ADVANCED PDE II - LECTURE 5 (PART 1)

PIETER BLUE AND OANA POCOVNICU

Warning: This is a first draft of the lecture notes and should be used with care!

1. Sobolev spaces and Sobolev embeddings

Definition 1.1. The *homogeneous Sobolev space* $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{\dot{H}^s} := \||\xi|^s \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}. \tag{1.1}$$

Similarly, the *inhomogeneous Sobolev space* $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{H^s} := \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

where $\langle\xi\rangle = \sqrt{|\xi|^2 + 1}$.

Remark 1.2. If $s \in \mathbb{N}$, then

$$\dot{H}^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| = s\}$$

and

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \le s\}.$$

Theorem 1.3 (Sobolev inequality)**.**
(i). Let $s > \frac{n}{2}$. There exists a constant $C = C(n, s)$ such that for all $f \in H^s(\mathbb{R}^n)$ the following holds:

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.3}$$

(ii). Let $s < \frac{n}{2}$ and $\frac{1}{p} \ge \frac{1}{2} - \frac{s}{n}$. There exists a constant $C = C(n, p)$ such that for all $f \in H^s(\mathbb{R}^n)$ the following holds:

$$\|f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.4}$$

(iii). Let $s < \frac{n}{2}$ and $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$. There exists a constant $C = C(n, p)$ such that for all $f \in \dot{H}^s(\mathbb{R}^n)$ the following holds:

$$\|f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.5}$$

Proof of (i). Let $s > \frac{n}{2}$. For $f \in H^s(\mathbb{R}^n)$, we have by the Fourier inversion formula, the definition of the inverse Fourier transform, inequality (**??**), Cauchy-Schwarz inequality, and using polar coordinates together with $2s - n + 1 > 1$:

$$\begin{aligned}\|f\|_{L^\infty} &= \|\mathcal{F}^{-1}(\hat{f})(x)\|_{L^\infty} = \|\mathcal{F}(\hat{f})(-x)\|_{L^\infty} \leq C\|\mathcal{F}(\hat{f})(x)\|_{L^\infty} \leq C\|\hat{f}\|_{L^1} \\ &= C\int |\hat{f}(\xi)|d\xi = C\int \frac{1}{\langle\xi\rangle^s}\langle\xi\rangle^s|\hat{f}(\xi)|d\xi \leq C\left(\int_{\mathbb{R}^n}\frac{1}{\langle\xi\rangle^{2s}}d\xi\right)^{\frac{1}{2}}\left(\int\langle\xi\rangle^{2s}|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\mathcal{S}^{n-1}}\int_0^\infty \frac{r^{n-1}}{\langle r\rangle^{2s}}drds\right)^{\frac{1}{2}}\|f\|_{H^s} \leq C\left(\int_{\mathcal{S}^{n-1}}\int_0^\infty \frac{1}{\langle r\rangle^{2s-n+1}}drds\right)^{\frac{1}{2}}\|f\|_{H^s} \\ &\leq C\|f\|_{H^s}.\end{aligned}$$

□

2. Gronwall's lemma

Theorem 2.1 (Gronwall's lemma)**.** Let $A > 0$ and let $\phi, B : [t_0, t_1] \to \mathbb{R}$ be continuous and non negative functions. Assume that ϕ satisfies

$$\phi(t) \leq A + \int_{t_0}^t B(s)\phi(s)ds \quad \text{for all} \quad t \in [t_0, t_1]. \tag{2.1}$$

Then,

$$\phi(t) \leq A\exp\left(\int_{t_0}^t B(s)ds\right). \tag{2.2}$$

Proof. Using (2.3), we notice that

$$\frac{d}{dt}\left(A + \int_{t_0}^t B(s)\phi(s)ds\right) = B(t)\phi(t) \leq B(t)\left(A + \int_{t_0}^t B(s)\phi(s)ds\right).$$

This implies

$$\frac{d}{dt}\log\left(A + \int_{t_0}^t B(s)\phi(s)ds\right) \leq B(t).$$

Integrating in t, we obtain

$$A + \int_{t_0}^t B(s)\phi(s)ds \leq A\exp\left(\int_{t_0}^t B(s)ds\right).$$

This together with (2.3) yields the conclusion. □

Theorem 2.2 (Gronwall's lemma - differential form)**.** Let $\phi : [t_0, t_1] \to \mathbb{R}$ be a C^1 function and $B : [t_0, t_1] \to \mathbb{R}$ be a continuous function. Assume that ϕ satisfies

$$\partial_t\phi(t) \leq B(t)\phi(t) \quad \text{for all} \quad t \in [t_0, t_1]. \tag{2.3}$$

Then,

$$\phi(t) \leq \phi(t_0)\exp\left(\int_{t_0}^t B(s)ds\right). \tag{2.4}$$

Proof. Define the C^1 function

$$\psi(t) = \phi(t)\exp\left(-\int_{t_0}^t B(s)ds\right)$$

and notice that $\psi'(t) \leq 0$. □