ADVANCED PDE II - LECTURE 1

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Warning: This is a first draft of the lecture notes and should be used with care![1]

1. Classification of PDEs

A PDE for a function $u(x_1, x_2, \ldots, x_n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a relation of the form

$$F(x_1, \ldots, x_n, u, \partial_{x_1} u, \ldots, \partial^2_{x_1} u, \partial^2_{x_1 x_2} u, \ldots) = 0,$$

where F depends on $x_1, \ldots, x_n$, u, and *finitely* many derivatives of u. We will sometimes use the shorthand notation

$$F(\vec{x}, u, Du, \ldots, D^m u) = 0. \tag{1.1}$$

The *order of a PDE* (1.1) is the order m of the highest derivative that occurs.

A *classical solution* of a PDE (1.1) of order m is a m-times differentiable function u satisfying (1.1).

Definition 1.1. (i) A PDE is *linear* if it is linear in u and its derivatives with coefficients depending only on $x_1, \ldots, x_n$:

$$\sum_{|\alpha| \leq m} a_\alpha(x_1, \ldots, x_n) D^\alpha u = f(x), \tag{1.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $f \equiv 0$, then (1.3) is called a homogeneous PDE. If a_α are constant, then (1.3) is called a constant-coefficient linear PDE.

(ii) A PDE of order m is called *semi-linear* if it is linear in the highest order derivatives with coefficients depending on $x_1, \ldots, x_n$ only:

$$\sum_{|\alpha| = m} a_\alpha(x_1, \ldots, x_n) D^\alpha u + \tilde{F}(\vec{x}, u, Du, \ldots, D^{m-1} u) = 0. \tag{1.3}$$

(iii) A PDE of order m is called *quasi-linear* if it is linear in the derivatives of order m with coefficients depending on $x_1, \ldots, x_n$ and derivatives of u of order less than m.

$$\sum_{|\alpha| = m} a_\alpha(x_1, \ldots, x_n, u, Du, \ldots, D^{m-1} u) D^\alpha u + \tilde{F}(\vec{x}, u, Du, \ldots, D^{m-1} u) = 0. \tag{1.4}$$

(iv) A *fully nonlinear* PDE is a PDE for which no special structure is assumed.

[1]Lecture 1 was inspired by Chapter 1 in Fritz John's PDE book and by Gustav Holzegel's lecture notes (weeks 1 and 2). We refer the interested readers to these two sources for more details. See also Evans' book (Chapter 3) and Alinhac's book on hyperbolic PDEs (Chapters 1-3) for slightly different perspectives.

Next, we consider a differential operator P of order m defined by

$$Pu(x_1,\ldots,x_n,t) := \sum_{|\alpha|+\ell\leq m} a_{\alpha\ell}(x_1,\ldots,x_n,t)D^\alpha\partial_t^\ell u(x_1,\ldots,x_n,t). \tag{1.5}$$

The operator

$$P_m := \sum_{|\alpha|+\ell=m} a_{\alpha\ell}(\vec{x},t)D^\alpha\partial_t^\ell \tag{1.6}$$

is called the "principal part" of the differential operator P. For fixed $\vec{x},\vec{\xi},t$, the solutions of the polynomial equation in τ:

$$\sum_{|\alpha|+\ell=m} a_{\alpha\ell}(\vec{x},t)\xi_1^{\alpha_1}\ldots\xi_n^{\alpha_n}\tau^\ell = 0,$$

are denoted by $-\lambda_1(\vec{x},t,\vec{\xi}),\ldots,-\lambda_m(\vec{x},t,\vec{\xi})$ and λ_i with $i=1,\ldots,m$ are called the *characteristic speeds* of the polynomial P.

Definition 1.2. We say that the differential operator P is *hyperbolic* in $\Omega\subset\mathbb{R}^n\times\mathbb{R}$ if for all $(\vec{x},t)\in\Omega$ and all $\xi\in\mathbb{R}^n$, the characteristic speeds $\lambda_i(\vec{x},t,\vec{\xi})$ are real. P is *strictly hyperbolic* if they are also distinct for any $\xi\neq 0$:

$$\lambda_1(\vec{x},t,\vec{\xi}) < \cdots < \lambda_m(\vec{x},t,\vec{\xi}).$$

Examples:
1). The linear advection equation:

$$\partial_t u + c\partial_x u = 0, \qquad x\in\mathbb{R}, c>0. \tag{1.7}$$

Here, the characteristic equation is $\tau + c\xi = 0$ with the real solution $\tau = -c\xi$.

2). The linear wave equation on $\mathbb{R}$:

$$\partial_t^2 u - \partial_x^2 u = 0, \qquad x\in\mathbb{R}. \tag{1.8}$$

The characteristic equation is in this case $\tau^2-\xi^2=0$ with real solutions $\tau=\pm\xi$. Notice that this equation is associated to the quadratic form $\tau^2-\xi^2$ whose level sets are hyperbolas, hence the term 'hyperbolic'.

3). We would like to consider second order hyperbolic PDEs on $\mathbb{R}^n$ that generalize the linear wave equation (1.8). Namely, PDEs of the form $Lu=f$, where

$$L = \partial_t^2 + 2\sum_{i=1}^n b_i(\vec{x},t)\partial_{x_i}\partial_t - \sum a_{ij}(\vec{x},t)\partial_{x_i}\partial_{x_j} + L_1, \tag{1.9}$$

with L_1 is of order one. According to Definition 1.2, L is *hyperbolic* in $\Omega\subset\mathbb{R}^n\times\mathbb{R}$ if for all $(\vec{x},t)\in\Omega$ and all $\xi\in\mathbb{R}^n$, the solutions $-\lambda_1(\vec{x},t,\vec{\xi}), -\lambda_2(\vec{x},t,\vec{\xi})$ of the characteristic equation

$$\tau^2 + 2\sum b_i(\vec{x},t)\tau\xi_i - \sum a_{ij}(\vec{x},t)\xi_i\xi_j = 0$$

are real, or equivalently,

$$(\vec{b}(\vec{x},t)\cdot\vec{\xi})^2 + \sum a_{ij}(\vec{x},t)\xi_i\xi_j \geq 0.$$

L is *strictly hyperbolic* if, in addition, $\lambda_1(\vec{x},t,\vec{\xi}) \neq \lambda_2(\vec{x},t,\vec{\xi})$ for all $(\vec{x},t) \in \Omega$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$.

We remark that if $\vec{b}(\vec{x},t) \equiv 0$, the strict hyperbolicity of L is equivalent to the quadratic form $\sum a_{ij}(\vec{x},t)\xi_i\xi_j$ being positive definite.

2. The linear advection equation

We start by noticing that along a line of the family $x - ct = const.$, any solution of the linear advection equation

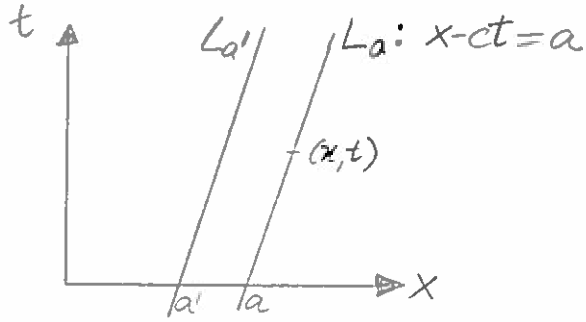
$$\partial_t u + c\partial_x u = 0, \qquad x \in \mathbb{R}, \quad c > 0$$

is constant. Indeed, for any $a \in \mathbb{R}$, we have

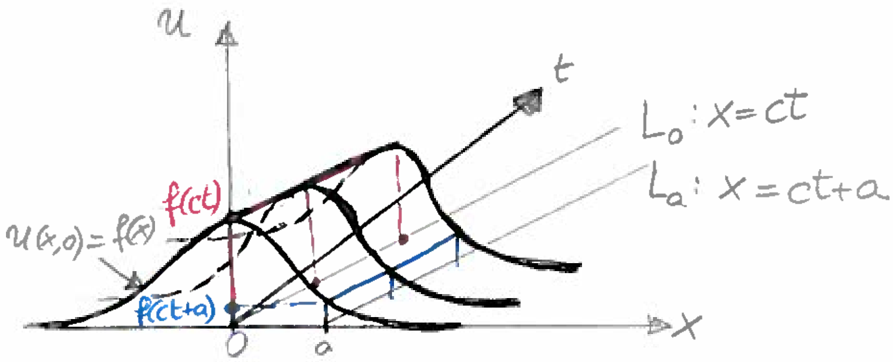
$$\frac{du}{dt} = \frac{d}{dt}u(ct+a,t) = c\partial_x u + \partial_t u = 0$$

and therefore $u(ct+a,t) = u(a,0)$. We thus obtained the following proposition.

Proposition 2.1. The unique classical solution of equation (1.7) with initial data $u(x,0) = f(x) \in C^1(\mathbb{R})$ is given by $u(t,x) = f(x-ct)$.



The line $L_a : x - ct = a$ is called a characteristic line of (1.7). For $(x,t) \in L_a$, the *domain of dependence* of the solution u on the initial value is the single point a. The *domain of influence* of the initial value at a on the solution u is just the characteristic line L_a.



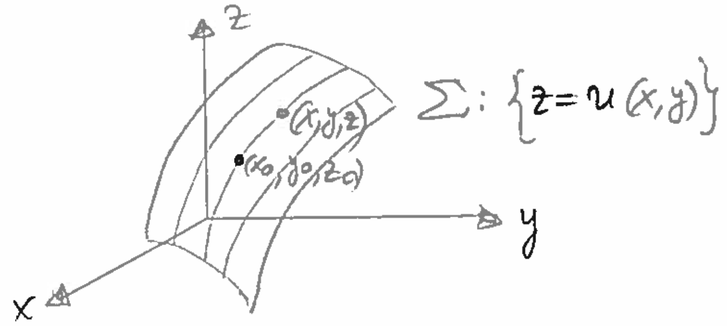
The initial condition $f(x)$ is transported along the characteristic lines L_a. For this reason, (1.7) is also known under the name of "transport equation". We can rephrase this by saying that the graph of a solution of (1.7) is a wave propagating to the right with velocity c, without changing shape.

3. The method of characteristics for first-order quasi-linear PDEs

For simplicity, we only consider quasi-linear PDEs in the two-dimensional case:

$$a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u), \tag{3.1}$$

where $a, b, c : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are C^1 functions.



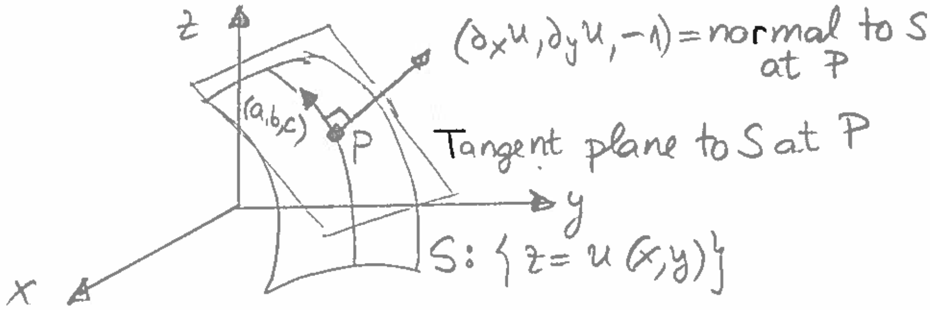
Loosely speaking, the idea of the method of characteristics is that for each (x, y, z) near a fixed point (x_0, y_0, z_0) we find characteristic curves going through (x, y, z) on which we can compute u: $z = u(x, y)$. These characteristic curves are solutions of a system of ODEs. Therefore, the method of characteristics consists in converting a first-order PDE into an appropriate system of ODEs.

Definition 3.1. If u is a C^1 function solving (3.1), we call it's graph $S = \{(x, y, z) : z = u(x, y)\}$ an *integral surface* of the PDE (3.1).

From multivariable calculus, we have that the normal to an integral surface S is proportional to $(\partial_x u, \partial_y u, -1)^t$. Then (3.1) is equivalent to

$$(a(x, y, z), b(x, y, z), c(x, y, x))^t \cdot (\partial_x u, \partial_y u, -1)^t = 0. \tag{3.2}$$

Thus, $(a(x, y, z), b(x, y, z), c(x, y, x))^t \perp (\partial_x u, \partial_y u, -1)^t$ or, in other words, the characteristic direction (a, b, c) is parallel to the tangent plane to the surface S. This suggests obtaining an integral surface via integral curves along the characteristic directions (a, b, c).



Given $(x_0, y_0, z_0) \in S$, we consider the system of ODEs

$$\begin{cases} \frac{dx}{dt} = a(x, y, z), \\ \frac{dy}{dt} = b(x, y, z), \\ \frac{dz}{dt} = c(x, y, z), \end{cases} \qquad (x, y, z)(0) = (x_0, y_0, z_0). \tag{3.3}$$

By ODE theory[2], this system has a unique solution that we call a *characteristic curve*.

An important element in the proof of Theorem 3.5 is the following proposition.

Proposition 3.2. *A C^1 surface S is an integral surface for the PDE* (3.1) *if and only if S is a union of characteristic curves.*

The proof of Proposition 3.2 follows from the two lemmas below.

Lemma 3.3. *If a C^1 surface is a union of characteristic curves, then it is an integral surface of* (3.1)

Proof. Through any point P of S we have a characteristic curve γ lying in S. The tangent line to γ at P necessarily lies in the tangent plane to S at P. Since the tangent line to γ has the characteristic direction (a, b, c), it follows that (a, b, c) is parallel to the tangent plane of S at P. In other words, with S given parametrically by $z = u(x, y)$, (a, b, c) is perpendicular to the normal $(\partial_x u, \partial_y u, -1)$ to the surface S. As in (3.2), it then follows that (3.1) is satisfied. □

Next, we show that any integral surface S is the union of characteristic curves, or that through any point of an integral surface S passes a characteristic curve contained in S.

Lemma 3.4. *Let S be an integral surface containing the point $P(x_0, y_0, z_0)$. Let γ be the characteristic curve through P. Then γ lies completely in S.*

Proof. We parametrize γ by $\gamma(t) = (x(t), y(t), z(t))$ such that $P = \gamma(0)$. We set $U(t) := z(t) - u(x(t), y(t))$. Since $P \in S$, we have $U(0) = 0$. We will show that $U(t) = 0$ for all t, which implies that $\gamma \subset S$. We have

$$\frac{dU}{dt} = (c - \partial_x u \cdot a - \partial_y u \cdot b)(x, y, z) = (c - \partial_x u \cdot a - \partial_y u \cdot b)\left(x(t), y(t), U(t) + u(x(t), y(t))\right).$$

This gives an ODE for $U(t)$, for which $U(t) \equiv 0$ is a particular solution. By the uniqueness theory for ODEs, it follows that this is the only solution. □

Let Γ be a curve in the three-dimensional space given parametrically by

$$x = f(s) \quad y = g(s) \quad z = h(s).$$

We are looking for a solution $u(x, y)$ of (3.1) such that

$$h(s) = u(f(s), g(s)).$$

This is the *Cauchy problem* for (3.1). We are only interested here in a *local* solution, defined for (x, y) near $(x_0 = f(s_0), y_0 = g(s_0))$ for some fixed s_0. The *initial-value problem* is the Cauchy problem with Γ of the form

$$x = s \quad y = 0 \quad z = h(s).$$

That is, we impose that $u(x, 0) = h(x)$.

The purpose of this section is to prove the following theorem using the method of characteristics.

[2]You can find a nice 'ODEs refresher' in the online lecture notes of Gustav Holzegel (week 1).

Theorem 3.5. *Let $P = (x_0, y_0, z_0)$ and Γ a C^1-curve parametrized by*

$$\Gamma : s \longmapsto (f(s), g(s), h(s)),$$

going through the point P at $s = s_0$. Assume that Γ is non-charcateristic, i.e.

$$f'(s_0)b(x_0, y_0, z_0) - g'(s_0)a(x_0, y_0, z_0) \neq 0. \tag{3.4}$$

Then, there exist a small neighbouhood $\mathcal{U} \subset \mathbb{R}^2$ of (x_0, y_0) and a unique C^1-function $u : \mathcal{U} \to \mathbb{R}$ solving (3.1) in $\mathcal{U}$ and satisfying

$$h(s) = u(f(s), g(s)) \quad along \quad \Gamma.$$

Proof. We are looking for an integral surface S of (3.1) given by $z = u(x, y)$ with (x, y) in a neighborhood of (x_0, y_0), passing through Γ. By Proposition 3.2, this will consist of characteristic curves passing through the various points of Γ .

More precisely, we construct a local solution near $(x_0, y_0) = (f(s_0), g(s_0))$ parametrized by

$$x = X(s, t) \quad y = Y(s, t) \quad z = Z(s, t), \tag{3.5}$$

where s is near s_0 and X, Y, Z satisfy the characteristic ODEs

$$\begin{cases} \frac{dX}{dt} = a\left(X(s,t), Y(s,t), Z(s,t)\right), \\ \frac{dY}{dt} = b\left(X(s,t), Y(s,t), Z(s,t)\right), \\ \frac{dZ}{dt} = c\left(X(s,t), Y(s,t), Z(s,t)\right), \end{cases} \tag{3.6}$$

with initial conditions

$$X(s, 0) = f(s) \quad Y(s, 0) = g(s) \quad Z(s, 0) = h(s).$$

From the ODE theory, there exists a unique solution of class C^1 of this system with (s, t) near $(s_0, 0)$. Moreover, this solution depends continuously on the parameter s.

Equation (3.5) represents a parametrized surface Σ. In order to represent this surface explicitly as $z = u(x, y)$, we need to invert $x = X(s, t)$, $y = Y(s, t)$ to $s = S(x, y)$, $t = T(x, y)$, since then

$$z = Z(s, t) = Z(S(x, y), T(x, y)) =: u(x, y).$$

By the implicit function theorem, the local inversion is possible provided that

$$\begin{vmatrix} \frac{dX}{ds}(s_0, 0) & \frac{dY}{ds}(s_0, 0) \\ \frac{dX}{dt}(s_0, 0) & \frac{dY}{dt}(s_0, 0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{vmatrix} \neq 0.$$

This is precisely the condition (3.4) that Γ is a non-characteristic curve. Therefore, (3.5) represents indeed a surface $z = u(x, y)$. In order to show that this is an integral surface, we notice from (3.6) that for any point $P \in \Sigma$, $(\partial_t X, \partial_t Y, \partial_t Z)^t$ gives the direction of the tangent to a curve $s = const.$ on Σ. Thus, the characteristic direction $(a, b, c)^t = (\partial_t X, \partial_t Y, \partial_t Z)^t$ is parallel to the tangent plane of Σ at P. In conclusion, Σ is an integral surface of the PDE (3.1).

So far, we have proved the existence of a local C^1-solution u. To prove the uniqueness of this solution, it suffices to apply Lemma 3.4. More precisely, any other integral surface

Σ' through Γ would have to contain all characteristic curves through Γ and hence would locally coincide with Σ.

□

Example: We apply the method of characteristics to the linear advection equation (1.7) with initial condition $u(x,0) = f(x)$.

In this case, the curve Γ is parametrized by

$$x = s \quad t = 0 \quad z = f(s),$$

where we relabeled y by t. The characteristic equations become:

$$\begin{cases} \frac{dx}{d\tau} = c, \\ \frac{dt}{d\tau} = 1, \\ \frac{dz}{d\tau} = 0, \end{cases} \qquad (x,y,z)(s,0) = (s,0,f(s)).$$

Thus,

$$\begin{cases} x(s,\tau) = c\tau + s \\ t(s,\tau) = \tau \\ z(s,\tau) = f(s). \end{cases}$$

We invert the first two equations obtaining $\tau = t$ and $s = x - ct$ and conclude that

$$u = z(s,\tau) = f(s) = f(x - ct).$$

We thus recovered the result in Proposition 2.1.

The method of characteristics can also be applied to first-order fully nonlinear PDEs, but we will not discuss this in this course. We refer the interested readers to [JohnPDE], [Holzegel], [Evans, Chapter 3].

4. The inviscid Burger's equation

The inviscid Burger's equation

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(x,0) = h(x), \end{cases} \quad x \in \mathbb{R}, \quad t \geq 0. \tag{4.1}$$

appears in fluid mechanics, where u represents the velocity field of a Newtonian fluid.

We start by applying the method of characteristics to (4.1). The curve Γ is in this case parametrized by

$$x = s \quad t = 0 \quad z = h(s),$$

while the characteristic equations become:

$$\begin{cases} \frac{dx}{d\tau} = z, \\ \frac{dt}{d\tau} = 1, \\ \frac{dz}{d\tau} = 0, \end{cases} \qquad (x,y,z)(s,0) = (s,0,f(s)).$$

Then,

$$\begin{cases} x(s,\tau) = z\tau + s \\ t(s,\tau) = \tau \\ z(s,\tau) = h(s), \end{cases}$$

where we used $\frac{dz}{d\tau} = 0$ to deduce the expression of $x(s,\tau)$. We invert the first two equations obtaining $\tau = t$ and $s = x - zt$, and thus

$$z(s,\tau) = h(s) = h(x - zt).$$

Therefore, $u = z(s,\tau)$ satisfies the implicit equation

$$u(x,t) = h(x - u(x,t) \cdot t).$$