30/1/2018 ①

Lecture 4:

Local well-posedness in $H^1(\mathbb{R}^3)$ for energy-subcritical & energy-critical NLW

$$(NLW)\begin{cases} -\partial_t^2 u + \Delta u = u^p & , \quad u(t,x)\in\mathbb{R}, \ p>1 \quad (t\in\mathbb{R},\ x\in\mathbb{R}^3) \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^3) \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^3) \end{cases}$$

Energy: $E(u,\partial_t u) := \int_{\mathbb{R}^3} \frac{(\partial_t u)^2 + |Du|^2}{2} + \frac{u^{p+1}}{p+1}\,dx$

$E(u(t),\partial_t u(t)) = E(u_0,u_1)$, $\forall t$ if u is a smooth solution of NLW

$$-\partial_t^2 u + \Delta u = u^p \ \Big|\cdot \partial_t u \iff -\frac{1}{2}\partial_t\left[(\partial_t u)^2\right] + \underbrace{\sum_i \partial_i^2 u \cdot \partial_t u}_{} = \partial_t\left(\frac{u^{p+1}}{p+1}\right)$$

$$\sum_i \partial_i(\partial_i u \cdot \partial_t u) - \sum_i \partial_i u \cdot \partial_i\partial_t u$$

$$= \operatorname{div}(\partial_t u\, Du) - \frac{1}{2}\partial_t(|Du|^2)$$

$$\Longleftrightarrow \partial_t\left[\frac{(\partial_t u)^2+|\nabla u|^2}{2}+\frac{u^{p+1}}{p+1}\right]-\operatorname{div}(\partial_t u\,\nabla u)=0 \;\Big|\int_{\mathbb{R}^3}$$

$$\Longleftrightarrow \partial_t\left[E(u(t),\partial_t u(t))\right]=0$$

$$\Longleftrightarrow E(u(t),\partial_t u(t))=C=E(u_0,u_1)$$

- <u>Scaling invariance of NLW:</u>

If u is sol of NLW $\Longrightarrow$ $u_\lambda(t,x):=\frac{1}{\lambda^{\frac{2}{p-1}}}\,u\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)$, $\forall\lambda>0$

also sol. of NLW

$$-\partial_t^2 u_\lambda^{(t,x)}=\frac{1}{\lambda^{\frac{2}{p-1}}}\cdot\frac{1}{\lambda^2}\cdot\left(\partial_t^2 u\right)\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)$$

$$\Delta u_\lambda^{(t,x)}=\frac{1}{\lambda^{\frac{2}{p-1}}}\cdot\frac{1}{\lambda^2}\,(\Delta u)\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)$$

$$u_\lambda^p(t,x)=\frac{1}{\lambda^{\frac{2p}{p-1}}}\,(u^p)\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)$$

$$\Longrightarrow\left(-\partial_t^2 u_\lambda+\Delta u_\lambda-u_\lambda^p\right)(t,x)=\frac{1}{\lambda^{\frac{2p}{p-1}}}\underbrace{\left(-\partial_t^2 u+\Delta u-u^p\right)\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)}_{=0}=0$$

$t=0$: $\|u_\lambda(0,x)\|_{\dot H^1} = \|\nabla(u_\lambda(0,x))\|_{L^2} = \left\| \frac{1}{\lambda^{\frac{2}{p-1}}} \cdot \frac{1}{\lambda^1} (\nabla u)\left(0, \frac{x}{\lambda}\right)\right\|_{L^2}$

$$= \frac{1}{\lambda^{\frac{p+1}{p-1}}} \left(\int_{\mathbb{R}^3} \Big|\nabla u\Big(0, \underbrace{\tfrac{x}{\lambda}}_{=y}\Big)\Big|^2 \underbrace{dx}_{=\lambda^3 dy} \right)^{\frac{1}{2}}$$

$$= \frac{1}{\lambda^{\frac{p+1}{p-1}}} \left(\lambda^3 \int_{\mathbb{R}^3} |\nabla u(0,y)|^2 dy \right)^{\frac{1}{2}}$$

$$= \frac{1}{\lambda^{\frac{p+1}{p-1}}} \lambda^{\frac{3}{2}} \|\nabla u(0)\|_{L^2}$$

$$= \frac{1}{\lambda^{\frac{5-p}{2(p-1)}}} \|u(0)\|_{\dot H^1}$$

$$\frac{p+1}{p-1} - \frac{3}{2} = \frac{2p+2-3p+3}{2(p-1)} = \frac{5-p}{2(p-1)}$$

$\boxed{\text{If } p=5} \to \|u_\lambda(0)\|_{\dot H^1} = \|u(0)\|_{\dot H^1}, \forall \lambda > 0.$

Actually, we also have

$$\boxed{E(u_\lambda(0), \partial_t u_\lambda(0)) = E(u(0), \partial_t u(0)), \forall \lambda > 0}$$

- NLW with $p=5$ is called energy-critical in $\mathbb{R}^3$
- NLW with $1<p<5$ — " — energy-subcritical — " —
- — " — $p>5$ — " — energy-supercritical — " —

Claim 1: Energy-subcritical is easier:

- $\|u_\lambda(0)\|_{\dot H^1} = \frac{1}{\lambda^{\frac{5-p}{2(p-1)}}} \|u(0)\|_{\dot H^1}$

$\lambda \to \infty$, $1<p<5 \Rightarrow \frac{1}{\lambda^{\frac{5-p}{2(p-1)}}} \to 0 \Rightarrow \|u_\lambda(0)\|_{\dot H^1} \to 0$

By taking $\lambda \to \infty$, I can make initial data small.

- $\underbrace{u_\lambda(t,x)}_{t = \lambda T} = \frac{1}{\lambda^{\frac{2}{p-1}}} u\Big(\underbrace{\frac{t}{\lambda}}_{= T}, \frac{x}{\lambda}\Big)$

If u exists up to time $T \Rightarrow$ u_λ exists up to time λT and $\lambda T > T$ if $\lambda \to \infty$

Claim 2: Energy-supercritical case is harder

- ill-posed in H^1, but locally well-posed in H^s, for some $s>1$
- but then, problems with global well-posedness because our energy-conservation only gives control on H^1-norm, so we know nothing about the H^s-norm of a solut.

Energy-subcritical cubic NLS on $\mathbb{R}^3$

Banach fixed point thm:

Let (X,d) be a nonempty complete metric space.

Let $T: X \to X$ be a contraction:

$$d(Tu, Tv) < d(u,v), \quad \forall u, v \in X$$

Then $\exists!$ fixed point $x^* \in X$ of T: $T(x^*) = x^*$.

Duhamel's formula:

If u solves NLW with $u(t_0) = u_0$, $\partial_t u(t_0) = u_1$,

then:

$$u(t,x) = \underbrace{\cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1 - \int_{t_0}^{t} \frac{\sin((t-s)|\nabla|)}{|\nabla|}(u^p)(s)\,ds}_{=\Gamma u}$$

Thm 1 (LWP in H^1 of cubic NLW on $\mathbb{R}^3$)

Let $t_0 \in \mathbb{R}$, $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$. Then $\exists\, T \leq \dfrac{1}{30 \|(u_0,u_1)\|^2_{\dot{H}^1 \times L^2}}$

s.t.

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ u|_{t=t_0} = u_0 \\ \partial_t u|_{t=t_0} = u_1 \end{cases}$$

admits a unique solution $(u, \partial_t u) \in C([t_0, t_0+T), (\dot{H}^1 \times L^2)(\mathbb{R}^3))$.

Moreover the $(u_0, u_1) \mapsto u$ is Lipschitz.

Notation: $\|(u_0,u_1)\|_{\dot{H}^1 \times L^2} = \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}$.

$S(t-t_0)(u_0, u_1)$

Proof: Denote $\Gamma u := \cos((t-t_0)|D|)\, u_0 + \dfrac{\sin((t-t_0)|D|)}{|D|} u_1$

$$- \int_{t_0}^{t} \frac{\sin((t-s)|D|)}{|D|} u^3(s)\, ds$$

Finding a sol u, boils down to finding a fixed point for Γ; $u = \Gamma u$, in $C([0,T); \dot{H}^1(\mathbb{R}^3))$

$B_R = \{ u \in L^\infty([0,T); \dot{H}^1(\mathbb{R}^3)) \; ; \; \|u\|_{L^\infty_T \dot{H}^1_x} \leq R \}$ closed ball in $L^\infty_T \dot{H}^1_x$ (complete metric subspace)

$R = 2 \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$

- We show $\underset{?}{\underline{\Gamma(B_R) \subset B_R}}$

(1)

$\|f\|_{\dot{H}^1} = \||\nabla| f\|_{L^2}$

Take $u \in B_R \Rightarrow \|\Gamma u\|_{L^\infty_T \dot{H}^1_x} \leq \|S(t-t_0)(u_0, u_1)\|_{L^\infty_T \dot{H}^1}$

$$+ \sup_t \int_{t_0}^{t} \left\| \cancel{|\nabla|} \frac{\sin((t-s)|\nabla|)}{\cancel{|\nabla|}} u^3 \right\|_{L^2} ds$$

- $\boxed{\|\sin((t-s)|\nabla|) f\|_{L^2}} \overset{\text{Plancherel}}{=} \| \underbrace{|\sin((t-s)|\xi|)|}_{\leq 1} \hat{f}(\xi) \|_{L^2}$

$$= \Big(\int \underbrace{|\sin((t-s)|\xi|)|^2}_{\leq 1} |\hat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}}$$

(2)

$$\leq \Big(\int |\hat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} = \|\hat{f}\|_{L^2} \overset{\text{Planch.}}{=} \boxed{\|f\|_{L^2}}$$

$$\bullet\ \| S(t-t_0)(u_0,u_1)\|_{L^\infty_T \dot H^1} = \left\| |D| \left[\cos((t-t_0)|D|) u_0 + \frac{\sin((t-t_0)|D|)}{|D|} u_1 \right] \right\|_{L^2}$$

$$\leq \| \cos((t-t_0)|D|)\, |D| u_0 \|_{L^2} + \| \sin((t-t_0)|D|)\, u_1 \|_{L^2} \quad (8)$$

(3)

$$\leq \| |D| u_0 \|_{L^2} + \| u_1 \|_{L^2} = \| (u_0,u_1) \|_{\dot H^1 \times L^2}$$

Then (1)+(2)+(3):

$$\| \Gamma u \|_{L^\infty_T \dot H^1} \leq \| (u_0,u_1) \|_{\dot H^1 \times L^2} + \sup_{t\in[t_0,t_0+T]} \int_{t_0}^{t} \| u^3 \|_{L^2}\, ds$$

$$\leq \text{---}\text{''}\text{---} + T \, \| \| u^3 \|_{L^2_x} \|_{L^\infty_T}$$

(4)

$$\leq \text{---}\text{''}\text{---} + T \, \| u \|^3_{L^\infty_T L^6_x}$$

$$\| u^3 \|_{L^2} = \left(\int |u^3|^2 dx \right)^{\frac{1}{2}} = \left(\int |u|^6 dx \right)^{\frac{1}{6}\cdot 3} = \| u \|^3_{L^6}$$

Sobolev embedding:

$$\dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$$

(5) $\iff \| f \|_{L^6} \leq C \| f \|_{\dot H^1}, \ \forall f \in \dot H^1(\mathbb{R}^3)$

(4)+(5): $\|\Gamma u\|_{L^\infty_T \dot H^1} \le \|(u_0,u_1)\|_{\dot H^1\times L^2} + \underbrace{T\|u\|^3_{L^\infty_T \dot H^1}}$

$$\le \frac{R}{2} + TR^3$$

$$= \frac{R}{2} + \underbrace{(TR^2)}_{\le \frac{1}{2}} \cdot R \le R.$$

Indeed, $TR^2 \le \frac{1}{30\|(u_0,u_1)\|^2_{\dot H^1\times L^2}} \cdot \left(2\|(u_0,u_1)\|_{\dot H^1\times L^2}\right)^2 < \frac{1}{2}$

So, indeed $\Gamma(B_R) \subset B_R$.

- Next, we show $\Gamma: B_R \to B_R$ is a contraction:

$$\|\Gamma u - \Gamma v\|_{L^\infty_T \dot H^1} = \left\| \int_{t_0}^{t} \frac{\sin((t-s)|D|)}{|D|}(u^3-v^3)(s)\,ds \right\|_{L^\infty_T \dot H^1}$$

$$\le \sup_t \int_{t_0}^{t} \left\| \cancel{|D|} \cdot \frac{\sin((t-s)|D|)}{\cancel{|D|}}(u^3-v^3)(s)\right\|_{L^2_x} ds$$

$\|u\|_{L^\infty_T \dot H^1} = \|u\|_{L^\infty([t_0,t_0+T),\dot H^1)} = \sup_{t\in[t_0,t_0+T]} \|u(t,\cdot)\|_{\dot H^1_x}$

$$\le T\,\|u^3 - v^3\|_{L^\infty_T L^2_x} = T\,\|(u-v)(u^2+uv+v^2)\|_{L^\infty_T L^2_x}$$

Hölder

$$\le T\,\|u-v\|_{L^\infty_T L^6_x}\left(\|u^2\|_{L^\infty_T L^3} + \|uv\|_{L^\infty_T L^3_x} + \|v^2\|_{L^\infty_T L^3_x}\right)$$

$\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$

Hölder

$$\le T\,\|u-v\|_{L^\infty_T L^6_x}\left(\|u\|^2_{L^\infty_T L^6} + \|v\|^2_{L^\infty_T L^6_x} + \|u\|_{L^\infty_T L^6_x}\|v\|_{L^\infty_T L^6_x}\right)$$

$\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$

Sobolev

$$\le T\,\|u-v\|_{L^\infty_T \dot H^1}\left(\|u\|^2_{L^\infty_T \dot H^1} + \|v\|^2_{L^\infty_T \dot H^1} + \|u\|_{L^\infty_T \dot H^1}\|v\|_{L^\infty_T \dot H^1}\right)$$

$u, v \in B_R$

$$\le \underbrace{T \cdot 3R^2}_{<1} \cdot \|u-v\|_{L^\infty_T \dot H^1}$$

$$< \|u-v\|_{L^\infty_T \dot H^1}$$

Here, $T \cdot 3R^2 \le \frac{1}{30\,\|(u_0,u_1)\|^2} \cdot 3 \cdot \left(2\|(u_0,u_1)\|\right)^2 = \frac{12}{30} < 1$

$\Longrightarrow$ By Banach fixed pt. thm, Γ has a unique fixed pt. in B_R.

Thm 2 (GWP in H^1 for cubic NLS on $\mathbb{R}^3$)

Let $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$. Then

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

has a unique global soln $(u, \partial_t u) \in C(\underbrace{[0,\infty)}_{\mathbb{R}}; \dot{H}^1 \times L^2(\mathbb{R}^3))$

Proof: We will iterate LWP.

Key: $\|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} \leq \underset{?}{C}(\|(u_0, u_1)\|_{\dot{H}^1 \times L^2})$, $\forall t \in [0,\infty)$

$$\frac{1}{2}\|(u,\partial_t u)\|^2_{\dot{H}^1 \times L^2} \leq E(u, \partial_t u) = \int_{\mathbb{R}^3} \frac{(\partial_t u)^2 + |\nabla u|^2}{2} + \cancel{\frac{u^4}{4}}\, dx$$

|| conserv of energy

(a) $$E(u_0, u_1) = \int \frac{u_1^2 + |\nabla u_0|^2}{2} + \frac{u_0^4}{4}\, dx$$

$$\int u_0^4\,dx = \int u_0 \cdot u_0^3\,dx \overset{C-S}{\leq} \left(\int u_0^2\,dx\right)^{\frac{1}{2}} \left(\int u_0^6\,dx\right)^{\frac{1}{2}}$$

b

$$= \|u_0\|_{L^2} \cdot \|u_0\|_{L^6}^3$$

$$\underset{\text{Sobolev}}{\leq} \|u_0\|_{L^2} \cdot \|u_0\|_{\dot{H}^1}^3 \leq \|u_0\|_{H^1}^4$$

Then (a)+(b):

$$\frac{1}{2}\|(u,\partial_t u)\|_{\dot{H}^1 \times L^2}^2 \leq \underbrace{\frac{1}{2}\|(u_0,u_1)\|_{\dot{H}^1 \times L^2}^2 + \frac{1}{4}\|u_0\|_{H^1}^4}_{= \tilde{C}\left(\|(u_0,u_1)\|_{H^1 \times L^2}\right)}$$

Recap:

$$\begin{cases} \|f\|_{\dot{H}^1} = \left(\int |\xi|^2 |\hat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} < \infty \\ \|f\|_{H^1} = \left(\int (1+|\xi|^2)|\hat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} = \left[\int |\hat{f}(\xi)|^2\,d\xi + \int |\xi|^2 |\hat{f}(\xi)|^2\,d\xi\right]^{\frac{1}{2}} \\ \overset{\text{Plancherel}}{=} \left(\|f\|_{L^2}^2 + \|f\|_{\dot{H}^1}^2\right)^{\frac{1}{2}} < \infty \end{cases}$$

$$t=0 \xrightarrow[{[0,T_1)}]{\text{LWP}} T_1 = \frac{1}{30\|(u_0,u_1)\|^2_{\dot H^1\times L^2}} \xrightarrow[{[T_1,T_2)}]{\text{LWP}} T_2 = T_1 + \frac{1}{30\underbrace{\|(u(T_1),\partial_t u(T_1))\|^2_{\dot H^1\times L^2}}} \underset{\substack{\text{for}\\ t=T_1}}{\overset{\text{Key}}{\geq}} \frac{1}{30\,C(\|(u_0,u_1)\|_{\dot H^1\times L^2})^2}$$

LWP $[T_2,T_3)$

$$\to T_3 = T_2 + \frac{1}{30\underbrace{\|(u(T_2),\partial_t u(T_2))\|^2_{\dot H^1\times L^2}}} \underset{t=T_2}{\overset{\text{Key}}{\geq}} \frac{1}{30\,C(\|(u_0,u_1)\|_{\dot H^1\times L^2})^2}$$

$$\geq T_1 + \frac{2}{30\,C(\|(u_0,u_1)\|_{\dot H^1\times L^2})^2}$$

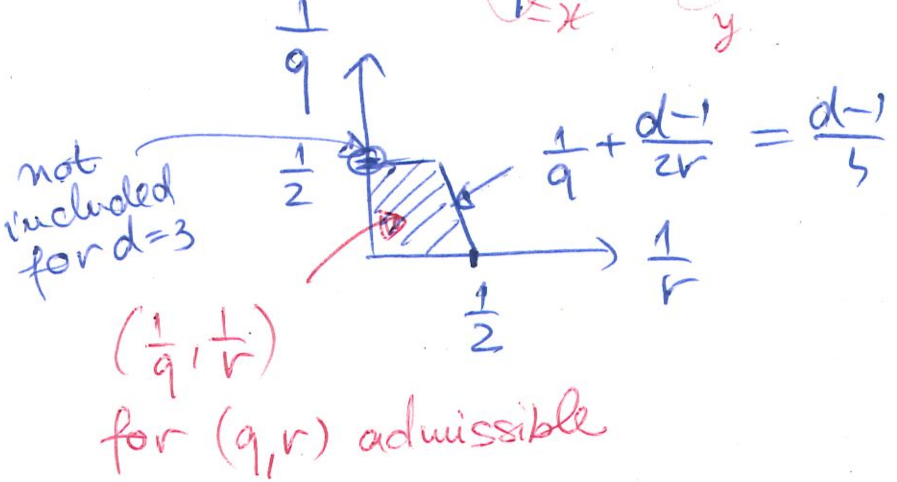
⋮

$$\xrightarrow[{[T_{n-1},T_n)}]{\text{LWP}} T_n \geq T_1 + \frac{n-1}{30\,C(\|(u_0,u_1)\|_{\dot H^1\times L^2})^2} \longrightarrow \infty \text{ as } n\to\infty$$

Strichartz estimates

Def: (q,r) is wave-admissible in $\mathbb{R}^d$ if $q, r \geq 2$, $(q,r,d) \neq (2,\infty,3)$

$$\underbrace{\frac{1}{q}}_{=x} + \underbrace{\frac{d-1}{2r}}_{y} \leq \frac{d-1}{4}.$$



Thm (Strichartz estimates)
Let (q,r), $(\tilde{q},\tilde{r})$ wave-admissible s.t. $r, \tilde{r} < \infty$,

$$\begin{cases} \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma \\ \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2 = \frac{d}{2} - \gamma \end{cases}$$

for some $\gamma > 0$ and $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1 = \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'}$.

Assume u solves:

$$\begin{cases} -\partial_t^2 u + \Delta u = F \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases} \quad \text{on } I \times \mathbb{R}^d.$$

Then, $\|(u, \partial_t u)\|_{L^\infty(I; \dot{H}^\sigma \times \dot{H}^{\sigma-1})} + \|u\|_{L^q(I; L^r_x)}$

$$\leq C\left[\|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{\dot{H}^{\sigma-1}} + \|F\|_{L^{\tilde{q}'}(I; L^{\tilde{r}'}_x)} \right]$$

where:

$$\|u\|_{L^q(I; L^r_x)} \overset{\text{def}}{=} \big\| \|u(t,x)\|_{L^r_x} \big\|_{L^q_t(I)} = \left(\int_I \left(\int_{\mathbb{R}^d} |u(t,x)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}$$

$$\|(u, \partial_t u)\|_{L^\infty(I; \dot{H}^\sigma \times \dot{H}^{\sigma-1})} \overset{\text{def}}{=} \sup_{t \in I} \|u(t,\cdot)\|_{\dot{H}^\sigma} + \sup_{t \in I} \|\partial_t u(t,\cdot)\|_{\dot{H}^{\sigma-1}}.$$