ADVANCED PDE II - LECTURE 10

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Warning: This is a first draft of the lecture notes and should be used with care! Again, this lecture's notes very closely follow C. Sogge's "Nonlinear Wave Equations".

1. Global existence for quasilinear equations in high dimensions with small data

Recall $\mathbb{\Gamma} = \{\Gamma_i\}_{i=1}^N$, $\mathcal{T}^i{}_j = \sum_{k=0}^n G^{ik}\partial_k u\partial_j u - \frac{1}{2}\delta^i_j \sum_{l,m=1}^n G^{lm}\partial_l u\partial_m u$ (no u^2 term) for the wave equation,

$$|f|^2_{\mathbb{\Gamma},k} = \sum_{0\leq j\leq k}\sum_{i_1=1}^N \dots \sum_{i_j=1}^N |\Gamma_{i_j}\dots\Gamma_{i_1} f|^2,$$
$$\|f\|^2_{\mathbb{\Gamma},k} = \int_{\mathbb{R}^n} |f|^2_{\mathbb{\Gamma},k}\mathrm{d}^n x,$$
$$E_{\partial_t,\mathbb{\Gamma},k}[u](t) = \sum_{0\leq j\leq k}\sum_{i_1=1}^N \dots \sum_{i_j=1}^N E_{\partial_t}[\Gamma_{i_j}\dots\Gamma_{i_1} u](t)$$
$$C^{-1}\|\partial u\|^2_{\mathbb{\Gamma},k}(t) \leq \|E_{\partial_t,\mathbb{\Gamma},k}[u](t) \leq C^{-1}\|\partial u\|^2_{\mathbb{\Gamma},k}(t).$$

Theorem 1.1. *Let $n \geq 4$. Assume condition $2Q(\mathbb{R}^{1+n})$, that $G^{00} = -1$, that G^{jk} and F depend only on ∂u, and that $F(0) = 0$ and $\delta F(0) = 0$.*

There is an $\epsilon > 0$ such that if $T > 0$ and u is a solution on $[0,T]\times\mathbb{R}^n$ of

$$\sum_{jk} G^{jk}(\partial u)\partial_i\partial_j u = F(\partial u)$$

and $\|u\|_{\mathbb{\Gamma},n+4}(0) < \epsilon$, then u can be extended to a solution of the PDE on $[0,\infty)\times\mathbb{R}^n$.

Lemma 1.2 (Commutators for Γ^α and ∂)**.** *Let $u \in C^\infty_c$. For $\Gamma \in \mathbb{\Gamma}$, $|[\Gamma,\partial]u| \leq |\partial u|$. Thus, for $\alpha \in \mathbb{Z}^N$,*

$$|\Gamma^\alpha \partial u| \leq |\partial\Gamma^\alpha u| + C\sum_{|\beta|\leq|\alpha|-1} |\partial\Gamma^\beta u|.$$

Proof. Each Γ has a coefficient that is at most linear in x, thus $[\Gamma,\partial]$ is either zero or a differential operator of order 0 with a constant coefficient. Thus, $|[\Gamma,\partial]u| \leq |\partial u|$.

Consider the second part of the claim.

$$\Gamma^\alpha\partial u = \partial\Gamma^\alpha u + [\Gamma^\alpha,\partial]u$$

For $|\alpha| = 1$, the first result implies the first. For $|\alpha| > 1$, by induction, ∂ can be commuted through Γ^α so that ∂ appears as the right most term, and the number of Γ's is $|\alpha| - 1$. Thus, the second result holds. $\square$

Lemma 1.3 (Derivation property of commutators)**.** *Let A, B, C be finite order differential operators and concatenation denote compostion (e.g. $BC = B \circ C$).*

$$[A, BC] = [A, B]C + B[A, C].$$

Proof.

$$[A, BC] = ABC - BCA = ABC - BAC + BAC - BAC = [A, B]C + B[A, C].$$

□

Proof of global existence for quasilinear equations. The key idea of the proof is to follow the strategy of the local well-posedness proof, but to use energies that have been strengthened with Γ instead of $\vec{\partial}$, to use the Klainerman-Sobolev inequality instead of the Sobolev inequality, and to use the additional gain in $(1+t)^{-n-1/2}$ to get integrability for $t \in [0, \infty)$ instead of merely in $[0, T]$ for a small T.

Step 1: Some preliminary reductions Consider $|\beta| \leq \frac{n+4}{2}$. Observe that for $t \in \mathbb{R}$, $\sup_x |\vec{\partial}^\beta u|(t, \vec{x}) \leq \|\vec{\partial}^\beta u\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}(t)$. Furthermore, $\|\vec{\partial}^\beta u\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}(0) = \|\vec{\partial}^\beta f\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\partial}^\beta u\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}(t)^2 \leq \int \sum_{\gamma \leq \frac{n+2}{2}} 2|\vec{\partial}^\gamma \vec{\partial}^\beta u||\partial_t \vec{\partial}^\gamma \vec{\partial}^\beta u| \mathrm{d}^n x,$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\partial}^\beta u\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}(t) &\leq \|\vec{\partial}^\beta \partial_t u\|_{H^{\lfloor \frac{n+2}{2} \rfloor}}(t) \\
&\leq \|\partial_t u\|_{H^{\lfloor \frac{n+2}{2} \rfloor + \lfloor \frac{n+4}{2} \rfloor}}(t) \\
&\leq \|\partial_t u\|_{H^{n+3}}(t) \\
&\leq \|\partial_t u\|_{\mathbb{\Gamma}, n+3}(t) \\
&\leq \|\partial u\|_{\mathbb{\Gamma}, n+3}(t) \\
&\leq \|\partial u\|_{\mathbb{\Gamma}, n+4}(t).
\end{aligned}$$

Integrating this, if $\|\partial u\|_{\mathbb{\Gamma}, n+4}(t)$ is uniformly bounded in t, then $\sup_x |\vec{\partial}^\beta u|(t, \vec{x})$ is finite, although it might grow linearly.

From the fact that translations are in $\mathbb{\Gamma}$, the Klainerman-Sobolev inequality, grouping Γ terms, and $n+3 \leq n+4$, we find

$$\begin{aligned}
&\sup_{t,x} \sum_{|\alpha| \leq \frac{n+4}{2}} |\partial \vec{\partial}^\alpha u| \\
&= \sup_{t,x} \sum_{|\alpha| \leq \frac{n+4}{2}} |\vec{\partial}^\alpha \partial u| \\
&\leq \sup_{t,x} \sum_{|\alpha| \leq \frac{n+4}{2}} |\Gamma^\alpha \partial u| \\
&\leq \sup_{t} \sum_{|\alpha| \leq \frac{n+4}{2}} |\Gamma^\alpha \partial u| \\
&\leq \sup_{t} \sum_{|\alpha| \leq \frac{n+4}{2}} \|\Gamma^\alpha \partial u\|_{\mathbb{\Gamma}, \lfloor \frac{n+2}{2} \rfloor}
\end{aligned}$$

$$\begin{aligned} &\leq \sup_t \|\partial u\|_{\mathbb{\Gamma},n+3} \\ &\leq \sup_t \|\partial u\|_{\mathbb{\Gamma},n+4}. \end{aligned}$$

Furthermore,

$$\sup_t \|\partial u\|_{\mathbb{\Gamma},n+4} \leq \sup_t C E_{\partial_t,\mathbb{\Gamma},n+4}[u](t)^{1/2}.$$

For simplicity in this proof, use the notation

$$E(t) = E_{\partial_t,\mathbb{\Gamma},n+4}[u](t).$$

By the continuation criterion in lecture 8, it is sufficient to show that

$$\sup_{t,x} \sum_{|\alpha|<1} \sum_{|\beta|\leq \frac{n+4}{2}} |\partial^\alpha \vec{\partial}^\beta u| < \infty.$$

From the arguments in the previous paragraph, it is therefore sufficient to show that $E(t)$ is uniformly bounded.

Remark 1.4. *This is an important trick in treating wave equations instead of Klein-Gordon equations. For global problems it is preferable to study wave equations, since the full set $\mathbb{\Gamma}$ are all symmetries for $\sum_{ij} \eta^{ij}\partial_i\partial_j u = 0$, whereas there is one fewer symmetry for the Klein-Gordon equation.*

Step 2: Set up the continuous induction on $E(t)$ We wish to show that there is an $\epsilon > 0$ such that if $E(0) \leq \epsilon/2$, then for all t, $E(t) \leq \epsilon$. We aim to use continuous induction/ a boot-strap argument. By standard energy estimates, $E(t)$ is continuous in t. If $E(0) < \epsilon/2$, then clearly $E(0) \leq \epsilon$. It remains to prove that if $\sup_t E(t) \leq 2\epsilon$, then $\sup_t E(t) \leq \epsilon$. In the following argument, there will be several implicit constants denoted C, and ϵ will depend on these, but not on u, f, or g.

From differentiating the quasilinear wave equation, for all α

$$\begin{aligned} &\sum_{ij} G^{ij}\partial_i\partial_j\Gamma^\alpha u \\ &\quad = [\Gamma^\alpha, \sum_{ij} G^{ij}\partial_i\partial_j]u + \Gamma^\alpha F \\ &\quad = [\Gamma^\alpha, \sum_{ij}(G^{ij}-\eta^{ij})\partial_i\partial_j]u + [\Gamma^\alpha, \sum_{ij}\eta^{ij}\partial_i\partial_j]u + \Gamma^\alpha F \\ &\quad = [\Gamma^\alpha, \sum_{ij}(G^{ij}-\eta^{ij})]\partial_i\partial_j u + \sum_{ij}(G^{ij}-\eta^{ij})[\Gamma^\alpha, \partial_i\partial_j]u + [\Gamma^\alpha, \sum_{ij}\eta^{ij}\partial_i\partial_j]u + \Gamma^\alpha F. \end{aligned}$$

From the energy estimate for $\Gamma^\alpha u$, we find

$$\begin{aligned}
E(t) &= E_{\partial_t,\Gamma,n+4}[u](t)\\
&= E_{\partial_t,\Gamma,n+4}[u](0)\\
&\quad+\sum_{|\alpha|\leq n+4}\int_0^t \Big(\|[\Gamma^\alpha,\sum_{ij}(G^{ij}-\eta^{ij})]\partial_i\partial_j u\|_{L^2}\\
&\qquad+\|\sum_{ij}(G^{ij}-\eta^{ij})[\Gamma^\alpha,\partial_i\partial_j]u\|_{L^2}\\
&\qquad+\|[\Gamma^\alpha,\sum_{ij}\eta^{ij}\partial_i\partial_j]u\|_{L^2}\\
&\qquad+\|\Gamma^\alpha F\|_{L^2}\Big)E(t)^{1/2}\mathrm{d}t'.
\end{aligned}$$

The goal now is to estimate each of the L^2 terms in the previous integral by

$$C\epsilon^{1/2}(1+t)^{-\frac{n-1}{2}}E(t)^{1/2}.$$

Consider $[\Gamma^\alpha, G^{ij}-\eta^{ij}]\partial_i\partial_j u$. Treat $\partial_j u$ as the relevant quantity instead of u and recall that $G^{ij}-\eta^{ij}$ is assumed to be a function of ∂u and vanishes linearly in $|\partial u|$. Thus, there are $|\alpha|$ derivatives to be distributed between $G^{ij}-\eta^{ij}$ and $\partial_j u$. Thus, one factor of ∂u can have at most $n+4$ Γ derivatives, and all other factors can have at most $\lfloor\frac{n+4}{2}\rfloor$ Γ derivatives. If $|\alpha|=0$, then the commutator vanishes. For $1\leq|\alpha|\leq n+4$, since $G^{ij}-\eta^{ij}$ has uniformly bounded derivatives and at most α derivatives are applied to $G^{ij}-\eta^{ij}$,

$$|[\Gamma^\alpha, G^{ij}-\eta^{ij}]\partial_i\partial_j u|\leq C(1+|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor})^{|\alpha|}|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor}|\partial u|_{\Gamma,n+4}.$$

Thus,

$$\begin{aligned}
\|[\Gamma^\alpha, G^{ij}-\eta^{ij}]\partial_i\partial_j u\|_{L^2}&\leq C\sup_x\left((1+|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor})^{|\alpha|}|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor}\right)\|\partial u\|_{\Gamma,n+4}\\
&\leq C\epsilon^{1/2}(1+t)^{-\frac{n-1}{2}}E(t)^{1/2}.
\end{aligned}$$

Consider $[\Gamma^\alpha,\partial_i\partial_j]=\partial_i[\Gamma^\alpha,\partial_j]+[\Gamma^\alpha,\partial_i]\partial_j$. $[\Gamma^\alpha,\partial_i]\partial_j u$ can be expanded in terms of $(|\alpha|-1)$ Γ derivatives on $\partial_j u$. $[\Gamma^\alpha,\partial_j]$ can be expanded as $(|\alpha|-1)$ Γ derivatives, so ∂_i can be commuted through $[\Gamma^\alpha,\partial_j]$ to give at most $|\alpha|-1$ derivatives applied to ∂_i, so $|\partial_i[\Gamma^\alpha,\partial_j]u|\leq|\partial u|_{\Gamma,|\alpha|-1}$. Since $G^{ij}-\eta^{ij}$ is bounded by $|\partial u|$,

$$\begin{aligned}
|(G^{ij}-\eta^{ij})[\Gamma^\alpha,\partial_i\partial_j]u|&\leq C|\partial u||\partial u|_{\Gamma,n+4}\\
\|(G^{ij}-\eta^{ij})[\Gamma^\alpha,\partial_i\partial_j]u\|_{L^2}&\leq C\|\partial u\|_{L^\infty}\|\partial u\|_{\Gamma,n+4}\\
&\leq C\epsilon^{1/2}(1+t)^{-\frac{n-1}{2}}E(t)^{1/2}.
\end{aligned}$$

Consider $\Gamma^\alpha F$. Since F is a function of ∂u that vanishes quadratically, if $|\alpha|$ Γ derivatives are applied, then there will be at most $|\alpha|$ factors of ∂u or its derivatives that are produced by the chain rule, the one with the highest number of derivatives will have at most $n+4$ Γ derivatives, and the rest will have at most $\lfloor\frac{n+4}{2}\rfloor$ Γ derivatives. Furthermore, if there are no factors of ∂u or its derivatives, then there is a factor of F, which vanishes quadratically in $|\partial u|$. If there is only one factor of ∂u or its derivatives, then there will be a factor of $\mathrm{d}F$, which vanishes linearly. Thus, $\Gamma^\alpha F$ vanishes at least quadratically in ∂u or its derivatives. Thus,

$$|\Gamma^\alpha F| \leq C(1+|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor})^{\min(0,|\alpha|-2)}|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor}|\partial u|_{\Gamma,n+4}$$
$$\begin{aligned}\|\Gamma^\alpha F\| &\leq C\sup_x\left((1+|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor})^{\min(0,|\alpha|-2)}|\partial u|_{\Gamma,\lfloor\frac{n+4}{2}\rfloor}\right)\|\partial u\|_{\Gamma,n+4}\\ &\leq C\epsilon^{1/2}(1+t)^{-\frac{n-1}{2}}E(t)^{1/2}.\end{aligned}$$

Finally, consider $[\Gamma^\alpha, \sum_{ij}\eta^{ij}\partial_i\partial_j]$ can be expanded in terms of constant multiples of $\Gamma^\beta\sum_{ij}\eta^{ij}\partial_i\partial_j$ with $|\beta| \leq |\alpha|-1$. Since $\sum_{ij}\eta^{ij}\partial_i\partial_j u = -\sum_{ij}(G^{ij}-\eta^{ij})\partial_i\partial_j u - F$, it follows that $[\Gamma^\alpha, \sum_{ij}\eta^{ij}\partial_i\partial_j]$ can be expanded as constant multiples of terms of the form $\Gamma^\beta\sum_{ij}(G^{ij}-\eta^{ij})\partial_i\partial_j u$ and $\Gamma^\beta F$. Such terms have already been shown to have the desired form.

We have thus shown that, under the assumption $\sup_t E(t) \leq 2\epsilon$,

$$E(t) \leq E(0) + C\epsilon^{1/2}\int_0^t (1+t')^{-\frac{n-1}{2}}E(t')\mathrm{d}t'.$$

Thus, by Gronwall's inequality, $E(t) \leq 2E(0) \leq \epsilon$ if

$$C\epsilon^{1/2}\int_0^t (1+t')^{-\frac{n-1}{2}}\mathrm{d}t' \leq \ln 2.$$

ϵ can be chosen small enough to enforce this condition, as long as the integral is uniformly bounded, which holds if $n-1 > 2$, i.e. $n \geq 4$. □

2. Derivative semilinear equations and null forms

Consider

$$\sum_{ij}\eta^{ij}\partial_i\partial_j u = F(\partial u),$$

with F smooth and satisfying, for ∂u small,

$$|F(\partial u)| \leq C|\partial u|^p.$$

From the smoothness of F, we may assume p is an integer.

If $p = 1$, then the equation may act like a linear equation, such as the Klein-Gordon equation, and thus may have very different decay, or or even no decay. (If u satisfies the wave equation, then $v = (x^i)^q u$ satisfies the wave equation with an additional first-order term, and, if q is large relative to $(n-1)/2$, then the growth in x may ovewhelm the decay in t, and v may grow linearly.) Thus, consider the situation for $p \geq 2$.

Applying the same arguments as in the previous section, we find

$$E(t) \leq E(0) + \int_0^t \epsilon^{\frac{p-1}{2}}\left((1+t)^{-\frac{n-1}{2}}\right)^{p-1}E(t')\mathrm{d}t',$$

so that Gronwall's inequality can be applied if

$$1 < (p-1)\frac{n-1}{2}.$$

This is automatically satisfied if $n \geq 4$. For $n = 3$, this condition is satisfied for $p \geq 3$. However, this integrability condition is not satisfied for quadratic nonlinearities, with $p = 2$.

However, consider $\sum_{ij} \eta^{ij}\partial_i\partial_j u = 0$ with u vanishing for $|\vec{x}|$ large when t is bounded and u uniformly small. Let $v = ln(1+u)$. Thus, $u = e^v - 1$, so

$$\begin{aligned} 0 &= \sum_{ij} \eta^{ij}\partial_i\partial_j u \\ &= e^v \left(\sum_{ij} \eta^{ij}\partial_i\partial_j v + \sum_{ij} \eta^{ij}(\partial_i v)(\partial_j v) \right), \\ \sum_{ij} \eta^{ij}\partial_i\partial_j v &= -\sum_{ij} \eta^{ij}(\partial_i v)(\partial_j v). \end{aligned} \tag{1}$$

Thus, although equation (1) appears to be a nonlinear equation with a quadratic nonlinearity, at least for small data, it can be transformed to the linear wave equation and has global, decaying solutions.

In $\mathbb{R}^{1+3}$, consider

$$K = \sum_i K^i\partial_i = (t^2 + r^2)\partial_t + 2rt\partial_r.$$

This satisfies $[\sum_{ij} \eta^{ij}\partial_i\partial_j, K] = 4t \sum_{ij} \eta^{ij}\partial_i\partial_j$, so K is a symmetry of the wave equation. The energy $\int \mathcal{P}^i \mathrm{d}\nu_i$ with $\mathcal{P}^i = \sum_j \mathcal{T}_j K^j$ is not constant in time, but if we take

$$\tilde{\mathcal{P}}^i = \sum_j \mathcal{T}_j K^j + \frac{4t}{2}\psi\partial_i\psi - \frac{1}{4}(\partial_i 4t)\psi^2,$$

then $\sum_i \partial_i \tilde{\mathcal{P}}^i = 0$, so $\int_{\{t\}\times\mathbb{R}^3} \tilde{\mathcal{P}}^i \mathrm{d}\nu_i$ is constant in t, and

$$\int_{\{t\}\times\mathbb{R}^3} \tilde{\mathcal{P}}^i \mathrm{d}\nu_i = \frac{1}{4}\int u_+^2 |L_+\psi|^2 + (u_+^2 + u_-^2)\sum_{i=1}^{3} \left|\frac{1}{r}R_i\psi\right|^2 + u_-^2|L_-\psi|^2 \mathrm{d}^3x$$

where

$$L_\pm = \partial_t \pm \partial_r, \qquad\qquad u_\pm = t \pm r,$$

and R_i is rotation around the x^i axis. Thus, at least in some averaged sense, $L_+\psi$ and $r^{-1}R_i\psi$ should be “good” with fast decay, like $(t+r)^{-1}$), unlike $L_-\psi$ which should be bad with decay, like $(t-r)^{-1}$.

Since

$$\sum_{ij} \eta^{ij}(\partial_i\psi)(\partial_j\psi) = (L_+\psi)(L_-\psi) + \sum_{i=1}^{3} \left|\frac{1}{r}R_i\psi\right|^2,$$

in this type of quadratic nonlinearity, it is impossible for two “bad” derivatives to combine. This allows one to prove small data global well-posedness using estimates, rather than using the transformation. This is convenient if there is a complicated nonlinearity that is a sum of terms.

Systems, with for $I \in \{1, \dots, N\}$,

$$\sum_{ij} (G^1)^{ij} \partial_i \partial_j \psi^1 = F^1(\psi^1, \dots, \psi^N),$$
$$\dots$$
$$\sum_{ij} (G^N)^{ij} \partial_i \partial_j \psi^N = F^I(\psi^1, \dots, \psi^N).$$

can be treated using energies in the same way. If $n \geq 4$, then the same results hold. The same can also be applied if $n \geq 3$, $G^I = \eta$ for all I, and the F^I are sums of terms that vanish cubicly or are of the form

$$Q_0(\partial\psi^I, \partial\psi^J) = \sum^{ij} (\partial_i \psi^I)(\partial_j \psi^J),$$
$$Q_{ij}(\partial\psi^I, \partial\psi^J) = (\partial_i \psi^I)(\partial_j \psi^J) - (\partial_j \psi^I)(\partial_i \psi^J).$$

The antisymmetry of Q_{ij} also prevents two “bad” derivatives from