ADVANCED PDE II - LECTURE 2

PIETER BLUE AND OANA POCOVNICU

Warning: This is a first draft of the lecture notes and should be used with care![1]

1. THE INVISCID BURGER'S EQUATION

We consider the initial-value problem for the inviscid Burger's equation

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(x,0) = h(x), \end{cases} \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.1}$$

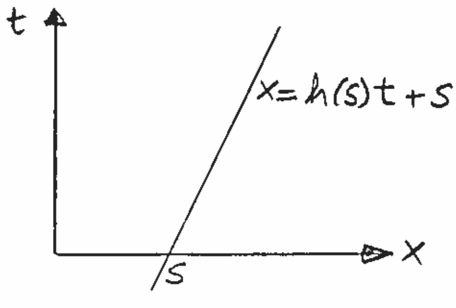
In the previous lecture we found the characteristic curves for this problem:

$$\begin{cases} x(s,\tau) = z\tau + s \\ t(s,\tau) = \tau \\ z(s,\tau) = h(s), \end{cases} \qquad \tau \geq 0.$$

In order to understand the solutions of the Burger's equation, we project the characteristic curves on the (x,t)-plane. Fix a point $(s,0)$. Then, $z = u(s,0) = h(s) = const$. The projected characteristic through $(s,0,h(s)) \in \Gamma$ is the line

$$\begin{cases} x(s,\tau) = z\tau + s \\ t(s,\tau) = \tau, \end{cases}$$

which can also be expressed by $x = h(s)t + s$ for $t \geq 0$. The solution is constant along this line since it is constant ($= h(s)$) on the characteristic through $(s,0,h(s))$.



Unless $h(s)$ is monotonically increasing, we notice that two projected characteristics $C_1 : x = h(s_1)t + s_1, t \geq 0$ and $C_2 : x = h(s_2)t + s_2, t \geq 0$, intersect at some point (x,t) where

$$t = -\frac{s_2 - s_1}{h(s_2) - h(s_1)}.$$

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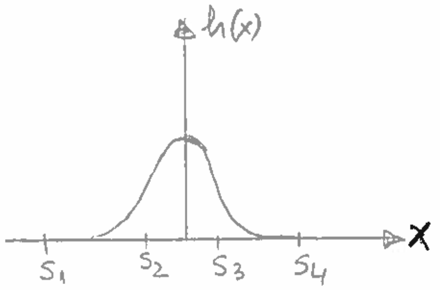
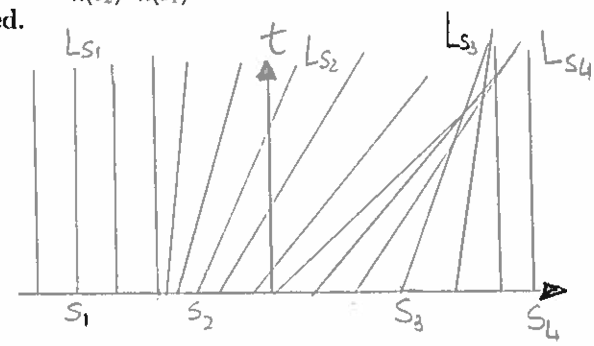
(Note that if h is monotonically increasing, then $\frac{s_2-s_1}{h(s_2)-h(s_1)} \leq 0$.) At the point of intersection, the solution can no longer be univalued.





In order to continue a solution of Burger's equation beyond the crossing of characteristics, we need to give up the requirement that it should be of class C^1 (classical). We define below a less restrictive notion of solution.

Assume for the moment that u is a C^1-solution of (1.1) with initial condition $u(x,0) = h(x)$. Let $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be smooth and compactly supported. We call such a function v a test function. Multiplying (1.1) by v and integrating by parts, we obtain:

$$0 = \int_0^\infty \int_{-\infty}^\infty (\partial_t u + \frac{1}{2}\partial_x(u^2))v dx dt = -\int_0^\infty \int_{-\infty}^\infty u\partial_t v + \frac{u^2}{2}\partial_x v dx dt - \int_{-\infty}^\infty h(x) v dx\Big|_{t=0}.$$

Thus,

$$\int_0^\infty \int_{-\infty}^\infty u\partial_t v + \frac{u^2}{2}\partial_x v dx dt + \int_{-\infty}^\infty h(x) v dx\Big|_{t=0} = 0. \tag{1.2}$$

We derived (1.2) by assuming that u is of class C^1, but (1.2) makes sense even if u is only bounded.

Definition 1.1. A function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an *integral solution* of (1.1) provided that (1.2) holds for any smooth compactly supported test function $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$.

The natural questions to ask is whether there exists an integral solutions of (1.1) and if so, whether it is unique. We will not answer these questions in full generality, but we will gain further insight by studying piecewise-smooth integral solutions and by looking at a few concrete examples. Before we proceed, we first recall Green's theorem that we will use shortly.

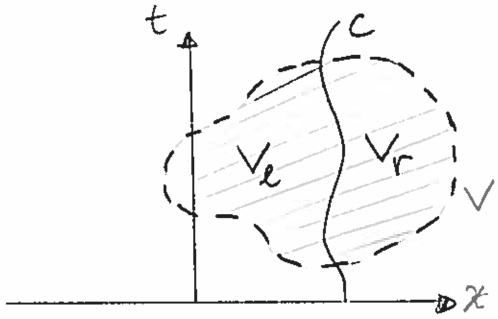
Theorem 1.2 (Green's theorem). Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. Denote by $\nu = (\nu_1, \nu_2)$ the outward unit normal to D. Let $\vec{F} = (P, Q)$ be a vector field on D. Then

$$\iint_D div \vec{F} dx dy = \int_C \vec{F} \cdot \nu ds,$$

or equivalently,

$$\iint_D \partial_x P + \partial_y Q dx dy = \int_C P\nu_1 + Q\nu_2 ds.$$

Let $V \subset \mathbb{R} \times (0, \infty)$ be some open region separated into V_ℓ and V_r by a smooth curve C.



We assume that u is an integral solution and that u and its derivatives are uniformly continuous in V_ℓ and V_r (i.e., u is a piecewise-smooth integral solution).

For any v compactly supported in V_ℓ, we have by integration by parts as above that

$$\int_0^\infty \int_{-\infty}^\infty u\partial_t v + \frac{u^2}{2}\partial_x v dx dt = 0.$$

Thus, $\partial_t u + u\partial_x u = 0$ on V_ℓ. Similarly, $\partial_t u + u\partial_x u = 0$ on V_r. On the other hand, for any v compactly supported in V, which does not necessarily vanish on C, we have:

$$0 = \int_0^\infty \int_{-\infty}^\infty u\partial_t v + \frac{u^2}{2}\partial_x v dx dt = \iint_{V_\ell} u\partial_t v + \frac{u^2}{2}\partial_x v dx dt + \iint_{V_r} u\partial_t v + \frac{u^2}{2}\partial_x v dx dt. \quad (1.3)$$

Let $\vec{\nu} = (\nu_1, \nu_2)$ be the unit normal to C pointing from V_ℓ into V_r. By applying Green's theorem 1.2 to the vector field $\vec{F} = (\frac{u^2}{2}v, uv)$ in the (x,t)-plane, we have that

$$\begin{aligned}\iint_{V_\ell} u\partial_t v + \frac{u^2}{2}\partial_x v dx dt &= -\iint_{V_\ell} (\partial_t u + u\partial_x u) v dx dt + \int_C (\frac{u_\ell^2}{2}\nu_1 + u_\ell \nu_2) v ds \\ &= \int_C (\frac{u_\ell^2}{2}\nu_1 + u_\ell \nu_2) v ds. \end{aligned} \quad (1.4)$$

Similarly, we obtain

$$\begin{aligned}\iint_{V_r} u\partial_t v + \frac{u^2}{2}\partial_x v dx dt &= -\iint_{V_r} (\partial_t u + u\partial_x u) v dx dt + \int_C (\frac{u_r^2}{2}\nu_1 + u_r \nu_2) v ds \\ &= \int_C (\frac{u_r^2}{2}\nu_1 + u_r \nu_2) v ds. \end{aligned} \quad (1.5)$$

By (1.3), (1.4), and (1.5), we obtain that

$$0 = \int_C \left(\frac{u_\ell^2 - u_r^2}{2}\nu_1 + (u_\ell - u_r)\nu_2 \right) v ds$$

for any test function v compactly supported in V. Therefore,

$$\frac{u_\ell^2 - u_r^2}{2}\nu_1 + (u_\ell - u_r)\nu_2 = 0 \quad \text{along} \quad C. \quad (1.6)$$

Suppose C is parametrized by $\gamma(t) = (x = s(t), t)$, where $s : [0, \infty) \to \mathbb{R}$. Since the tangent to C is parallel to $(\dot{s}, 1)$, we have $\vec{\nu} = \frac{1}{\sqrt{\dot{s}^2+1}}(1, -\dot{s})$. Then, (1.6) becomes

$$\frac{u_\ell^2 - u_r^2}{2} = \dot{s}(u_\ell - u_r). \tag{1.7}$$

This equality relating the jump of the function $\frac{u^2}{2}$, the jump of u, and the speed $\dot{s}$ of the curve C is called the *Rankine-Hugoniot condition.*

We have thus deduced that any piecewise-smooth integral solution of (1.1) in V must be a classical solution on both V_ℓ and V_r and must also satisfy the Rankine-Hugoniot condition. This leads us to the following definition.

Definition 1.3. Let V, V_ℓ, V_r, and C be as above. An integral solution u of (1.1) with the following properties:

(1) u is discontinuous across C,
(2) u is a classical solution of (1.1) in V_ℓ and V_r,
(3) u satisfies the Rankine-Hugoniot condition (1.7)

is called a *shock solution.*

We discuss next two examples of initial data for the inviscid Burger's equation (1.1) which lead to crossing of characteristics, shock solutions, rarefaction solutions.

Example 1: We consider the initial condition

$$h_1(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Given $x_0 \in \mathbb{R}$, we know by the method of characteristics that u takes the constant value $h_1(x_0)$ on the projected characteristic $x = h_1(x_0)t + x_0$. Consequently,

- If $x_0 \leq 0$, then $u \equiv 1$ on $x = t + x_0$,
- If $x_0 > 1$, then $u \equiv 0$ on $x = x_0$,
- If $0 \leq x_0 \leq 1$, then $u \equiv 1 - x_0$ on $x = (1 - x_0)t + x_0$.

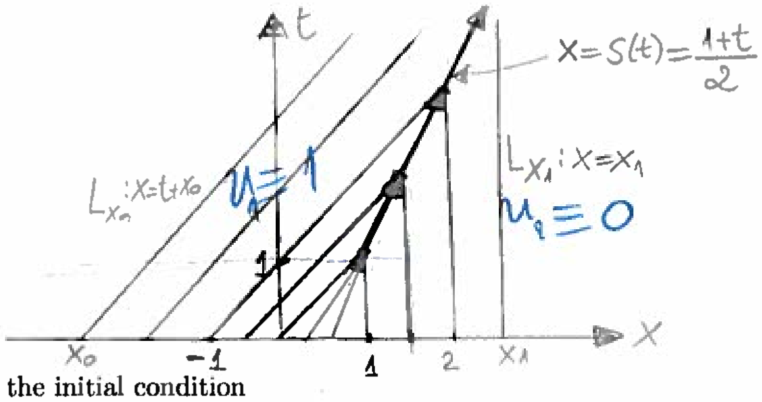
From this, it follows easily that for $0 \leq t \leq 1$ the solution of (1.1) with initial condition $u(x, 0) = h_1(x)$ is:

$$u_1(x,t) = \begin{cases} 1, & \text{if } x \leq t, \\ \frac{1-x}{1-t}, & \text{if } t < x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

For $t > 1$ the method of characteristics breaks down since the projected characteristics cross. How do we then define a solution for $t > 1$? The Rakine-Hugoniot condition (1.7) suggests considering a curve with speed $\dot{s} = \frac{u_\ell + u_r}{2} = \frac{1}{2}$. So, we set $s(t) = \frac{1+t}{2}$ and, for $t \geq 1$, we consider

$$u_1(x,t) := \begin{cases} 1, & \text{if } x < s(t) \\ 0, & \text{if } x > s(t). \end{cases}$$

One can easily verify that this a (*compressible*) shock solution of (1.1).



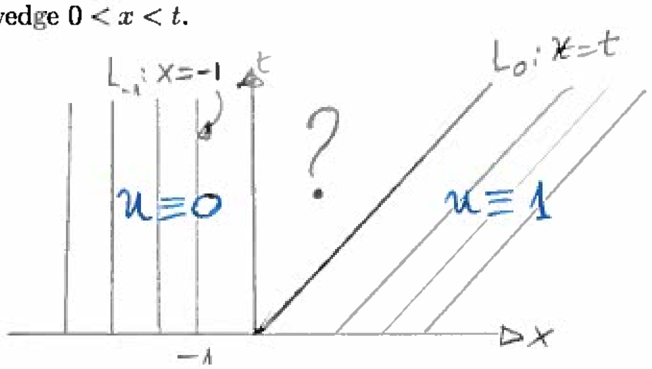
Example 2: We consider the initial condition

$$h_2(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Given $x_0 \in \mathbb{R}$, again by the method of characteristics we have that u takes the constant value $h_2(x_0)$ on the projected characteristic $x = h_2(x_0)t + x_0$. Consequently,

- If $x_0 < 0$, then $u \equiv 0$ on $x = x_0$,
- If $x_0 \geq 0$, then $u \equiv 1$ on $x = t + x_0$.

Therefore, the method of characteristics does not give us any information about the solution $u(x,t)$ for (x,t) in the wedge $0 < x < t$.



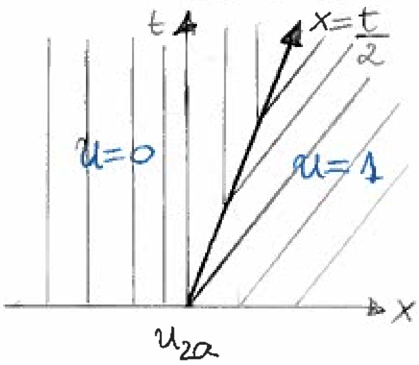
In this case, we can find at least two integral solutions of (1.1). First,

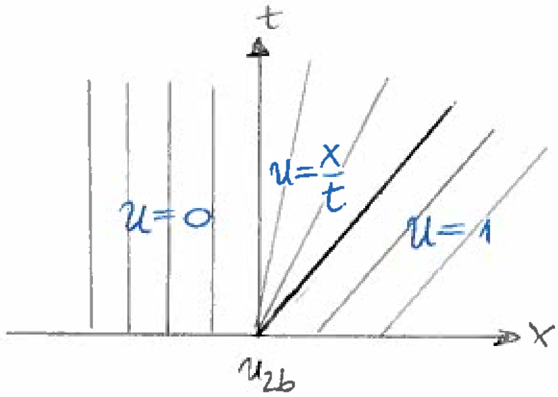
$$u_{2a}(x,t) = \begin{cases} 0, & \text{if } x < \frac{t}{2}, \\ 1, & \text{if } x > \frac{t}{2}, \end{cases}$$

which can be easily checked to be a compressible shock solution, and secondly

$$u_{2b}(x,t) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 \leq t \leq x, \\ 1, & \text{if } x > t. \end{cases}$$

The solution u_{2b} is called a *rarefaction wave*.





Example 2 shows that integral solutions are in general not unique. To ensure uniqueness one needs to impose a so-called *entropy condition*.

Let us briefly discuss the entropy condition in the context of piecewise-smooth integral solutions of (1.1). Starting with a point P in the (x,t)-plane, we go backwards in time along a characteristic and hope not to cross any other characteristic. If $P \in C$ and if a projected characteristic from the left ($x = u_\ell t + s$, or equivalently, $t = \frac{x}{u_\ell} - \frac{s}{u_\ell}$ if $u_\ell \neq 0$) and one from the right ($x = u_r t + s$) meet at P, then $u_\ell > \dot{s} > u_r$. This is the entropy condition yielding uniqueness of integral solutions in this case. In particular, this shows that the shock solution u_1 constructed in Example 1 is the unique integral solution of (1.1) with initial condition $u_1(x,0) = h_1(x)$.

For more on entropy solutions see for example [Evans, Chapter 3].

ADVANCED PDE II - LECTURE 2, PART 2

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1. The linear wave equation on $\mathbb{R}$

The wave equation on $\mathbb{R}^n$

$$\partial_t^2 u - \Delta u = F, \tag{LW}$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is Laplace's operator, appears as a model in various physical contexts, including:

- **Vibrating string.** With $n = 1$ and $F \equiv 0$, u represents the normal displacement of a vibrating string.
- **Light in vacuum.** From Maxwell's equations in electromagnetism, it can be seen that each component of the electric and magnetic fields satisfies (LW) with $n = 3$ and $F \equiv 0$.
- **Propagation of sound.** Equation (LW) arises as the linear approximation of the compressible Euler equations, which describe the behaviour of compressible fluids, such as air.
- **Gravitational waves.** A suitable geometric generalization of (LW) turns out to be the linear approximation of Einstein's equation, which is the basic equation of the theory of general relativity.

In this section, we discuss the simplest wave equation, namely the linear wave equation on $\mathbb{R}$ ($n = 1$ and $F \equiv 0$), also known as the equation of the vibrating string:

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \\ \partial_t u(x,0) = g(x), \end{cases} \tag{1.1}$$

The equation can be rewritten as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0.$$

This suggests that, instead of using (x,t), we use the so-called characteristic coordinates

$$\xi = x + ct \quad \eta = x - ct.$$

In these coordinates, (1.1) has the form $\partial_\xi \partial_\eta u = 0$. Integrating successively with respect to ξ and η, we easily obtain that $u = F(\xi) + G(\eta)$, or equivalently that

$$u(x,t) = F(x + ct) + G(x - ct). \tag{1.2}$$

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Therefore, we have two families of characteristic lines $x \pm ct = const$. The graph of $u(x,t)$ consists of two waves propagating with velocity c in opposite directions.

[Figure 11]

Next, we determine F and G in terms of the initial conditions f and g. By differentiating the first equation in the following system:

$$\begin{cases} F(x) + G(x) = f(x) \\ cF'(x) - cG'(x) = g(x), \end{cases}$$

we obtain easily that

$$F'(x) = \frac{cf'(x) + g(x)}{2c}, \qquad G'(x) = \frac{cf'(x) - g(x)}{2c}.$$

Integrating these from 0 to x, we have

$$\begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(x')dx' + (F(0) - \frac{f(0)}{2}) \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(x')dx' + (G(0) - \frac{f(0)}{2}). \end{cases}$$

Combining these with (1.2), we obtain

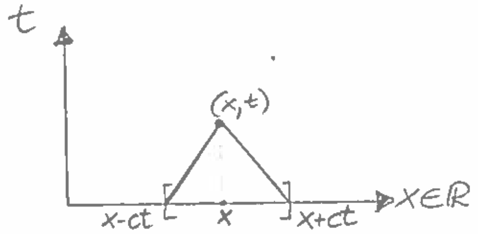
$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')dx'.$$

This is called *D'Alembert's formula* for the solution of the linear wave equation on $\mathbb{R}$. We list below some of the consequences of this formula.

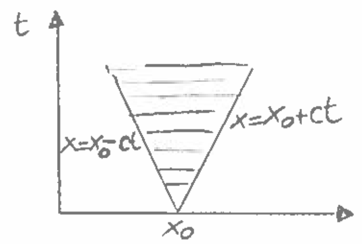
Consequences:

1). If $g \equiv 0$, then u breaks up into two traveling waves with speed c, with the same profile $\frac{f}{2}$, one traveling to the left and the other one to the right.

2). *The domain of dependence* of the solution u at (x,t) is the interval $[x-ct, x+ct]$



3). *The domain of influence* of a point x_0 is the interior of the "light cone" with vertex x_0: $x_0 - ct < x < x_0 + ct$.



4). [*Finite speed of propagation*] For the linear wave equation, "signals" only travel with speed c.

5). [*Conservation of energy*] We introduce the energy of a solution u at time t:

$$E(u(t)) := \int_{-\infty}^{\infty} \left((\partial_t u)^2 + c^2(\partial_x u)^2\right)(x,t)dx.$$

Then, $E(u(t)) = E(u(0))$ for all $t \in \mathbb{R}$. In order to prove this fact, we note by (1.2) that

$$(\partial_t u)^2 + c^2(\partial_x u)^2 = 2c^2\left[\left(F'(x+ct)\right)^2 + \left(G'(x-ct)\right)^2\right].$$

Then, by simple changes of variables, we obtain that

$$\begin{aligned} E(u(t)) &= 2c^2 \int_{-\infty}^{\infty} \left(F'(x+ct)\right)^2 + \left(G'(x-ct)\right)^2 dx = 2c^2 \int_{-\infty}^{\infty} \left(F'(x)\right)^2 + \left(G'(x)\right)^2 dx \\ &= E(u(0)). \end{aligned}$$