ADVANCED PDE II - LECTURE 3

PIETER BLUE AND OANA POCOVNICU

Warning: This is a first draft of the lecture notes and should be used with care![1]

1. The linear wave equation in higher dimensions

In this section we first consider the linear wave equation

$$\partial_t^2 u - \Delta u = 0 \tag{LW}$$

on $\mathbb{R}^n$ with n odd. Our goal is to find an explicit formula for classical solutions (of class C^2). We start with the case $n = 3$.

Proposition 1.1 (Kirchhoff's formula for the solution of (LW) on $\mathbb{R}^3$). *Let $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$. Then, the initial value problem*

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \end{cases} \qquad x \in \mathbb{R}^3, \quad t \geq 0 \tag{1.1}$$

has a unique solution $u \in C^2(\mathbb{R}^3)$ given by the formula

$$u(x,t) = \frac{1}{4\pi t^2} \int_{|x-y|=t} tg(y) + f(y) - \nabla_y f(y) \cdot (x-y) d\sigma(y). \tag{1.2}$$

Our argument will involve spherical means of a function $h(x)$, $x \in \mathbb{R}^3$:

$$(A_r h)(x) = \frac{1}{4\pi} \int_{S^2} h(x+ry) d\sigma(y), \tag{1.3}$$

where $r > 0$, S^2 is the unit sphere in $\mathbb{R}^3$ (whose area is 4π).

Assume that u is a C^2-solution of (LW) at time t. Then we notice by the above definition that $A_0(u(\cdot,t))(x) = u(x,t)$. In the following, we will obtain an explicit formula for $A_r(u(\cdot,t))(x)$ and we will make $r \to 0$ to obtain a formula for $u(x,t)$. The explicit formula for $A_r(u(\cdot,t))(x)$ is obtained by showing that a certain function $v(r,t)$ related to it, satisfies the linear wave equation $\partial_t^2 v - \partial_r^2 v = 0$ on $\mathbb{R}$ and by applying D'Alembert's formula.

Before proving Kirchhoff's formula, let us recall the Divergence theorem that we'll be using shortly.

Theorem 1.2 (Divergence theorem). Let S be an oriented, piecewise smooth, closed surface enclosing the compact subset V of $\mathbb{R}^3$. Denote by ν the outward unit normal to S. Let $\vec{F}$ be a vector field on V. Then

$$\iiint_V \text{div}\vec{F} dxdydz = \iint_S \vec{F} \cdot \nu ds.$$

[1]Lecture 3 was inspired by Chapter 1 in Sogge's book and by Gustav Holzegel's lecture notes (week 3).

Proof of Kirchhoff's formula. We start with some calculations for spherical means $A_r h(x)$. By noticing that $\nu(y) = y$ is the outer unit normal to the sphere $\mathcal{S}^2$, applying the Divergence theorem 1.2, and making the change of variables $x + ry \mapsto y$, we have that

$$\begin{aligned}\partial_r(A_r h)(x) &= \frac{1}{4\pi}\int_{\mathcal{S}^2} \nabla_x h(x+ry)\cdot y d\sigma(y) = \frac{1}{4\pi}\int_{|y|<1} \mathrm{div}_y\left(\nabla_x h(x+ry)\right) dy\\ &= \frac{r}{4\pi}\int_{|y|<1} (\mathrm{div}_x \nabla_x)\, h(x+ry) dy = \frac{r}{4\pi}\int_{|y|<1} (\Delta_x h)(x+ry) dy\\ &= \frac{1}{4\pi r^2}\Delta_x \int_{|y-x|<r} h(y) dy.\end{aligned}$$

On the other hand, using polar coordinates we have

$$\begin{aligned}\frac{1}{4\pi}\int_{|y-x|<r} h(y)dy = \frac{1}{4\pi}\int_{|y|<r} h(x+y)dy &= \frac{1}{4\pi}\int_0^r\int_{\mathcal{S}^2} h(x+\rho y)\rho^2 d\sigma(y) d\rho\\ &= \int_0^r \rho^2 A_\rho h(x) d\rho.\end{aligned}$$

The above two calculations show that

$$\partial_r(A_r h)(x) = \frac{1}{r^2}\Delta_x\int_0^r \rho^2 A_\rho h(x) d\rho,$$

from which we easily obtain that

$$\partial_r\left(r^2 \partial_r(A_r h)(x)\right) = \Delta_x r^2 A_r h(x).$$

If we set $H(r,x) := A_r h(x)$, then this shows that H solves Darboux's equation

$$\left(\partial_r^2 + \frac{2}{r}\partial_r\right) H(r,x) = \Delta_x H(r,x). \tag{1.4}$$

We notice that $H(0,x) = A_0 h(x) = h(x)$. Also, notice that $r \mapsto A_r h(x)$ is an even function. Since the derivative at zero of any differentiable even function is zero, we conclude that $\partial_r H(0,x) = 0$.

In the following, let us assume that u is a C^2-solution of (LW). We set

$$U(r,t,x) := (A_r u(\cdot,t))(x) = \frac{1}{4\pi}\int_{\mathcal{S}^2} u(x+ry,t) d\sigma(y).$$

Then, by (1.4), we have

$$\Delta_x U = \left(\partial_r^2 + \frac{2}{r}\partial_r\right) U = \frac{1}{r}\partial_r^2(rU).$$

On the other hand, since $\partial_t^2 u = \Delta_x u$, we have

$$\Delta_x U = \frac{1}{4\pi}\int_{\mathcal{S}^2}\Delta_x u(x+ry,t) d\sigma(y) = \frac{1}{4\pi}\int_{\mathcal{S}^2}\partial_t^2 u(x+ry,t) d\sigma(y) = \partial_t^2 U.$$

The last two calculations show that $\partial_t^2(rU) = \partial_r^2(rU)$. Therefore, if we set $v(r,t) := rU(t,r,x)$ (think of x as a parameter here), we have that v solves the one-dimensional wave equation:

$$\begin{cases}\partial_t^2 v = \partial_r^2 v\\ v(r,0) = rA_r f(x), \quad \partial_t v(r,0) = rA_r g(x).\end{cases}$$

By D'Alembert's formula, it follows that

$$v(r,t)=\frac{1}{2}\left[(r+t)A_{r+t}f(x)+(r-t)A_{r-t}f(x)\right]+\frac{1}{2}\int_{r-t}^{r+t}\rho A_\rho g(x)d\rho.$$

Using $U=\frac{v}{r}$ together with the fact that $r\mapsto A_rf$ and $r\mapsto A_rg$ are even functions, we deduce that

$$U(r,t,x)=\frac{1}{2r}\left[(t+r)A_{r+t}f(x)-(t-r)A_{t-r}f(x)\right]+\frac{1}{2r}\int_{t-r}^{t+r}\rho A_\rho g(x)d\rho.$$

Letting $r\to 0$ and recalling that $U(0,t,x)=(A_0u(t,\cdot))(x)=u(t,x)$, we finally obtain

$$u(x,t)=\partial_t(tA_tf(x))+tA_tg(x)=\frac{1}{4\pi}\int_{S^2}tg(x+ty)+f(x+ty)+\nabla_xf(x+ty)\cdot tyd\sigma(y)$$
$$=\frac{1}{4\pi t^2}\int_{|x-y|=t}tg(y)+f(y)-\nabla_yf(y)\cdot(x-y)d\sigma(y).$$

This shows that any C^2-solution of the Cauchy problem (1.1) must be given by the above formula and, in particular, it must be unique. Conversely, if $f\in C^3(\mathbb{R}^3)$ and $g\in C^2(\mathbb{R}^3)$, it can be easily verified that $u(t,x)$ given by the above formula solves (1.1). □

We can generalize the above strategy to obtain an explicit formula for solutions of (LW) on $\mathbb{R}^n$ for any n odd.

Proposition 1.3 (Explicit formula for the solution of (LW) on $\mathbb{R}^n$ with n odd). *Let n be odd, $f\in C^{\frac{n+3}{2}}(\mathbb{R}^n)$, and $g\in C^{\frac{n+1}{2}}(\mathbb{R}^n)$. Then, the initial value problem*

$$\begin{cases}\partial_t^2u-\Delta_xu=0, \\ u(x,0)=f(x),\quad \partial_tu(x,0)=g(x),\end{cases}\qquad x\in\mathbb{R}^n,\quad t\geq 0 \tag{1.5}$$

has a unique solution $u\in C^2(\mathbb{R}^n)$ given by the formula

$$u(x,t)=\frac{1}{1\cdot 3\cdot\dots\cdot(n-2)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}t^{n-2}A_tf(x)+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}t^{n-2}A_tg(x)\right]$$
$$=\frac{1}{1\cdot 3\cdot\dots\cdot(n-2)\omega_{n-1}}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\frac{1}{t}\int_{|x-y|=t}f(y)d\sigma(y)\right.$$
$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\frac{1}{t}\int_{|x-y|=t}g(y)d\sigma(y).\right] \tag{1.6}$$

Sketch of proof. If $n>3$ odd, we use a similar construction to that for $n=3$. More precisely, we consider the spherical means

$$A_rh(x):=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}h(x+ty)d\sigma(y),$$

where ω_{n-1} denotes the area of the unit sphere $S_{n-1}\subset\mathbb{R}^n$. If $n=2k+1$, we consider

$$v(r,t):=\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}r^{2k-1}A_ru(t,x) \tag{1.7}$$

and it turns out that v solves the linear wave equation on $\mathbb{R}$, $\partial_t^2 v - \partial_r^2 v = 0$. Also, the initial values of v are

$$v(r,0) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} A_r f(t,x) =: \phi(r), \quad \partial_t v(r,0) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} A_r g(t,x) =: \psi(r).$$

By D'Alembert's formula, it follows that

$$v(r,t) = \frac{1}{2}\left[\phi(r+t) - \phi(r-t)\right] + \frac{1}{2}\int_{r-t}^{r+t} \psi(s)ds. \tag{1.8}$$

One can show that there exist constants c_j with

$$c_0 = 1\cdot 3\cdot 5\cdot \dots \cdot (2k-1) = 1\cdot 3\cdot 5\cdot \dots \cdot (n-2)$$

such that

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\theta(r)) = \sum_{j=0}^{k-1} c_j r^{j+1}\frac{\partial^j}{\partial r^j}\theta(r).$$

Combining this with the definition (1.7) of v, it follows that

$$v(r,t) = \left(c_0 r + c_1 r^2\frac{\partial}{\partial r} + \dots c_{k-1} r^k \frac{\partial^k}{\partial r^k}\right) A_r u(t,x).$$

Therefore, the arguments for $n=3$ and (1.8) yield

$$u(x,t) = \lim_{r\to 0} A_r u(x,t) = \lim_{r\to 0}\frac{1}{c_0 r} v(r,t) = \frac{1}{c_0}\partial_r\phi\Big|_{r=t} + \frac{1}{c_0}\psi(t).$$

We then obtain formula (1.6) by plugging in the above equation the definitions of ϕ and ψ. □

Next, we obtain an explicit formula for solutions of (LW) on $\mathbb{R}^n$ for n even.

Proposition 1.4 (Explicit formula for the solution of (LW) on $\mathbb{R}^n$ with n even). *Let n be even, $f \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$, and $g \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$. Then, the initial value problem*

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \end{cases} \qquad x\in\mathbb{R}^n, \quad t\geq 0$$

has a unique solution $u \in C^2(\mathbb{R}^n)$ given by the formula

$$\begin{aligned} u(x,t) &= \frac{2}{1\cdot 3\cdot \dots \cdot (n-1)\omega_n}\Big[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|<1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}dy \\ &\qquad + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|<1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}dy\Big] \\ &= \frac{2}{1\cdot 3\cdot \dots \cdot (n-1)\omega_n}\Big[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\int_{|x-y|<t}\frac{f(y)}{\sqrt{t^2-|x-y|^2}}dy \\ &\qquad + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\int_{|x-y|<t}\frac{g(y)}{\sqrt{t^2-|x-y|^2}}dy\Big] \end{aligned} \tag{1.9}$$

Proof. We will use *Hadamard's method of descent.* The idea here is that if u solves the linear wave equation on $\mathbb{R}^n$, then it is also a solution of the linear wave equation on $\mathbb{R}^{n+1}$ which happens to be independent of the variable x_{n+1}. Therefore, if n is even, we can obtain a formula for $u(t,x)$ by using the formula (1.6) for the solution on $\mathbb{R}^{n+1}$ and then integrating out the redundant variable.

Let

$$\begin{aligned}\bar{u}(x_1,\dots,x_{n+1},t) &:= u(x_1,\dots,x_n,t),\\ \bar{f}(x_1,\dots,x_{n+1}) &:= f(x_1,\dots,x_n),\\ \bar{g}(x_1,\dots,x_{n+1}) &:= g(x_1,\dots,x_n).\end{aligned}$$

If u solves the Cauchy problem (1.5) on $\mathbb{R}^n$, then $\bar{u}$ solves the Cauchy problem for (LW) on $\mathbb{R}^{n+1}$ with initial data $\bar{f}$ and $\bar{g}$. By applying formula (1.6) on $\mathbb{R}^{n+1}$, it follows that

$$\begin{aligned}&\bar{u}(x,x_{n+1},t) =\\ &= \frac{1}{1\cdot 3\cdot\dots\cdot(n-1)\omega_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1}\bar{f}(x+ty,x_{n+1}+ty_{n+1})d\sigma(y,y_{n+1})\right.\\ &\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1}\bar{g}(x+ty,x_{n+1}+ty_{n+1})d\sigma(y,y_{n+1})\right],\end{aligned}$$

where we used the notations $(x_1,\dots,x_n,x_{n+1}) =: (x,x_{n+1})$ and $(y_1,\dots,y_n,y_{n+1}) =: (y,y_{n+1})$. In view of the definitions of $\bar{u},\bar{f},\bar{g}$, this can be rewritten as

$$\begin{aligned}u(x,t) =&\frac{1}{1\cdot 3\cdot\dots\cdot(n-1)\omega_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1}f(x+ty)d\sigma(y,y_{n+1})\right.\\ &\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1}g(x+ty)d\sigma(y,y_{n+1})\right].\end{aligned}\tag{1.10}$$

Next, we apply [2]

$$\int_{\substack{y_{n+1}=h(y)\\ y\in D}} F(y,y_{n+1})d\sigma(y,y_{n+1}) = \int_D F(y,h(y))\sqrt{1+|\nabla h(y)|^2}dy,$$

with the domain of integration being one of the two hemispheres of S^n (for which we have $h(y)=\pm\sqrt{1-|y|^2}$ and $D=\{|y|<1\}$) and $F(y,y_{n+1})=f(x+ty)$ or $F(y,y_{n+1})=g(x+ty)$. This yields

$$\begin{aligned}\int_{|y|^2+y_{n+1}^2=1} f(x+ty)d\sigma(y,y_{n+1}) &= \int_{|y|<1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}dy\\ \int_{|y|^2+y_{n+1}^2=1} g(x+ty)d\sigma(y,y_{n+1}) &= \int_{|y|<1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}dy.\end{aligned}$$

Combining these with (1.10) yields the desired formula (1.9). □

[2]This is simply a generalization of the formula for a surface integral over a surface S given explicitly by $S: z=h(x,y)$ with $(x,y)\in D\subset\mathbb{R}^2$. Namely, $\iint_S f(x,y,z)d\sigma = \iint_D f(x,y,h(x,y))\sqrt{1+(\partial_x h)^2+(\partial_y h)^2}dxdy$.

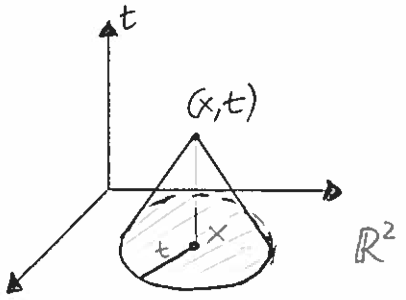
Remark 1.5. In the case $n = 2$, formula (1.9) simplifies to

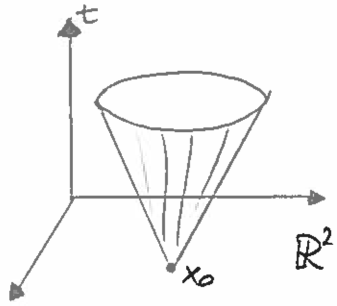
$$u(x,t) = \frac{2}{4\pi}\int_{|y|<1} \frac{tg(x+ty) + f(x+ty) + \nabla_x f(x+ty)\cdot ty}{\sqrt{1-|y|^2}}dy$$
$$= \frac{1}{2\pi}\int_{|x-y|<t} \frac{tg(y) + f(y) - \nabla_x f(y)\cdot(x-y)}{t\sqrt{t^2-|x-y|^2}}dy. \tag{1.11}$$

This is called *Poisson's formula* for the solution of the linear wave equation on $\mathbb{R}^2$.

From formulas (1.6) and (1.9), we deduce the following:

(1) The domain of dependence of a solution u of (LW) at (x,t) on the initial data f and g is contained in the ball centered at x of radius t.
(2) (Finite speed of propagation) The domain of influence of a point x_0 on the solution is contained in the "light cone" with vertex $(x_0, 0)$: $\{(x,t) : |x - x_0| \leq t,\ t \geq 0\}$.
(3) (*Strong Huygens Principle*) For n odd, the domain of influence of a point x_0 on the solution is the *boundary* $\{(x,t) : |x - x_0| = t,\ t \geq 0\}$ of the "light cone" $\{(x,t) : |x - x_0| \leq t,\ t \geq 0\}$.





Moreover, from Propositions 1.3 and 1.4, we obtain the following theorem.

Theorem 1.6. *Let $k = 2, 3, \ldots$, $f \in C^{[n/2]+k}(\mathbb{R}^n)$, and $g \in C^{[n/2]+k-1}(\mathbb{R}^n)$. Then, the Cauchy problem* (1.5) *has a unique solution $u \in C^k(\mathbb{R}^n)$, given by formulas* (1.6) *or* (1.9) *depending on whether u is odd or even.*

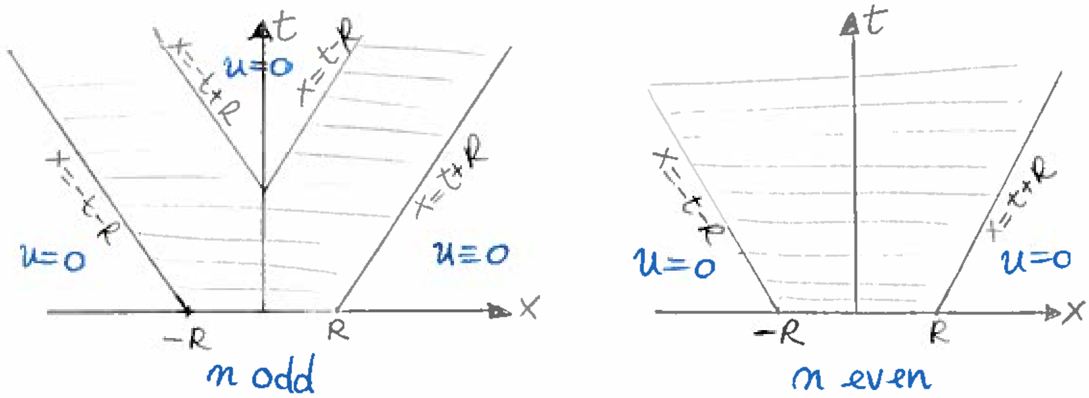
If n is odd and f, g are supported on $\{x : |x| < R\}$, then u is supported on $||x| - t| < R$. Moreover, there exists a constant $C > 0$ such that

$$|u(x,t)| \leq \frac{C}{\langle t\rangle^{\frac{n-1}{2}}}. \tag{1.12}$$

If n is even and f, g are supported on $\{x : |x| < R\}$, then u is supported on $|x| \leq t + R$. Moreover, there exists a constant $C > 0$ such that

$$|u(x,t)| \leq \frac{C}{\langle t\rangle^{\frac{n-1}{2}}\langle |x| - t\rangle^{\frac{n-1}{2}}}. \tag{1.13}$$

Here, $[n/2]$ *denotes the integer part of* $n/2$ *and* $\langle t \rangle := \sqrt{t^2+1}$.



Sketch of proof. The proof follows from formulas (1.6) and (1.9).

For n odd and f, g supported on $\{x : |x| < R\}$, we obtain the solution u in (1.6) by integrating in y over $|x-y| = t$, with the additional constraint $|y| < R$ (this is clear in Kirchhoff's formula (1.2)). Then, u is supported on

$$||x| - t| = ||x| - |y-x|| < |x + (y-x)| = |y| < R.$$

The decay (1.12) also follows from formula (1.6). In particular, for $n = 3$, we see immediately from Kirchhoff's formula (1.2) that $|u(t,x)| \le \frac{C}{\langle t \rangle}$.

For n even and f, g supported on $\{x : |x| < R\}$, we obtain the solution u in (1.6) by integrating in y over $|x-y| < t$, with the additional constraint $|y| < R$ (this is clear in Poisson's formula (1.11)). Then, u is supported on

$$|x| \le |x-y| + |y| < t + R.$$

The decay (1.13) is more subtle then (1.12), but it does follow from formula (1.9) [Exercise]. □

Remark 1.7. Even though for n even we don't have a strong Huygens principle, the decay estimate (1.13) compensates for it. More precisely, it says that for compactly supported initial data, u decays more and more rapidly as one goes away from the light cone $\{(x,t) : |x| = t,\ t \ge 0\}$.

Both decay estimates (1.12) and (1.13) can also be derived from the Klainerman-Sobolev inequality that we'll see later in the course, without using formulas (1.6) and (1.9).

2. Cauchy-Kowalevski theorem

The Cauchy-Kowalevski theorem concerns the existence and uniqueness of a real analytic solution of a Cauchy problem for the case of real analytic data and equations.

We first recall the definition of a real analytic function.

Definition 2.1. A function $f:\mathbb{R}^n\to\mathbb{R}$ is called *real analytic* near x_0 if there exists $r>0$ and constants f_α such that

$$f(x)=\sum_\alpha f_\alpha(x-x_0)^\alpha \quad \text{for} \quad |x-x_0|<r,$$

where the sum is taken over all $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and we used the notation $x^\alpha := x_1^{\alpha_1}\dots x_n^{\alpha_n}$.

Theorem 2.2 (Cauchy-Kowalevski theorem). *Let $k\geq 0$. Let $g_j:\mathbb{R}^n\to\mathbb{R}$, $j=0,\dots,k-1$, be real analytic functions on a neighborhood of $\bar{x}\in\mathbb{R}^n$, and G be real analytic on a neighborhood of $\big(\bar{x},0,(D_x^\alpha g_j(\bar{x}))_{j+|\alpha|\leq k, j\leq k-1}\big)$.*

Then, the Cauchy problem

$$\begin{cases}\partial_t^k u=G\big(x,t,(D_x^\alpha\partial_t^j u)_{j+|\alpha|\leq k, 0\leq j\leq k-1}\big)\\ \partial_t^j u(x,0)=g_j(x), \quad j=0,\dots,k-1 \quad \textit{near} \quad x=\bar{x}\end{cases} \tag{2.1}$$

admits a real analytic solution u defined on a neighborhood of $(\bar{x},0)\in\mathbb{R}_x^n\times\mathbb{R}_t$. This solution is unique in the class of real analytic solutions.

Sketch of proof. Step 1: *Transformation to zero Cauchy data and formulation as a first order system* for the new unknown:

$$\underline{u}=(u,\partial_{x_1}u,\dots,\partial_{x_n}u,\partial_t u,\dots).$$

with components $(\underline{u}^k)_{k=1}^m$.

Step 2: *Computation of the Taylor series near the origin.*

Since we are looking for a real analytic solution, we expect that $\underline{u}$ has the power series expansion

$$\underline{u}^k(x,t)=\sum_\alpha\frac{D^\alpha\underline{u}^k(0,0)}{\alpha!}x^{\alpha'}t^{\alpha_t}, \tag{2.2}$$

where $\alpha=(\alpha',\alpha_t)$.

Step 3: *Convergence of the power series* (2.2).

In this step we intend to use the *method of majorants* to show that the power series (2.2) converges for $|x|+|t|<r$ and r sufficiently small. Once we do this, the existence of a real analytic solution $\underline{u}$ of our first order system follows. Indeed, by construction we have that the power series at $(0,0)$ of the left and right hand-side of the system agree, and now they also converge. Hence, the left and right hand-side of the system agree on a neighborhood of $(0,0)$, or in other words $\underline{u}$ is a real analytic solution in a neighborhood of $(0,0)$.

□

We will give the details of the above proof in Lecture 4. We will also discuss the different settings in which the Cauchy-Kowalevski theorem can be applied.