Lecture 5

$$(NLW)\begin{cases} -\partial_t^2 u + \Delta u = u^p, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad t_0 \in \mathbb{R} \\ u(t_0) = u_0 \in H^1(\mathbb{R}^d) \\ \partial_t u(t_0) = u_1 \in L^2(\mathbb{R}^d) \end{cases}$$

- $1 < p < 1 + \frac{4}{d-2}$ $(d \geq 3)$: NLW is called energy-subcritical
  - LWP in $H^1(\mathbb{R}^d)$: Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$

    $\|u\|_{L^{\frac{2d}{d-2}}} \leq \|\nabla u\|_{L^2}$, $\forall u \in \dot{H}^1$
  - GWP in $H^1(\mathbb{R}^d)$: energy conservation + iteration of LWP

- $p = 1 + \frac{4}{d-2}$ $(d \geq 3)$: NLW is called energy-critical
  - LWP in $H^1(\mathbb{R}^d)$: Strichartz estimates
  - GWP in $H^1(\mathbb{R}^d)$: . . .

Strichartz estimates: Stri. '77, Ginibre-Velo '89, '95, Keel-Tao '98

GWP for energ-crit. NLW in $H^1(\mathbb{R}^d)$: Shatah-Struwe { '93 Annals of Math
'94 IMRN

→ here: (we will also use a proof by Tao (book Nonlin Disp. Eqns.)

GWP for the energy-crit. nonlinear Schrödinger Equation in $H^1(\mathbb{R}^d)$
Colliander-Keel-Staffilani-Takaoka-Tao '08 Annals of Math

LWP of energy-critical NLW in $H^1(\mathbb{R}^3)$

$d=3, \quad p = 1 + \frac{4}{d-2} = 5$

$$\begin{cases} -\partial_t^2 u + \Delta u = u^5 & (t,x) \in \mathbb{R}\times\mathbb{R}^3 \\ u(0) = u_0 \in H^1(\mathbb{R}^3) \\ \partial_t u(0) = u_1 \in L^2(\mathbb{R}^3) \end{cases}$$

$$L^q_t L^r_x(I\times\mathbb{R}^3 \to \mathbb{R}) = L^q_t(I \to L^r_x(\mathbb{R}^3\to\mathbb{R}))$$

$$t \longmapsto u(t,\cdot) \in L^r_x(\mathbb{R}^3)$$

- $\|u\|_{L^q_t L^r_x(I\times\mathbb{R}^3\to\mathbb{R})} \overset{\text{def}}{=} \big\| \|u(t,\cdot)\|_{L^r_x} \big\|_{L^q_t} = \left[\int_I \left(\int_{\mathbb{R}^3} |u(t,x)|^r dx \right)^{\frac{q}{r}} dt \right]^{\frac{1}{q}}$

mixed Lebesgue norms

- $(q,r) = (4,12)$ wave-admissible, $\gamma = 1$, $d=3$

$$\frac{1}{q} + \frac{d-1}{2r} = \frac{1}{4} + \frac{\cancel{2}}{\cancel{2}\cdot 12} \leq \frac{1}{2} = \frac{d-1}{4}$$

- $\frac{1}{q} + \frac{d}{r} = \frac{1}{4} + \frac{3}{12} = \frac{1}{2} = \frac{3}{2} - 1 = \frac{d}{2} - \gamma$ for $\gamma = 1$

- $(\tilde{q},\tilde{r}) = (\infty, 2)$ wave-admissible, $\gamma = 1$, $d=3$

$$\frac{1}{\tilde{q}} + \frac{d-1}{2\tilde{r}} = \frac{1}{\infty} + \frac{2}{2\cdot 2} = \frac{1}{2} \leq \frac{1}{2} = \frac{d-1}{4}$$

$$\tilde{q}' = 1, \quad \tilde{r}' = 2 \qquad : \begin{cases} \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = \frac{1}{\infty} + \frac{1}{1} = 1 \\ \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = \frac{1}{2} + \frac{1}{2} = 1 \end{cases}$$

$$\frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2 = 1 + \frac{3}{2} - 2 = \frac{1}{2} = \frac{3}{2} - 1 = \frac{d}{2} - \gamma$$

Strichartz estimates: $\mathbb{R}^3$, $\gamma = 1$;

$$\boxed{\|(u,\partial_t u)\|_{L^\infty(I;\dot{H}^1\times L^2)} + \|u\|_{L^4_t(I;L^{12}_x)} \leq C\left[\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|F\|_{L^1_I L^2_x}\right]}$$

Thm 1 (LWP of quintic NLW in $\dot H^1(\mathbb{R}^3)$) ⑤

GWP for small energy initial data

① Let $(u_0,u_1)\in \dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ with $\|(u_0,u_1)\|_{\dot H^1\times L^2}$ suff. small then $\exists!$ solution u to NLW s.t. $\begin{cases}(u,\partial_t u)\in C([0,\infty);(\dot H^1\times L^2)(\mathbb{R}^3)) \\ u\in L^4_t(\mathbb{R};L^{12}_x(\mathbb{R}^3))\end{cases}$

& $\|(u,\partial_t u)\|_{L^\infty(\mathbb{R},\dot H^1\times L^2)}, \|u\|_{L^4(\mathbb{R},L^{12}_x)} \le 2\|(u_0,u_1)\|_{\dot H^1\times L^2}$.

② Let $(u_0,u_1)\in \dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, I time interval, $t_0\in I$.

Then $\exists$ $\delta>0$ small such that if $\|S(t-t_0)(u_0,u_1)\|_{L^4_t(I;L^{12}_x)}<\delta$ then $\exists!$ solution u defined on $I\times\mathbb{R}^3$ with

$$\begin{cases}(u,\partial_t u)\in C(I;\dot H^1\times L^2), \\ \|u\|_{L^4_t(I;L^{12}_x)}\le 2\delta, \\ \|(u,\partial_t u)\|_{L^\infty(I;\dot H^1\times L^2)}\le 2\|(u_0,u_1)\|_{\dot H^1\times L^2}.\end{cases}$$

Proof 1) $R = 2\,\|(u_0,u_1)\|_{\dot H^1\times L^2}$ small

$$B_R = \left\{ u \in L^\infty(\mathbb{R};\dot H^1),\ \ \|(u,v)\|_{L^\infty(\mathbb{R};\dot H^1)} \le R,\ \ \|u\|_{L^4_{\mathbb{R}}L^{12}} \le R \right\}$$

Duhamel's formula:

$$u(x,t) = \underbrace{\underbrace{\cos((t-t_0)|\nabla|)u_0 + \frac{\sin((t-t_0)|\nabla|)}{|\nabla|}u_1}_{\overset{\text{def}}{=} S(t-t_0)(u_0,u_1)} - \int_{t_0}^{t} \frac{\sin((t-s)|\nabla|)}{|\nabla|}(u^5)(s)\,ds}_{\Gamma u}$$

linear solution

Finding a solution is equivalent to finding a fixed point for operator Γ: $u = \Gamma u$

$\Gamma: B_R \to B_R$ is a contraction

Take $u \in B_R$. Want to show $\Gamma u \in B_R$.

$$\|\Gamma u\|_{L^\infty(\mathbb{R};\dot H^1)} + \|\Gamma u\|_{L^4_{\mathbb{R}} L^{12}_x} \overset{\text{Strichartz}}{\leq} \|(u_0,u_1)\|_{\dot H^1 \times L^2} + C\|u^5\|_{L^{①}_{\mathbb{R}} L^{②}}$$

$$\overset{\text{Hölder}}{\leq} \|(u_0,u_1)\|_{\dot H^1\times L^2} + C\underline{\|u\|_{L^\infty_t L^6_x}}\,\|u\|^4_{L^4_t L^{12}_x}.$$

$\frac{1}{\infty} + \frac{4}{4} = 1$

$\frac{1}{6} + \frac{4}{12} = \frac{1}{2}$

$$\overset{\text{Sobolev}}{\leq} \|(u_0,u_1)\|_{\dot H^1\times L^2} + C\|u\|_{L^\infty_t \dot H^1_x}\,\|u\|^4_{L^4_t L^{12}_x}$$

$$\leq \frac{R}{2} + \underbrace{CR^5}_{CR^4\cdot R} \leq \frac{R}{2} + \frac{R}{2} = R$$

$CR^4 \leq \frac{1}{2}$ ← smalness of $R = 2\|(u_0,u_1)\|_{\dot H^1\times L^2}$

$\Longrightarrow \Gamma u \in B_R$, in other words $\Gamma(B_R) \subset B_R$

$$\|\Gamma u - \Gamma v\|_{L^\infty_{\mathbb{R}} \dot H^1} + \|\Gamma u - \Gamma v\|_{L^4_{\mathbb{R}} L^{12}_x} \le c\|u^5 - v^5\|_{L^1_t L^2_x}$$

$$= c\|(u-v)(\underline{u^4} + u^3 v + u^2 v^2 + u v^3 + \underline{v^4})\|_{L^1 L^2}$$

can be controlled by u^4 & v^4, so we'll disregard them

Hölder

$$\le C\|u-v\|_{L^\infty L^6}\left(\|u\|^4_{L^4 L^{12}} + \|v\|^4_{L^4 L^{12}}\right)$$

$$\le C\|u-v\|_{L^\infty \dot H^1}\Big(\underbrace{\|u\|^4_{L^4 L^{12}}}_{\le R^4} + \underbrace{\|v\|^4_{L^4 L^{12}}}_{\le R^4}\Big)$$

$$\le \underbrace{2CR^4}_{<1}\, \|u-v\|_{L^\infty \dot H^1}$$

smallness of R

$$< \|u-v\|_{L^\infty \dot H^1}$$

$\Rightarrow \Gamma$ is a contraction on $B_R \Rightarrow$ global solution

Note: By Strichartz, we also have:

$$\|\partial_t u\|_{L^\infty(\mathbb{R};L^2)} \le \|(u_0,u_1)\|_{\dot H^1 \times L^2} + C\|u\|_{L^\infty(\mathbb{R},\dot H^1)}\|u\|^4_{L^4(\mathbb{R};L^{12})} \le \frac{R}{2} + CR^4 \le \underset{\underset{2\|(u_0,u_1)\|_{\dot H^1 \times L^2}}{\shortparallel}}{R}$$

(2) Strichartz for linear wave equation ($F=0$):

$$\|S(t-t_0)(u_0,u_1)\|_{(L^4_t;L^{12}_x)} \leq \|(u_0,u_1)\|_{\dot{H}^1\times L^2}$$

Rem: In 1) we had $\|(u_0,u_1)\|_{\dot{H}^1\times L^2}$ small, so in particular $\|S(t-t_0)(u_0,u_1)\|_{L^4_t(\mathbb{R};L^{12}_x)}$ small

In 2) we require $\|S(t-t_0)(u_0,u_1)\|_{L^4_t(I;L^{12}_x)}$ to be small, but $\|(u_0,u_1)\|_{\dot{H}^1\times L^2}$ could be large.

$A := 2\|(u_0,u_1)\|_{\dot{H}^1\times L^2}$.

$$B_{A,2\delta} = \Big\{ (u,v) \,;\, \|(u,v)\|_{(L^\infty_I \dot{H}^1\times L^2)} \leq \underset{\substack{\text{large}\\ \text{norm}}}{A} \,,\, \|u\|_{L^4(I;L^{12}_x)} \leq \underset{\substack{\text{small}\\ \text{norm}}}{2\delta} \Big\}$$

Show Γ has a fixed point in $B_{A,2\delta}$.

First, let's show $\underline{\Gamma(B_{A,2\delta}) \subset B_{A,2\delta}}$?

Take $(u,v) \in B_{A,2\delta}$

$$\|\Gamma u\|_{L^4_I L^{12}_x} \overset{\text{Strich}}{\leq} \|S(t-t_0)(u_0,u_1)\|_{L^4(I;L^{12}_x)} + C\|u^5\|_{L^1(I;L^{12}_x)}$$

$$\overset{\text{as before}}{\leq} \|S(t-t_0)(u_0,u_1)\|_{L^4_I L^{12}} + C\|u\|_{L^\infty(I;\dot H^1)} \|u\|^4_{L^4_I L^{12}}$$

$$\leq \delta + CA\cdot(2\delta)^4$$

$$\leq \delta + \underbrace{(16CA\delta^3)}_{\leq 1}\cdot\delta \leq 2\delta$$

$$\|\Gamma u\|_{L^\infty_I \dot H^1} \overset{\text{strich}}{\leq} \|(u_0,u_1)\|_{\dot H^1\times L^2} + C\|u^5\|_{L^1(I;L^2)}$$

$$\underset{\text{above}}{\overset{\text{as}}{\leq}} \underbrace{\|(u_0,u_1)\|_{\dot H^1\times L^2}}\ + \underbrace{16C\delta^4}_{<\frac{1}{2}} A \qquad A := 2\|(u_0,u_1)\|_{\dot H^1\times L^2}$$

$$\leq \frac{A}{2} + \frac{A}{2} = A$$

So: $\Gamma(B_{A,2\delta}) \subset B_{A,2\delta}$. Similary, one can prove it's a contraction.

Thm 2) <u>Finite time blowup criterion:</u>

Let u be $\dot{H}^1 \times L^2$-solution of NLW defined on a maximal time interval $[t_0, t_0 + T_*)$.

If $T_* < \infty$, then

$$\|u\|_{L^4([t_0, t_0+T_*), L^{12}_x(\mathbb{R}^3))} = \infty \quad \boxed{}$$

<u>Proof</u>: By contradiction, assume $\|u\|_{L^4([t_0, t_0+T_*), L^{12})} < \infty$.

$\exists\, t_n \nearrow t_0 + T_*$ s.t.

$$\|u\|_{L^4([t_n, t_0+T_*), L^{12}_x)} < \frac{\delta}{4} \quad \text{for } n \text{ suff. large.}$$

<u>Duhamel</u>: $u(t) = S(t-t_0)(u_0, u_1) - \int_{t_0}^{t} \frac{\sin((t-s)|\nabla|)}{|\nabla|}(u^5)(s)\,ds$

$\iff S(t-t_0)(u_0, u_1) = u(t) + \int_{t_0}^{t} -"- \quad \text{———}$

$\Rightarrow \|S(t-t_0)(u_0,u_1)\|_{L^4([t_n, t_0+T_*), L^{12})} \leq \|u\|_{L^4([t_n, t_0+T_*), L^{12})} + C\|\nabla u\|_{L^\infty L^2}\|u\|^4_{L^4([t_n, \ldots), L^{12}_x)}$

$\leq \frac{\delta}{4} + C\|\nabla u\|_{L^\infty L^2} \cdot \left(\frac{\delta}{4}\right)^4$

By energy conserv. : $\|\nabla u\|_{L^\infty L^2} \leq [E(u_0, u_1)]^{\frac{1}{2}}$

$$\Rightarrow \|S(t-t_0)(u_0,u_1)\|_{L^4([t_n, t_0+T_*), L^{12})} \leq \frac{\delta}{4} + \underbrace{C(E(u_0,u_1))\left(\frac{\delta}{4}\right)^4}_{= C(E)\cdot\frac{\delta^3}{4^4}\cdot\delta,\ C(E)\cdot\frac{\delta^3}{4^4} < \frac{1}{4}} \leq \frac{\delta}{2}$$

$$\boxed{\|S(t-t_0)(u_0,u_1)\|_{L^4(\mathbb{R}; L^{12}_x)} \leq \|(u_0,u_1)\|_{\dot{H}^1 \times L^2} < \infty}$$

$\Rightarrow \exists\, \varepsilon > 0$ s.t.

$$\|S(t-t_0)(u_0,u_1)\|_{L^4([t_n, t_0+T_*+\varepsilon), L^{12}_x)} < \delta$$

Then by 2) in LWP theorem:

$\exists!$ solution u defined on $[t_n, t_0+T_*+\varepsilon)$.

This contradicts the maximality of $[t_0, t_0+T_*)$ as time interval of existence for u.

Lemma: Let $u \in \dot{H}^1(\mathbb{R}^3)$, $\psi \in \mathcal{S}(\mathbb{R}^3)$. Then

$$\boxed{\| u(x)\psi\left(\tfrac{x}{R}\right) \|_{\dot{H}^1} \leq C \cdot \| u \|_{\dot{H}^1},}$$ where $C = C(\psi)$ indep of R. $\forall R$.

Proof: $\| u\psi\left(\tfrac{\cdot}{R}\right) \|_{\dot{H}^1} = \| \nabla\left(u\psi\left(\tfrac{x}{R}\right)\right)\|_{L^2}$

$$\leq \|\nabla u \cdot \psi\left(\tfrac{x}{R}\right)\|_{L^2} + \| u \cdot \tfrac{1}{R}(\nabla\psi)\left(\tfrac{x}{R}\right)\|_{L^2}$$

$$\overset{\text{Hölder}}{\leq} \|\nabla u\|_{L^2} \cdot \|\psi\left(\tfrac{x}{R}\right)\|_{L^\infty} + \underline{\|u\|_{L^6}} \cdot \frac{1}{R} \cdot \|(\nabla\psi)\left(\tfrac{x}{R}\right)\|_{L^3}$$

$$\frac{1}{2} = \frac{1}{2} + \frac{1}{\infty}$$

$$\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$$

$$\overset{\text{Sobolev}}{\leq} C\|\nabla u\|_{L^2} \cdot \|\psi\|_{L^\infty} + C\|u\|_{\dot{H}^1} \cdot \frac{1}{R}\left(\int |\nabla\psi\left(\tfrac{x}{R}\right)|^3 dx\right)^{\frac{1}{3}}$$

$$\tfrac{x}{R} = y$$

$x = Ry$

$dx = R^3 dy$

$$\leq C\|\nabla u\|_{L^2} \cdot \|\psi\|_{L^\infty} + C\|u\|_{\dot{H}^1} \cdot \cancel{\frac{1}{R}} \cdot \left(\cancel{R^3} \int |\nabla\psi(y)|^3 dy\right)^{\frac{1}{3}}$$

$$= C\|u\|_{\dot{H}^1} \left(\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^3}\right)$$