ADVANCED PDE II - LECTURE 4

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Warning: This is a first draft of the lecture notes and should be used with care![1]

1. Cauchy-Kowalevski theorem

The Cauchy-Kowalevski theorem concerns the existence and uniqueness of a real analytic solution of a Cauchy problem for the case of real analytic data and equations.

1.1. Preliminaries.

We recall the definition of a real analytic function.

Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *real analytic* near x_0 if there exists $r > 0$ and constants f_α such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \quad \text{for} \quad |x - x_0| < r,$$

where the sum is taken over all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and we used the notation $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Note that if f is real analytic near x_0, then f is C^∞ near x_0 and, moreover, $f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}$. Here we are using the notations $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ and $\alpha! = \alpha_1! \dots \alpha_n!$.

Example: For $r > 0$, we consider

$$f(x) := \frac{r}{r - (x_1 + \dots x_n)} \quad \text{for} \quad |x| < \frac{r}{\sqrt{n}}.$$

Then, f is real analytic near zero and moreover

$$f(x) = \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} x^\alpha \quad \text{for} \quad |x| < \frac{r}{\sqrt{n}}.$$

To prove this, we first notice by Cauchy-Schwarz inequality that $|x| < \frac{r}{\sqrt{n}}$ yields

$$|x_1 + \dots + x_n| \leq \sqrt{n}\sqrt{x_1^2 + \dots x_n^2} = \sqrt{n}|x| < r.$$

Then, for $|x| < \frac{r}{\sqrt{n}}$ we have that

$$f(x) = \frac{1}{1 - \frac{x_1 + \dots + x_n}{r}} = \sum_{k=0}^\infty \left(\frac{x_1 + \dots + x_n}{r} \right)^k = \sum_{k=0}^\infty \frac{1}{r^k} \sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!} x^\alpha = \sum_\alpha \frac{|\alpha|!}{r^{|\alpha|}\alpha!} x^\alpha.$$

Before discussing the Cauchy-Kowalevski theorem, we first introduce some preliminary notions and results.

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[1] Lecture 4 was inspired by Section 4.6 in Evans' book, by Chapter 1, Section D in Folland's PDE book, by Gustav Holzegel's lecture notes (week 3), and by Jonathan Luk's lecture notes (Section 2).

Definition 1.2. Let $f = \sum_\alpha f_\alpha x^\alpha$ and $g = \sum_\alpha g_\alpha x^\alpha$ be two power series. We say that g *majorizes* f ($g \gg f$) if $g_\alpha \geq |f_\alpha|$ for all α.

Lemma 1.3. If $g \gg f$ and g converges for $|x| < r$, then f also converges for $|x| < r$.

Lemma 1.4. If $f = \sum_\alpha f_\alpha x^\alpha$ converges for $|x| < r$ and if s is such that $0 < s\sqrt{n} < r$, then there exists $C > 0$ such that

$$g(x) := \frac{Cs}{s - (x_1 + \dots x_n)} = \sum_\alpha \frac{C|\alpha|!}{s^{|\alpha|}\alpha!} x^\alpha$$

majorizes f for $|x| < s\sqrt{n}$.

Proof. Let $y = s(1, 1, \dots, 1)$. Then $|y| = s\sqrt{n} < r$. By the hypothesis, it follows that $\sum_\alpha f_\alpha y^\alpha$ converges. Thus, there exists $C > 0$ such that $|f_\alpha y^\alpha| < C$ for all α. In particular, $|f_\alpha| \leq \frac{C}{s^{|\alpha|}} \leq \frac{C|\alpha|!}{s^{|\alpha|}\alpha!}$. This shows that indeed $g \gg f$. $\square$

1.2. **The general setting.** We consider the following k-th order fully nonlinear equation on $\mathbb{R}^{n+1}$:

$$F(x, (D^\alpha u)_{|\alpha| \leq k}) = 0, \tag{1.1}$$

where F is real analytic. Let Γ be a hypersurface of class C^k. Fix $x^0 \in \Gamma$. We denote by $\nu = (\nu_1, \dots, \nu_{n+1})$ the unit normal to Γ at x^0.

Definition 1.5. *Given u, the j-th normal derivative of u at x^0 is defined by*

$$\partial_\nu^j u := \sum_{|\alpha| = j} \nu^\alpha D^\alpha u = \sum_{\alpha_1 + \dots + \alpha_n = j} \nu_1^{\alpha_1} \dots \nu_{n+1}^{\alpha_{n+1}} \frac{\partial^j u}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_{n+1}}^{\alpha_{n+1}}}.$$

The *Cauchy problem* for (1.1) consists in solving (1.1) with prescribed data on Γ:

$$\partial_\nu^j u(x, 0) = g_j(x), \quad j = 0, \dots, k-1, \quad \text{for} \quad x \in \Gamma.$$

All our considerations will be restricted to a neighborhood of a given point x^0 on Γ. Using a change of coordinates, we may assume that x^0 is the origin and that near the origin Γ coincides with the hypersurface $x_{n+1} = 0$. (This procedure is called flattening of the boundary. See Appendix C.1 in Evans' book for details.) It will then be convenient to make the change of notations $(x_1, \dots, x_{n+1}) =: (x, t)$, where $x := (x_1, \dots, x_n) \in \mathbb{R}^n$. Then, $\partial_\nu^j u = \partial_t^j u$ and we can restate the Cauchy problem for (1.1) as

$$\begin{cases} F\big(x, t, (D^\alpha \partial_t^j u)_{|\alpha| + j \leq k}\big) = 0, \\ \partial_t^j u(x, 0) = g_j(x), \quad j = 0, \dots, k-1, \quad |x| < r, \end{cases} \tag{1.2}$$

for some small r.

We observe that if u is a function of class C^r with $r \geq k$, then the Cauchy data $(g_j)_{j=0}^{k-1}$ determine the derivatives of u on Γ:

$$D^\alpha \partial_t^j u(x, 0) = D^\alpha g_j(x)$$

with $j \leq k-1$ and $|\alpha|+j \leq r$. The only quantity in (1.2) which is unknown is $\partial_t^k u(0,0)$. In order for the Cauchy problem to be well-behaved, we need to assume that the equation $F=0$ can be solved for $\partial_t^k u$. In the linear case

$$F\big(x,t,(D^\alpha \partial_t^j u)_{|\alpha|+j\leq k}\big) = \sum_{|\alpha|+j\leq k} a_{\alpha j}(x,t) D_x^\alpha \partial_j^t u - f(x,t),$$

this assumption just means that

$$a_{0k}(x,0) \neq 0, \tag{1.3}$$

and hence by continuity $a_{0k}(x,t) \neq 0$ for small t.

In the quasi-linear case

$$F\big(x,t,(D^\alpha \partial_t^j u)_{|\alpha|+j\leq k}\big) = \sum_{|\alpha|+j\leq k} a_{\alpha j}\big(x,t,(D_x^\beta \partial_t^i u)_{|\beta|+i\leq k-1}\big) D_x^\alpha \partial_j^t u - b\big(x,t,(D_x^\beta \partial_t^i u)_{|\beta|+i\leq k-1}\big),$$

this assumption means that

$$a_{0k}\big(x,0,(D_x^\beta g_i(x))_{|\beta|+i\leq k-1}\big) \neq 0. \tag{1.4}$$

In the fully nonlinear case, the equation

$$F\big(x,0,\partial_t^k u(x,0),(D^\alpha \phi_j(x))_{|\alpha|+j\leq k,\, j\leq k-1}\big) = 0,$$

can be locally and uniquely solved for $\partial_t^k u(x,0)$ (using the implicit function theorem) provided that

$$F\big(0,0,\gamma,(D^\alpha \phi_j(0))_{|\alpha|+j\leq k,\, j\leq k-1}\big) = 0, \tag{1.5}$$

$$\frac{\partial F}{\partial s}\big(0,0,s,(D^\alpha \phi_j(0))_{|\alpha|+j\leq k,\, j\leq k-1}\big)\Big|_{s=\gamma} \neq 0 \tag{1.6}$$

for some $\gamma \in \mathbb{R}$. Then, we can locally write our equation as

$$\partial_t^k u = G\big(x,t,(D^\alpha \partial_t^j u)_{|\alpha|+j\leq k,\, j\leq k-1}\big), \tag{1.7}$$

where G is an analytic function.

If conditions (1.3), (1.4), or (1.5) hold, then we say that the Cauchy problem for (1.1) is *non-characteristic*. In this case, if the Cauchy data $(g_j)_{j=0}^{k-1}$ are real analytic, then $(g_j)_{j=0}^{k-1}$ together with (1.7) determine all the derivatives of $D^\alpha \partial_t^j u(x,0)$, for all α, j.

1.3. **The Cauchy-Kowalevski theorem.**

Theorem 1.6 (Cauchy-Kowalevski theorem)**.** *Let $k \geq 0$. Let $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 0,\dots,k-1$, be real analytic functions on a neighborhood of $\bar{x} \in \mathbb{R}^n$, and G be real analytic on a neighborhood of $\big(\bar{x},0,(D_x^\alpha g_j(\bar{x}))_{j+|\alpha|\leq k,\, j\leq k-1}\big)$.*

Then, the Cauchy problem

$$\begin{cases} \partial_t^k u = G\big(x,t,(D_x^\alpha \partial_t^j u)_{j+|\alpha|\leq k,\, 0\leq j\leq k-1}\big) \\ \partial_t^j u(x,0) = g_j(x), \quad j = 0,\dots,k-1 \quad \textit{near} \quad x = \bar{x} \end{cases} \tag{1.8}$$

admits a real analytic solution u defined on a neighborhood of $(\bar{x},0) \in \mathbb{R}_x^n \times \mathbb{R}_t$. This solution is unique in the class of real analytic solutions.

Sketch of proof. By a translation, we easily reduce to the case $\bar{x} = 0$ (which coincides with the setting discuss above). We will prove the theorem for the case of a second order quasi-linear equation of the form

$$\begin{cases} \partial_t^2 u = \sum_{|\alpha|=2,\, \alpha_t \leq 1} G_\alpha(t, x, u, \partial_t u, \partial_{x_1} u, \dots, \partial_{x_n} u) D^\alpha u + \tilde{G}(t, x, u, Du) \\ u(x, 0) = g_0(x), \quad \partial_t u(x, 0) = g_1(x) \quad \text{for} \quad |x| < r, \end{cases} \tag{1.9}$$

with $\alpha = (\alpha', \alpha_t) = (\alpha_1, \dots, \alpha_n, \alpha_t)$, $G_\alpha, \tilde{G}, g_0, g_1$ real analytic, r sufficiently small. The general case (1.8) can be treated similarly, but we omit it here because of the more involved notations. See Chapter 1.D in Folland's book for a proof of Theorem 1.6 in the general case.

<u>Step 1</u>: *Transformation to zero Cauchy data and formulation as a first order system.*

First, we notice that $\tilde{u} = u - g_0(x) - t g_1(x)$ satisfies a Cauchy problem of the form (1.9) (but with different $G_\alpha, \tilde{G}$) and with zero initial data. Therefore, in the following we may assume the initial data are identically zero.

Next, we write (1.9) as a system of first order equations. We set

$$\underline{u} = (u, \partial_{x_1} u, \dots, \partial_{x_n} u, \partial_t u). \tag{1.10}$$

This is a vector with $m = n + 2$ components satisfying $\underline{u}(x, t = 0) = 0$. We notice that the first $m - 1$ components of the vector $\partial_t \underline{u}$, $\partial_t u$, $\partial_t \partial_{x_i} u$, $i = 1, \dots, n$ are determined by the the vectors $\{\partial_{x_i} \underline{u}\}_{i=1}^n$. The last component $\partial_t^2 u$ of $\partial_t \underline{u}$ is determined by the same vectors in view of the PDE (1.9). More precisely, we obtain the first order system

$$\begin{cases} \partial_t \underline{u} = \sum_{j=1}^n \underline{B}_j(\underline{u}, x) \partial_{x_j} \underline{u} + \underline{c}(\underline{u}, x) \\ \underline{u}(x, 0) = \underline{0} \quad \text{for} \quad |x| < r, \end{cases} \tag{1.11}$$

where $\underline{B}_j : \mathbb{R}^m \times \mathbb{R}^n \to \mathrm{Mat}(m \times m)$, $j = 1, \dots, n$, and $\underline{c} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. We denote the components of $\underline{B}_j$ by $(b_j^{k\ell})_{k,\ell=1}^m$ and $\underline{c} = (c^1, \dots, c^m)$. In components, the system (1.11) reads

$$\begin{cases} \partial_t \underline{u}^k = \sum_{j=1}^n \sum_{\ell=1}^m b_j^{k\ell}(\underline{u}, x) \partial_{x_j} \underline{u}^\ell + c^k(\underline{u}, x), \\ \underline{u}^k(x, 0) = 0, \end{cases} \qquad k = 1, \dots, m, \quad |x| < r, \tag{1.12}$$

where we denoted the components of $\underline{u}$ by $(\underline{u}^k)_{k=1}^m$.

Notice that we assumed above that $\underline{B}_j$ and $\underline{c}$ do not depend on t. This can always be achieved by adding an additional component $\underline{u}^{m+1}$ to the vector $\underline{u}$ and by imposing the additional equation $\partial_t \underline{u}^{m+1} = 1$.

Notice that here we were not precise about the matrices $\underline{B}_j$ and the vector $\underline{c}$. For a rigorous proof, one would need to write these explicitly and show that if a vector $\underline{u}$ is a solution of the corresponding system involving precisely these $\underline{B}_j$ and $\underline{c}$, then the first component u of $\underline{u}$ indeed satisfies equation (1.9).

<u>Step 2</u>: *Computation of the Taylor series near the origin.*

Since we are looking for a real analytic solution of (1.9), we expect that $\underline{u}$ has the power series expansion

$$\underline{u}^k(x,t) = \sum_{\alpha} \frac{D^\alpha \underline{u}^k(0,0)}{\alpha!} x^{\alpha'} t^{\alpha_t}, \tag{1.13}$$

where $\alpha = (\alpha', \alpha_t)$, for . In the following, we compute the coefficients $D^\alpha \underline{u}^k(0,0)$. By the analyticity of $\underline{B}_j$ and $\underline{c}$, we have that

$$\begin{aligned} \underline{B}_j(z,x) &= \sum_{\gamma,\delta} \underline{B}_{j,\gamma,\delta} z^\gamma x^\delta, \quad j = 1, \ldots, n, \\ \underline{c}(z,x) &= \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma x^\delta \end{aligned}$$

for $|z| + |x| < s$ and s small, where

$$\underline{B}_{j,\gamma,\delta} = \frac{D_z^\gamma D_x^\delta \underline{B}_j(0,0)}{(\gamma+\delta)!}, \quad \underline{c}_{\gamma,\delta} = \frac{D_z^\gamma D_x^\delta \underline{c}(0,0)}{(\gamma+\delta)!}.$$

For $\alpha = (\alpha', 0)$, we have $D^\alpha \underline{u}^k(0,0) = 0$ for all $k = 1, \ldots, m$. This follows by taking the D^α derivative of the initial conditions $\underline{u}^k(x,0) = 0$.

For $\alpha = (\alpha', 1)$, we have $D^\alpha \underline{u}^k(0,0) = D_x^{\alpha'} c^k(0,0)$. This follows by taking the $D^{\alpha'}$ derivative of both sides of (1.12) and by using $D^{(\alpha',0)} \underline{u}^k(0,0) = 0$ for all α' and $k = 1, \ldots, m$.

For $\alpha = (\alpha', 2)$, we take the $D^{\alpha'} \partial_t$ derivative of both sides of (1.12):

$$D^\alpha \underline{u}^k(0,0) = D^{\alpha'} \left(\sum_{j=1}^n \sum_{\ell=1}^m b_j^{k\ell}(\underline{u}, x) \partial_{x_j} \partial_t \underline{u}^\ell + \sum_{j=1}^m \partial_{z_j} c^k \partial_t \underline{u}^j \right) \Bigg|_{(\underline{u},x)=(0,0)}.$$

The right hand side is a polynomial in the derivatives $D^\beta \underline{u}(0,0)$ with $\beta = (\beta', \beta_t)$, $\beta_t \leq 1$ and $|\beta'| \leq |\alpha'| + 1$, and in finitely many Taylor coefficients $\underline{B}_{j,\gamma,\delta}$ and $\underline{c}_{\gamma,\delta}$. Moreover, this polynomial has only positive integer coefficients since this is all the chain rule and the product rule can produce.

Continuing in this way, we see that for general $\alpha = (\alpha', \alpha_t)$, we have

$$D^\alpha \underline{u}^k(0,0) = q_\alpha^k(\ldots, \underline{B}_{j,\gamma,\delta}, \ldots, \underline{c}_{\gamma,\delta}, \ldots D^\beta \underline{u}(0,0)), \tag{1.14}$$

where q_α^k is a polynomial with nonnegative coefficients, $\beta_t \leq \alpha_t - 1$, and $|\beta'| \leq |\alpha'| + 1$.

In conclusion, we obtain by induction that all the coefficients $D^\alpha \underline{u}^k(0,0)$ can be determined in terms of $B_{j,\gamma,\delta}$ and $c_{\gamma,\delta}$ alone.

Step 3: *Convergence of the power series* (1.13).

In this step we intend to use the *method of majorants* to show that the power series (1.13) converges for $|x| + |t| < r$ and r sufficiently small. Once we do this, the existence of a real analytic solution $\underline{u}$ of (1.11) follows. Indeed, by construction we have that the power series at $(0,0)$ of the left and right hand-side of (1.11) agree, and now they also converge. Hence, the left and right hand-side of (1.11) agree on a neighborhood of $(0,0)$, or in other words $\underline{u}$ is a real analytic solution of (1.11) in a neighborhood of $(0,0)$.

Suppose that we can find majorizing $\underline{B}_j^* \gg \underline{B}_j$ and $\underline{c}^* \gg \underline{c}$, that is

$$0 \leq |\underline{B}_{j,\gamma,\delta}| \leq \underline{B}^*_{j,\gamma,\delta}, \qquad 0 \leq |\underline{c}_{\gamma,\delta}| \leq c_{\gamma,\delta}.$$

Given these majorants, we consider the system

$$\begin{cases} \partial_t \underline{u}^* = \sum_{j=1}^n \underline{B}_j^*(\underline{u}^*, x)\partial_{x_j}\underline{u}^* + \underline{c}^*(\underline{u}^*, x) & \text{for} \quad |x| + |t| < r \\ \underline{u}^*(x, 0) = \underline{0} \quad \text{for} \quad |x| < r. \end{cases} \tag{1.15}$$

If we can find a convergent power series for $\underline{u}^*$ which solves (1.15):

$$(\underline{u}^*)^k(x,t) = \sum_\alpha \frac{D^\alpha(\underline{u}^*)^k(0,0)}{\alpha!} x^{\alpha'} t^{\alpha_t}, \tag{1.16}$$

then we claim that $\underline{u}^* \gg \underline{u}$. This follows by induction. By (1.14), the general step of the induction is:

$$\begin{aligned} |D^\alpha \underline{u}^k(0,0)| &= |q^k_\alpha(\dots, \underline{B}_{j,\gamma,\delta}, \dots, \underline{c}_{\gamma,\delta}, \dots D^\beta \underline{u}(0,0))| \\ &\leq q^k_\alpha(\dots, |\underline{B}_{j,\gamma,\delta}|, \dots, |\underline{c}_{\gamma,\delta}|, \dots |D^\beta \underline{u}(0,0)|) \\ &\leq q^k_\alpha(\dots, \underline{B}^*_{j,\gamma,\delta}, \dots, \underline{c}^*_{\gamma,\delta}, \dots D^\beta \underline{u}^*(0,0)) = D^\alpha(\underline{u}^*)^k(0,0). \end{aligned}$$

Since we assumed that the power series of $\underline{u}^*$ is convergent for $|x|+|t| < r$ and since $\underline{u}^* \gg \underline{u}$, it follows by Lemma 1.3 that the power series (1.13) of $\underline{u}$ also converges for $|x| + |t| < r$.

In conclusion, by the method of majorants, it suffices to find majorants $\underline{B}_j^*$ and $\underline{c}^*$ such that (1.15) has a real analytic solution $\underline{u}^*$ near $(0,0)$. By Lemma 1.4, we have the following simple majorants for $\underline{B}_j$ and $\underline{c}$ for $|x| + |z| < r$ with r sufficiently small:

$$\underline{B}_j^* = \frac{Cr}{r - (x_1 + \dots + x_n) - (z_1 + \dots + z_m)} \begin{pmatrix} 1 & 1 & \dots \\ 1 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\underline{c}^* = \frac{Cr}{r - (x_1 + \dots + x_n) - (z_1 + \dots + z_m)} (1, \dots, 1).$$

The system (1.15) then becomes:

$$\begin{cases} \partial_t (\underline{u}^*)^k = \frac{Cr}{r - (x_1 + \dots + x_n) - ((\underline{u}^*)^1 + \dots + (\underline{u}^*)^m)} \left(\sum_{j,\ell} \partial_{x_j} (u^*)^\ell + 1 \right) & \text{for} \quad |x| + |t| < r, \quad k = 1, \dots, m, \\ \underline{u}^*(x,0) = \underline{0} \quad \text{for} \quad |x| < r. \end{cases}$$

The right hand side being independent of k, this suggests considering $\underline{u}^* = v^*(1, \dots, 1)$ with v^* satisfying

$$\partial_t v^* = \frac{Cr}{r - (x_1 + \dots + x_n) - mv^*} \left(m \sum_{j=1}^n \partial_{x_j} v^* + 1 \right).$$

The ansatz $v^*(x_1, \dots, x_n, t) = v^*(s := x_1 + \dots + x_n, t)$ leads to the equation

$$\partial_t v^* = \frac{Cr}{r - s - mv^*} (mn \partial_s v^* + 1)$$

with initial condition $v^*(s,0) = 0$. Using the method of characteristics, one can solve this (exercise) and obtains

$$v^*(x,t) = \frac{r - (x_1 + \cdots + x_n) - \sqrt{\big(r - (x_1 + \cdots + x_n)\big)^2 - 2m(n+1)Crt}}{m(n+1)},$$

which is analytic for $|x| + |t| < \tilde{r}$ and $\tilde{r}$ sufficiently small. □

1.4. **Remarks on the Cauchy-Kowalevski theorem.** The Cauchy-Kowalevski theorem gives local existence of a unique analytic solution near one point of the hypersurface Γ. However, given analytic Cauchy data on an analytic hypersurface Γ, there is an analytic solution near any point of Γ, and by uniqueness any two of these solutions must agree on their common domain. Hence, we can patch them together and obtain a solution near a neighborhood of Γ.

Many physical problems lead to analytic PDEs. The restriction to analytic Cauchy data and solutions, however, is unrealistic. (It would imply, in particular, that a solution is determined globally by local conditions near one point.)

The Cauchy-Kowalevski theorem is local in character and applies only to analytic solutions of analytic Cauchy problems. In particular, it does not give any information on global existence of solutions, it does not exclude the possibility that other non-analytic solutions exist, nor the possibility that an analytic solution becomes non-analytic away from the initial hypersurface.

An important application of the Cauchy-Kowalevski theorem is Holmgren's uniqueness theorem. It asserts the uniqueness of solutions of class C^k for *linear* equations with analytic coefficients. In particular, in the linear case, analytic Cauchy problems do not admit other non-analytic solutions.

Theorem 1.7 (Holmgren's uniqueness theorem)**.** *Let $k \geq 0$ and let $P = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ be a k^{th}-order differential operator with analytic coefficients in a neighborhood of $\bar{x} \in \mathbb{R}^n$. Let Γ be an analytic hypersurface which is non-characteristic at $\bar{x}$. If u is a solution of class C^k of the following Cauchy problem in a neighborhood of $\bar{x}$:*

$$\begin{cases} Pu = 0 \\ D^\alpha u = 0, \quad |\alpha| \leq k-1, \quad \text{on} \quad \Gamma, \end{cases}$$

then $u \equiv 0$ on a neighborhood of $\bar{x}$.

2. The Fourier transform and the linear wave equation

Definition 2.1. For $f \in L^1(\mathbb{R}^n)$, we define it's *Fourier transform* by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \tag{2.1}$$

and its inverse Fourier transform by

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi = \frac{1}{(2\pi)^n}\mathcal{F}(f)(-x). \tag{2.2}$$

Theorem 2.2 (Properties of the Fourier transform)**.**
(1) (Fourier inversion formula) Let f be such that $f, \hat{f} \in L^1(\mathbb{R}^n)$. Then

$$\mathcal{F}^{-1}(\hat{f}) = f \tag{2.3}$$

almost everywhere.

(2) Let $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in L^\infty(\mathbb{R}^n)$ and, moreover, we have

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.4}$$

(3) (Plancherel's identity) Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{f} \in L^2(\mathbb{R}^n)$ and

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{\frac{n}{2}}} \|\hat{f}\|_{L^2}. \tag{2.5}$$

(This identity allows one to define the Fourier transform for functions $f \in L^2(\mathbb{R}^n)$. More precisely, one considers $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^2(\mathbb{R}^n)$, and defines $\hat{f}$ as the limit $\hat{f}_n \to \hat{f}$ in $L^2(\mathbb{R}^n)$.)

In the following, let $f, g \in L^2(\mathbb{R}^n)$.

(4) (Parseval's identity)

$$\int_{\mathbb{R}^n} f\overline{g}dx = \int_{\mathbb{R}^n} \widehat{f}\widehat{\overline{g}}d\xi. \tag{2.6}$$

(5) Let $f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ denote the *convolution* of f and g. Then,

$$\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2.7}$$

(6)

$$\mathcal{F}(fg)(\xi) = (\hat{f} * \hat{g})(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2.8}$$

(7) Let α be a multiindex such that $\partial^\alpha f \in L^2(\mathbb{R}^n)$. Then

$$\mathcal{F}(\partial^\alpha f)(\xi) = (i\xi)^\alpha \widehat{f}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2.9}$$

(8) Let α be a multiindex such that $x^\alpha f \in L^2(\mathbb{R}^n)$. Then

$$\mathcal{F}(x^\alpha f)(\xi) = i^\alpha \partial_\xi^\alpha \widehat{f}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{2.10}$$

(9) Let $S(\mathbb{R}^n)$ be the *Schwarz space* of rapidly decreasing functions defined by

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}\}. \tag{2.11}$$

Then the Fourier transform maps the Schwarz space into itself.

(10) (Translation) Let $g(x) = f(x-a)$ with $a \in \mathbb{R}^n$. Then,

$$\hat{g}(\xi) = e^{-ia\cdot\xi}\hat{f}(\xi). \tag{2.12}$$

(11) (Modulation) Let $g(x) = e^{ia\cdot x}f(x)$ with $a \in \mathbb{R}^n$. Then,

$$\hat{g}(\xi) = \hat{f}(\xi - a). \tag{2.13}$$

(12) (Scaling/Dilation) Let $g(x) = f(ax)$ with $a > 0$. Then,

$$\hat{g}(\xi) = \frac{1}{a^n}\hat{f}(\frac{\xi}{a}). \tag{2.14}$$

Proof. For a proof of the above properties of the Fourier transform, see Section 2.2 in the book of Grafakos entitled "Classical Fourier Analysis". Alternatively, see Chapter I in the book of Stein and Weiss entitled "Introduction to Fourier Analysis on Euclidean Spaces". See also Section 4.3.1 in Evans' book. □

We now return to the following initial value problem for the linear wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \end{cases} \qquad x \in \mathbb{R}^n, \quad t \geq 0.$$

By the property (2.9) of the Fourier transform, we remark that

$$\mathcal{F}(\Delta u) = \mathcal{F}(\sum_{i=1}^{n} \partial_{x_i}^2 u) = \sum_{i=1}^{n}(i\xi_i)^2 \hat{u}(\xi) = -|\xi|^2\hat{u}(\xi).$$

Therefore, applying the Fourier transform to both sides of the linear wave equation, we obtain the ODE:

$$\partial_t^2 \hat{u}(\xi) + |\xi|^2\hat{u}(\xi) = 0. \tag{2.15}$$

The characteristic equation associated to this is $\lambda^2 + |\xi|^2 = 0$ with solutions $\lambda = \pm i|\xi|$. Therefore, the general solution of this ODE is

$$\hat{u}(\xi, t) = A(\xi)\cos(t|\xi|) + B(\xi)\sin(t|\xi|). \tag{2.16}$$

Using the initial conditions, we have that

$$\hat{f}(\xi) = \hat{u}(\xi, 0) = A(\xi)$$
$$\hat{g}(\xi) = \partial_t \hat{u}(\xi, 0) = B(\xi)|\xi|.$$

Thus, we obtain the following formula for the solution of the linear wave equation

$$\hat{u}(\xi, t) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi). \tag{2.17}$$

Remark 2.3. (i) Taking the inverse Fourier transform of both sides of (2.17), we recover the explicit formulas for a solution of the linear wave equation (D'Alembert, Poisson, Kirchhoff, etc.) that we obtained earlier.

(ii) Even though formula (2.17) is not explicit on the physical side, it easily gives us information on the Sobolev norms of a solution u. For example, we can easily use (2.17) together with the property (2.9) of the Fourier transform to prove the conservation of energy

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 dx = E(u(0)). \tag{2.18}$$

Exercise. Hint: By Plancherel's identity notice that

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\partial_t \hat{u}|^2 + |\xi|^2|\hat{u}|^2\right)(\xi, t)d\xi$$

and then use formula (2.17) for $\hat{u}(\xi, t)$.

Definition 2.4. Given any function $a : \mathbb{R}^n \to \mathbb{C}$, define the *Fourier multiplier operator* $a(D)$ by the formula

$$a(D)f := \mathcal{F}^{-1}\left(a(\xi)\hat{f}(\xi)\right).$$

We say that $a(\xi)$ is the *symbol* of $a(D)$.

In view of this definition, (2.17) can be rewritten as

$$u(x,t) = \cos(t|D|)f + \frac{\sin(t|D|)}{|D|}g.$$

Remark 2.5. In the case of a non homogeneous wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = F, \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \end{cases} \qquad x \in \mathbb{R}^n, \quad t \geq 0, \tag{2.19}$$

one considers the non autonomous ODE

$$\partial_t^2 \hat{u}(\xi) + |\xi|^2 \hat{u}(\xi) = \hat{F}(\xi, t)$$

and using the method of variation of constants one obtains

$$\hat{u}(\xi,t) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi) + \int_0^t \frac{\sin((t-s)|\xi|)}{|\xi|}\hat{F}(\xi,s)ds$$

or, equivalently,

$$u(x,t) = \cos(t|D|)f + \frac{\sin(t|D|)}{|D|}g + \int_0^t \frac{\sin((t-s)|D|)}{|D|}F(s)ds.$$

This is called *Duhamel's formula* for the solution of the wave equation (2.19).