ADVANCED PDE II - LECTURE 5 (PART 2)

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Warning: This is a first draft of the lecture notes and should be used with care!

1. Well-posedness

Informal Definition 1.1. *A PDE is* ***well-posed*** *(in the sense of Hadamard) if*

1. *For each choice of data, a solution exists* ***in some sense****.*
2. *For each choice of data, the solution is unique* ***in some space****.*
3. *The map from data to solutions is continuous* ***in some topology****.*

Definition 1.2. *Let $n, m \in \mathbb{Z}^+$ and $s \in \mathbb{R}$. Let $H^s = H^s(\mathbb{R}^n, \mathbb{R}^m)$. For $u_0 \in H^s$ and $r > 0$, let $B(u_0, r; H^s) = \{v_0 \in H^s : \|v_0 - u_0\|_{H^s}\}$ be a metric space with the H^s norm.*

Let $G : \mathbb{R} \times H^s \to H^{s-1}$ and $t_0 \in \mathbb{R}$. Consider the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = G(t, u(t)), \tag{1a}$$

$$u(t_0) = u_0. \tag{1b}$$

In this case, u is called the solution, and u_0 is the initial data.

The initial-value problem (1a)-(1b) *is* ***well-posed in*** *H^s (in the sense of Kato*[1]*) if for all $w_0 \in H^s$ there are $T > t_0$ and $r > 0$ such that, with $B = B(w_0, r; H^s)$ and $Z = C^0([t_0, T]; H^s) \cap C^1([t_0, T]; H^{s-1})$*

1. *[Existence] $\forall v_0 \in B : \exists S(v_0) \in Z$ such that $(u, u_0) = (S(v_0), v_0)$ solves* (1a)-(1b)*.*
2. *[Uniqueness] $\forall v_0 \in B : \forall v_1, v_2 \in Z$: if $(u, u_0) = (v_1, v_0)$ and $(u, u_0) = (v_2, v_0)$ are solution of* (1a)-(1b)*, then $v_1 = v_2$.*
3. *[Continuity] The map S taking data to solutions given by (1)-(2) is continuous from $B(w_0, r; H^s)$ to $C^0([t_0, T]; H^s)$.*

Well-posedness is also called ***local well-posedness****. If $\forall w_0 \in H^s$ the result holds for all $T > 0$, then* (1a)-(1b) *is* ***globally well-posed****.*

Remark 1.3.
1. *Wave equations can be written as first-order systems. For a second-order PDE, well-posedness is equivalent to showing the map $(u(0), \partial_t u(u)) \mapsto (u, \partial_t u)$ from $H^k \times H^{k-1}$ to $C^0([0, T], H^k) \times C^0([0, T], H^{k-1})$ exists uniquelly and is continuous.*
2. *Hyperbolic PDE are time symmetric, so we can solve both forward and backward in t.*

[1] Technically, Kato [**?**] merely requires for continuity that, $\forall t \in [t_0, T]$, the map $v_0 \mapsto S(v_0)(t)$ is continuous from $B(w_0, r; H^s)$ to H^s.

(3) *The above definition is adapted to hyperbolic problems. For other problems, H^s and H^{s-1} can be replaced by Banach spaces Y and X, with Y embedding densely and continuously into X. For Schrödinger or heat equations, one would take $Y = H^s$ and $X = H^{s-2}$. The heat equation $\partial_t u - \sum_{i=1}^n \partial_i^2 u = 0$ is not time symmetric and can only be solved forward in time.*

(4) *If A is Banach, $t_1, t_2 \in \mathbb{R}$, and $k \in \mathbb{N}$, then $C^k([t_1,t_2];A)$ is also a Banach space. On $\mathbb{R}$, $C^k(\mathbb{R};A)$ is not Banach, but we can introduce $C_0^k(\mathbb{R};A)$ and $C_b^k(\mathbb{R},A)$, the set of compactly supported C^k functions and the space of C^k functions with bounded partials of order k; the latter is a Banach space.*

(5) *In some cases, one can only establish uniqueness in $Z \cap A$ where A is some auxilliary Banach space, such as $L^p_{t,x}$. This is **conditional uniqueness**.*

(6) *T is lower semicontinuous in w_0.*

(7) *By density, it is sufficient (but not necessary) to check the above holds for all w_0 in some dense subset of H^s (such as Schwarz functions) and $T - t_0$ depends only on $\|w_0\|_{H^s}$.*

Proving local and global well-posedness for hyperbolic PDE will be the main focus of the remainder of the course. The following theorem seems useful in establishing such results. However, it is likely that in any application, the theorem slightly fails and slight modification of the proof is required.

Definition 1.4. *Let U be a metric space and V be a complete metric space. Let $\Phi : U \times V \to V$. Φ is **uniformly continuous** in U if*

$$\forall x_1 \in U, \epsilon > 0 : \exists \delta > 0 : \forall x_2 \in U, y \in V : \quad \|x_2 - x_1\|_U < \delta \implies \quad \|\Phi(x_2,y) - \Phi(x_1,y)\|_V < \epsilon.$$

*Φ is a **uniform contraction mapping** in V if*

$$\exists r \in [0,1) : \forall x \in U; y_1, y_2 \in V : \qquad \|\Phi(x,y_2) - \Phi(x,y_1)\|_V \le r\|y_2 - y_1\|_V.$$

Theorem 1.5. *Let U be a metric space and V be a complete metric space.*

If $\Phi : U \times V \to V$ is uniformly continuous in U and a uniform contraction mapping in V, then there is a map $S : U \to V$ such that

(1) *$\forall x \in U : \Phi(x, S(x)) = S(x)$;*
(2) *If $\Phi(x,y) = y$, then $y = S(x)$; and*
(3) *The map $S : U \to V$ is continuous.*

Proof. Exercise. (Use the ideas of the contraction mapping theorem and lots of $\epsilon - \delta$.) $\square$

2. Preparation for the vector-field method

2.1. **Illustrative examples.** In this section, we will use the following notation: the energy (of u at time t, if u is sufficiently regular and decaying at infinity) is

$$E[u](t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^n} \left(|\partial_t u|^2 + \sum_{i=1}^n |\partial_i u|^2 + |u|^2 \right) \mathrm{d}^n x.$$

Theorem 2.1 (Basic energy estimate)**.** *Let $n \in \mathbb{Z}^+$. Let $u \in C^2([0,T] \times \mathbb{R}^n, \mathbb{R})$ be a function that vanishes for x sufficiently large.*

If u is solution of the Klein-Gordon equation $-\partial_t^2 u + \sum_{i=1}^n \partial_i^2 u - u = 0$, then $\forall t \in [0,T]$

$$E[u](t) = E[u](0).$$

Proof. Let $N = -\partial_t$. Consider

$$\begin{aligned}
0 &= (Nu)\left(-\partial_t^2 u + \sum_{i=1}^n \partial_i^2 u - u\right) \\
&= (-\partial_t u)\left(-\partial_t^2 u + \sum_{i=1}^n \partial_i^2 u - u\right) \\
&= \partial_t u \partial_t^2 u - \sum_{i=1}^n \partial_i\left(\partial_t u \partial_i u\right) + \sum_{i=1}^n \partial_t \partial_i u \partial_i u + \partial_t u u \\
&= \partial_t\left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\sum_{i=1}^n (\partial_i u)^2 + \frac{1}{2}u^2\right) - \sum_{i=1}^n \partial_i\left(\partial_t u \partial_i u\right).
\end{aligned}$$

We now integrate by parts over $[0,t] \times B(\vec{0}, R)$ where R is sufficiently large that $u(s, \vec{x}) = 0$ for $|\vec{x}| > R$. This gives the desired result. □

Theorem 2.2 (Translation symmetries)**.** *Let $n \in \mathbb{Z}^+$. Let $k > 2$. Let $u \in C^k([0,T] \times \mathbb{R}^n, \mathbb{R})$.*

If u is a solution of $\Box u = 0$, then for any multiindex $|\alpha| < k-2$, $v = \partial^\alpha u$ is a $C^{k-|\alpha|}$ solution of $\Box v = 0$.

Proof. Since $\partial_j\left(-\partial_t^2 + \sum_{i=1}^n \partial_i^2\right) w = \left(-\partial_t^2 + \sum_{i=1}^n \partial_i^2\right)\partial_j w$ holds for all $w \in C^3$, the result holds by induction. □

Theorem 2.3 (Uniform bound)**.** *Let $n \in \mathbb{Z}^+$. Let $k > n/2$.*

There is a constant C such that if f, g are test functions in $\mathbb{R}^n$, and u is the unique C^∞ solution to the initial value problem

$$-\partial_t^2 u + \sum_{i=1}^n \partial_i^2 u - u = 0, \tag{2a}$$

$$u(0, \vec{x}) = f(\vec{x}), \tag{2b}$$

$$\partial_t u(0, \vec{x}) = g(\vec{x}), \tag{2c}$$

then

$$\|u\|_{L^\infty_{t,x}} \leq C\left(\|f\|_{H^k}^2 + \|g\|_{H^{k-1}}^2\right).$$

Proof. Let $T > 0$.

Since f and g are compactly supported. From the explicit representation formula for solutions of the wave equation, this means that a C^∞ solution u exists and vanishes for sufficiently large $|\vec{x}|$ uniformly in $[0,T] \times \mathbb{R}^n$.

Observe that

$$\begin{aligned}
E[u](0) &= \frac{1}{2}\int \left(|\partial_t u(0,\vec{x})|^2 + \sum_{i=1}^{n} |\partial_i u(0,\vec{x})|^2 + |u(0,\vec{x})|^2 \right) \mathrm{d}^n x \\
&= (1/2)\left(\|g\|_{L^2}^2 + \|f\|_{H^1}^2 \right),
\end{aligned}$$

$$\begin{aligned}
\sum_{|\alpha|<k-1} E[u](0) &= \int_{\{0\}\times\mathbb{R}} \left(\sum_{|\alpha|<k-1} |\partial^\alpha g|^2 + \sum_{|\alpha|<k-1}\sum_{i=1}^{n} \left(|\partial^\alpha \partial_i f|^2 + |\partial^\alpha f|^2 \right) \right) \mathrm{d}^n x \\
&\le C' \left(\|g\|_{H^{k-1}}^2 + \|f\|_{H^k}^2 \right).
\end{aligned}$$

We now apply the basic energy estimate to the solution $\partial^\alpha u$, the relation between the energy and H^k norm, and Sobolev embedding, to observe that

$$\begin{aligned}
\sum_{|\alpha|<k-1} E[\partial^\alpha u](0) &= \sum_{|\alpha|<k-1} E[\partial^\alpha u](T) \\
&\ge C'' \|u|_{t=T}\|_{H^k}^2 \\
&\ge C^{-1} \|u(T,\vec{x})\|_{L^\infty_x}.
\end{aligned}$$

The constant C is independent of T, which gives the desired result. □

Corollary 2.4 (Density argument)**.** *Let $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$.*

(1) *There is a unique, linear map $S : H^k \times H^{k-1} \to C^0(H^k) \cap C^1(H^{k-1})$ that takes test function initial data f, g to solutions $u = S(f,g)$ of the initial value problem* (2)
(2) *Furthermore, if $k > n/2$, then $S : H^k \times H^{k-1} \to C^0(H^k) \cap C^1(H^{k-1}) \cap L^\infty_{t,x}$.*

In particular, the Klein-Gordon equation is globally well-posed in H^k.

2.2. **Generalising from the examples.** In the above, there have been four crucial ideas:

(1) Multiplying the PDE by some factor and applying integration by parts gave us the basic energy estimate. Many other energy-like quantities can be considered by changing the multiplier. This is the **method of multipliers**. If $Nu = a\partial_t u + \vec{B} \cdot \vec{\nabla} u + cu$, this is **Friedrich's** *abc* **method**.
(2) Differentiating the equation and the translation symmetry allowed us to control higher Sobolev norms. We could differentiate in angular directions or differentiate along position dependent directions.
(3) An estimate, in this case the uniform bound, followed from our control of a higher energy.
(4) The density argument showed that by working in a very regular class (Schwarz!), we could then obtain results in a Banach space to get well-posedness.

The **vector-field method** (also called Klainerman's commuting vector-field method) uses geometric ideas to choose both the multipliers and the directions in which to differentiate the equation. The remaining two points are then used to prove estimates that are used to study nonlinear problems.