k \left(\lambda_n^k\right)^2 |}{\lambda_n^j \lambda_n^k} \to \bar{t}^1$ and $\dfrac{|x_n^j - x_n^l|^2}{\lambda_n^j \lambda_n^l} \to \infty$.

We note that

$$\tau_{\frac{x_n^j - x_n^l}{\lambda_n^l}} \delta_{\bar{\lambda}^1} S\left(\bar{t_1} \bar{\lambda}^1\right) \phi^l$$

converges weakly to 0. This follows from Lemma 5.4 that

$$\frac{x_n^j - x_n^l}{\lambda_n^l} = \frac{x_n^j - x_n^l}{\sqrt{\lambda_n^j \lambda_n^l}} \sqrt{\frac{\lambda_n^j}{\lambda_n^l}} \to \infty.$$

Similar to Case 2, this implies that I_n converges weakly to 0.

We finally show that our decomposition may be altered if necessary to satisfy (vii). By passing to a subsequence, we may assume $\lim_{n\to\infty} t_n^j$ exists and lies in $[-\infty,\infty]$. We are done if this limit is either ∞ or $-\infty$. Suppose that the limit is a real number t^j. Then we set each $t_n^j = 0$ and replace each profile ϕ^j by $S(t^j)\phi^j$. More precisely, we rewrite f_n as

$$f_n = \left(\sum_{j=1}^{J} \tau_{x_n^j} \delta_{\lambda_n^j} \nu^j \right) + w_n^J$$

where $\nu^j = S(t^j)\phi^j$ and $w_n^J = \sum_{j=1}^J \tau_{x_n^j}\delta_{\lambda_n^j}\left(S(t_n^j) - S(t^j)\right)\phi^j + r_n^J$. It remains to show that this change does not cause any trouble for (i) to (vi). Indeed, we note that

$$\left\| \tau_{x_n^j}\delta_{\lambda_n^j}\left(S(t_n^j) - S(t^j)\right)\phi^j \right\|_{\dot H_x^1} \to 0 \quad \text{as } n \to \infty \tag{5.19}$$

since $S(t_n^j)$ converges to $S(t^j)$ in operator norm. Thus (ii) holds immediately, (iii) and (iv) also hold by applying (5.10) and Sobolev inequality respectively. (i) also holds by the convergence of $S(t_n^j)$ to $S(t^j)$. (v) also holds since $\delta_{\lambda_n^j}^{-1}\tau_{x_n^j}^{-1} w_n^j = \sum_{j=1}^J \left(S(t_n^j) - S(t^j)\right)\phi^j + \delta_{\lambda_n^j}^{-1}\tau_{x_n^j}^{-1} r_n^J$ and both terms converge weakly to 0 by (v) for the original decomposition and that t_n^j converges to t^j. Finally for (i), we have nothing to prove unless one of $\{t_n^j\}_{n=1}^\infty$ or $\{t_n^k\}_{n=1}^\infty$ is bounded, so assume the former is bounded. If

$$\frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} + \frac{|x_n^k - x_n^j|^2}{\lambda_n^j\lambda_n^k} \not\to \infty,$$

then

$$\frac{|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2|}{\lambda_n^j\lambda_n^k} \to \infty$$

would imply that $\frac{|t_n^k|\lambda_n^k}{\lambda_n^j} \to \infty$. Hence (v) holds. This completes the proof of the theorem. □

A Appendix

In this appendix, we gather some important theorems from harmonic analysis which are used throughout the report. Most of the proofs can be found in the *MIGSAA Dispersive Equations* course in [**?**].

We first record some estimates related to Sobolev norms.

Lemma A.1 (Gagliardo-Nirenberg inequality)**.** *Let $s > 0$ and $1 < p < q \leq \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{q} + \frac{\theta s}{d}$$

for some $0 < \theta < 1$. Then for any $f \in W^{s,p}$,

$$\|f\|_{L^q} \lesssim_{p,q,s} \|f\|_{\dot{W}^{s,p}}^{\theta} \|f\|_{L^p}^{1-\theta}.$$

One can deduce Lemma A.1 using the interpolation property of homogeneous Sobolev spaces. Alternatively, one can prove this directly using Littlewood-Paley theory (see [17, Appendix]).

Theorem A.2 (Hardy-Littlewood-Sobolev inequality [12])**.** *Suppose that $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q} + 1$ for some $\alpha \in (0, d)$. Then for any $f \in L^p$, we have*

$$\left\| f * \frac{1}{|\cdot|^{\alpha}} \right\|_{L^q} \lesssim_{p,q} \|f\|_{L^p}.$$

Theorem A.3 (Sobolev Inequalities)**.** *Let $1 < p < q < \infty$ and $s > 0$.*

(i) ***Homogeneous Sobolev inequality:*** *Suppose that $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$. Then*

$$\|f\|_{L^q} \lesssim_{p,q} \|f\|_{\dot{W}^{s,p}}.$$

(ii) ***Inhomogeneous Sobolev inequality:*** *Suppose that $\frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}$. Then*

$$\|f\|_{L^q} \lesssim_{p,q,s} \|f\|_{W^{s,p}}.$$

Note that the inhomogeneous Sobolev inequality follows directly from the homogeneous version since $\|f\|_{\dot{W}^{s,p}} \leq \|f\|_{W^{s,p}}$. One can also show (see [**?**]) that the homogeneous Sobolev inequality is in fact equivalent to Theorem A.2.

In our analysis, we often have to decompose our functions into different frequencies and analyse each piece separately. This is achieved by considering the *Paley-Littlewood projections.*

To continue, we fix a radial[4] bump function $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfying[5]

$$\begin{cases} \varphi(\xi) = 1 & \text{if} \quad |\xi| \leq 1; \\ \varphi(\xi) = 0 & \text{if} \quad |\xi| \geq 2. \end{cases}$$

For each dyadic interger $N \in 2^{\mathbb{Z}} = \{2^z : z \in \mathbb{Z}\}$, we also define $\varphi_N := \varphi\left(\frac{\cdot}{N}\right)$ and $\psi_N := \varphi_N - \varphi_{N/2}$. The point here is that φ_N and ψ_N are supported on frequencies $|\xi| \leq 2N$ and $\frac{N}{2} \leq \xi \leq 2N$ respectively. We now define the Fourier multipliers

(i) $\widehat{P_{\leq N} f} := \varphi_N \hat{f}$;

(ii) $\widehat{P_N f} := \psi_N \hat{f}$;

(iii) $\widehat{P_{>N} f} := (1 - \varphi_N)\hat{f}$.

Thus $P_{\leq N} f$, $P_N f$ and $\widehat{P_{>N} f}$ are essentially frequency localisations onto $|\xi| \leq N$ and $|\xi| \sim N$ and $|\xi| > N$ respectively.

Remark A.4. *The following are immediate consequences of the definitions*

(i) $P_N f = P_{\leq N} f - P_{\frac{N}{2}} f$;

(ii) $P_{\leq N} f = \sum_{M \leq N} P_M$;

(iii) $f = \lim_{N \to \infty} P_{\leq N} f = \sum_N P_N f$.

Theorem A.5 (Bernstein Inequalities [?])**.** *For* $1 \leq p \leq q \leq \infty$,

(i) $\|P_{\leq N} |\nabla|^s f\|_{L^p} \lesssim N^s \|P_{\leq N} f\|_{L^p}$ *provided* $s \geq 0$;

(ii) $\|P_N |\nabla|^s f\|_{L^p} \sim N^s \|P_N f\|_{L^p}$ *for all* $s \in \mathbb{R}$;

(iii) $\|P_{\leq N} f\|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N} f\|_{L^p}$;

(iv) $\|P_N f\|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L^p}$.

These Bernstein inequalities are essentially proved by writing out the Paley Littlewood projections as convolutions and then apply Young's inequality.

Theorem A.6 (Square Function Estimate)**.** *Let* $p \in (1, \infty)$. *Then*

$$\|f\|_{L^p} \sim \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |P_N f|^2 \right)^{1/2} \right\|_{L^p}.$$

[4]A function $f : \mathbb{R}^d \to \mathbb{C}$ is radial if there exists $f_0 : \mathbb{R}_0^+ \to \mathbb{C}$ such that $f(x) = f_0(|x|)$ for all $x \in \mathbb{R}^d$. In other words, the $f(x)$ only depends on $|x|$.

[5]We need not specify the exact values of φ, as this turns out to be unimportant

The function $\left(\sum_N |P_N f|^2\right)^{1/2}$ is called the *square function* of f. The special case $p = 2$ can be easily proved with Plancherel's Theorem. The proof of the general case requires Calderon-Zygmund theory; see for example [**?**, Chapter 7].

Lemma A.7 (Schur's test [12])**.** *Let (X, μ) and (Y, ν) be $\sigma-$finite measure spaces and that $K : X \times Y \to \mathbb{R}$ be a locally integrable function. Suppose that*

$$\sup_{x\in X} \int_Y |K(x,y)|\, d\nu(y) < \infty;$$
$$\sup_{y\in Y} \int_Y |K(x,y)|\, d\mu(x) < \infty.$$

Then T defined by

$$T(f)(x) = \int_Y K(x,y) f(y)\, d\nu(y)$$

is a bounded linear operator from $L^p(Y)$ to $L^p(X)$.

An extremely useful tool in proving profile decompositions is the following lemma due to Brézis-Lieb, which is essentially an improvement of Fatou's Lemma.

Lemma A.8 (Brézis-Lieb [2])**.** *Let $p \in [1, \infty)$, and $\{f_n\}_{n=1}^\infty \subseteq L^p$ is a bounded sequence. If f is some measurable function and either*

(i) *$p = 2$ and $f_n \rightharpoonup f$ in L^2; or*

(ii) *f_n converges to f almost everywhere.*

Then

$$\int_{\mathbb{R}^d} ||f_n|^p - |f_n - f|^p - |f|^p| \to 0$$

as $n \to \infty$. This in particular implies that $\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p \to \|f\|_{L^p}^p$ as $n \to \infty$.

The Kolmorogorov-Riesz Theorem is a useful characterisation of precompactness in L^p space; this is essentially the L^p version of the Arzela-Ascoli Theorem. We will only need the special case in L^2, which we give below:

Theorem A.9 (Kolmorogorov-Riesz, L^2 version [8])**.** *A family $\mathcal{F}$ of functions is precompact in L^2 if and only if we have the uniform bound $\sup_{f\in\mathcal{F}} \|f\|_{L^2} < \infty$ and that for every $\epsilon > 0$, there exists $R = R(\epsilon)$ such that*

$$\int_{|x|>R} |f(x)|^2\, dx + \int_{|\xi|>R} |\hat{f}(\xi)|\, d\xi < \epsilon.$$

When dealing with an L^p norm of the derivative of a product of functions, one might want to use product rule then use Hölder to bound each individual part. The fractional Liebniz rule essentially says that one can also use this trick for fractional derivatives:

Lemma A.10 (Fractional Liebniz rule)**.** *Let $s \in (0,1]$. If $1 < r, p_1, p_2, r_1, r_2 < \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, then*

$$\left\| |\nabla|^s (fg) \right\|_{L^r} \lesssim \|f\|_{L^{p_1}} \left\| |\nabla|^s g \right\|_{L^{q_1}} + \left\| |\nabla|^s f \right\|_{L^{p_2}} \|g\|_{L^{q_2}} .$$

Lemma A.11. *Let H_1, H_2 be Hilbert spaces. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a sequence of bounded operators from H_1 to H_2 convergent to $T \in \mathcal{B}(H_1, H_2)$. If $\{v_n\}_{n=1}^{\infty} \subseteq H_1$ is weakly convergent to $v \in H_1$, then $\{T_n v_n\}_{n=1}^{\infty}$ is weakly convergent to $Tv \in H_2$.*

Proof. Let $u \in H_2$. By writing

$$\langle T_n v_n, u \rangle_{H_2} = \langle v_n, T_n^* u \rangle_{H_1} = \langle v_n, T^* u \rangle_{H_1} + \langle v_n, (T_n - T)^* u \rangle_{H_1} ,$$

we have that

$$\left| \langle T_n v, u \rangle_{H_2} - \langle Tv, u \rangle_{H_2} \right| \le \left| \langle v_n, T^* u \rangle_{H_1} - \langle v, T^* u \rangle_{H_1} \right| + \|v_n\|_{H_1} \|(T_n - T)^*\|_{H_1} ,$$

which tends to 0 as $n \to \infty$, since $v_n \rightharpoonup v$ in H_1, $\|v_n\|_{H^1}$ is uniformly bounded, and $\|(T_n - T)^*\|_{H_1} = \|T_n - T\|_{H_1} \to 0$. ☐

Lemma A.12. *Let $\{v_n\}_{n=1}^{\infty} \subseteq \dot{H}^s$ be a bounded sequence and $v \in \dot{H}^s$. Suppose that $\langle v_n - v, \varphi \rangle_{\dot{H}^s} \to 0$ for every $\varphi \in C_c^{\infty}$. Then $v_n \rightharpoonup v$ in $\dot{H}^s$.*

Proof. Let $\phi \in \dot{H}^s$ and $\varphi \in \mathcal{S}$. Then

$$\begin{aligned} |\langle v_n - v, \phi \rangle| &\le |\langle v_n - v, \varphi \rangle| + |\langle v_n - v, \phi - \varphi \rangle| \\ &\le |\langle v_n - v, \varphi \rangle| + \|v_n - v\|_{\dot{H}^s} \|\phi - \varphi\|_{\dot{H}^s} \end{aligned}$$

Since $\{v_n\}_{n=1}^{\infty}$ is bounded in $\dot{H}^s$, $\|v_n - v\|_{\dot{H}^s}$ is uniformly bounded. The hypothesis and the density of $\mathcal{S}$ in $\dot{H}^s$ show that the right-hand side can be made arbitrarily small. ☐

Lemma A.13 (A crude estimate)**.** *For any $u, v \in \mathbb{C}$, and $p > 1$, we have*

$$\left| |u|^{p-1} u - |v|^{p-1} v \right| \lesssim_p |u - v| (|u|^{p-1} + |v|^{p-1}).$$

Proof. We may assume without loss of generality that $|u| > |v|$. By dividing the inequality by $|v|^{p-1}$ and rotating u and v by the same factor so that $v = |v|$ (i.e. multiply the inside of each modulus by some $e^{i\theta}$), it suffices to prove the inequality

$$\left| |w|^{p-1} w - 1 \right| \le (p-1) |w - 1| (|w|^{p-1} + 1).$$

for $|w| > 1$. Now,

$$\left| |w|^{p-1} w - 1 \right| \le \left| |w|^{p-1} w - |w|^{p-1} \right| + \left| |w|^{p-1} - 1 \right| .$$

For the first term, we have $|w|^{p-1}|w-1|$ which is clearly bounded above by the right-hand side. For the second term, we let $r = |w|$ and note that by mean-value theorem, and the fact that the function $f(r) = r^{p-1}$ is increasing, we have

$$\frac{r^{p-1}-1}{r-1} \leq (p-1)r^{p-2} \leq (p-1)r^{p-1},$$

that is, we have

$$|w|^{p-1} - 1 \leq (p-1)(|w|-1)|w|^{p-1} \leq (p-1)|w-1||w|^{p-1},$$

which is also bounded by our right-hand side. This proves the complex inequality. $\square$

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