ADVANCED PDE II - LECTURE 6

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Warning: This is a first draft of the lecture notes and should be used with care!

1. Some geometry

1.1. **Tensor notation without manifolds.** Consider $\mathbb{R}^N$. We use the notation

$$\vec{x} = (x^1, \dots, x^N),$$
$$\partial_i = \frac{\partial}{\partial x^i},$$
$$\vec{\partial} = (\partial_1, \dots, \partial_N).$$

At this stage, we do not consider an inner product on $\mathbb{R}^N$.

Let Ω be an open subset of $\mathbb{R}^N$. Recall that a vector field is a map $X : \Omega \to \mathbb{R}^{1+n}$ that defines a vector at each point in the set Ω. In this perspective, it is useful to distinguish between points and vectors. We denote the components of X by $\{X^i\}_{i=1}^N$.

Recall that the directional derivative of a C^1 function, $f : \Omega \to \mathbb{R}$, is given by $\nabla_X f = \sum_i X^i \partial_i f$. Summations from 1 to N of this type are so common that we use the Einstein summation convention, that when there is exactly one subscript and one superscript in a formula, they are understood to be summed over. Such summations are called contractions over the index or simply contraction. Thus, we write,

$$\nabla_X f = (\partial_i f) X^i.$$

Typically, we identify a vector field with a differential operator by

$$X = X^i \partial_i.$$

Crucial in the use of this notation is the geometric fact that in a formula a single index should never appear more than twice and that when it appear twice, it appears once a subscript and once as a superscript. If you are unfamiliar with this notation, it may help to think of vectors as column vectors, the gradient as a row vector, and the components of a matrix A which takes vectors to vectors as being $A^j{}_i$. Moving the indices on a tensor radically changes the nature of the tensor!

A 1-form is an object with one subscript and no superscripts. The gradient of a function f is the 1-form defined to have components

$$\mathrm{grad} f = \vec{\partial} f = (\partial_1 f, \dots, \partial_N f),$$
$$(\mathrm{grad} f)_i = \partial_i f.$$

Observe that the gradient is not a vector field.

A tensor is an object with indices in $\{1,\dots,N\}$. A particularly important tensor is the Kronecker delta, given by

$$\delta_i^j = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j \end{cases}.$$

Occasionally, we will consider the tensors defined by $\delta_{ij}=\delta^{ij}=\delta_i^j$, but be aware that these are radically different objects; δ_i^j is a map from vectors to vectors, where as δ_{ij} is a map from vectors to 1-forms and defines an inner product on vectors. Another is the Levi-Civita tensor $\epsilon_{j_1\dots j_N}$, which is uniquelly defined by the conditions that $\epsilon_{j_1\dots j_N}$ is antisymmetric in every pair of indices and $\epsilon_{1\dots N}=1$.

1.2. **The divergence theorem.** We define smooth to mean C^∞.

We define a nondegenerate parameterisation of a smooth hypersurface to be a smooth map $f:R\to\mathbb{R}^{1+n}$ such that R is an open subset of $\mathbb{R}^n$ and such that, at each $p\in\mathbb{R}^n$, the $n\times(n+1)$ matrix of partial derivatives has rank n. We define $\Sigma\subset\mathbb{R}^{1+n}$ to be a smooth hypersurface if for all $x\in\mathbb{R}^{1+n}$, there is a neighbourhood N_x of x and a nondegenerate parameterisation of a hypersurface $f:R\to\mathbb{R}^{1+n}$, such that $f(R)=\Sigma\cap N_x$. Given a nondegenerate parameterisation of a smooth hypersurface, $f:R\to\mathbb{R}^{1+n}$, we define the normal 1-form to be the map $\nu:R\to\mathbb{R}^{1+n}$ with components

$$\nu_i=\epsilon_{ii_1\dots i_n}\frac{\partial f^{i_1}}{\partial y^1}\frac{\partial f^{i_n}}{\partial y^n}.$$

Recall that if R be an open subset of $\mathbb{R}^n$ and $f:R\to\mathbb{R}^{1+n}$ be a nondegenerate parametrisation of a smooth hypersurface, and F a C^0 map from $f(R)$ to $\mathbb{R}^{1+n}$

$$\int_{f(R)}F^i\mathrm{d}\nu_i=\int_R F(y)^i\nu_i\mathrm{d}y^1\dots\mathrm{d}y^n.$$

(Typically F will be a smooth vector field on a neighbourhood of $f(R)$.)

We define a region to be a connected open set such that the boundary is a finite union of closures of smooth hypersurfaces. From the implicit function theorem and the nondegeracy condition, around any point in a hypersurface, it can be written locally as a graph over one of the coordinate hyperplanes, i.e. one of the coordinates can be written as a function of the others. Given a region Ω and a point x on its boundary, a vector T is defined to leave Ω if there is a sufficiently small ϵ such that $\forall s\in(-\epsilon,0):x+sT\in\Omega$ and $\forall s\in(0,\epsilon):x+sT\notin\Omega$. Given a region Ω with boundary Σ and a point $x\in\Sigma$ with a neighbourhood N_x, a nondegenerate parameterisation $f:\Sigma\cap N_x$ is defined to be outward pointing if, for every outward vector T at x, $\nu_iT^i\geq 0$. Recall that if Ω is a region, if its boundary is a finite union of closures of smooth hypersurfaces all of which have parameterisations with outward normal 1-forms[1]. If Ω is a region with boundary $\partial\Omega$, $M\in\mathbb{N}$, $\{f_\alpha:R_\alpha\to\mathbb{R}^{1+n}\}_{\alpha=1}^M$ is a collection of nondegenerate parameterisations of smooth

[1]Be aware that, in Minkoswski space, if ν is an outward 1-form, it does not mean that the vector with components $\nu^i=\eta^{ij}\nu_j$ is an outward vector. This fact is an unavoidable consequence of the fact that $\eta_{00}=-1$ but $\eta_{ii}=1$ for $i\neq 0$.

hypersurfaces such that the closure of $\bigcup_{\alpha=1}^{M} f_\alpha(R_\alpha)$ is the boundary of Ω and each normal is outward pointing, and $F : \partial\Omega \to \mathbb{R}^{1+n}$, then

$$\int_{\partial\Omega} F^i \mathrm{d}\nu_i = \sum_{i=1}^{M} \int_{f_\alpha(R_\alpha)} F^i \mathrm{d}\nu_i.$$

The right-hand side in the previous formula is independent of the choice of parameterisation on the left. In $\mathbb{R}^{1+n}$, we typically use dvol for $\mathrm{d}x^0 \ldots \mathrm{d}x^n$.

Theorem 1.1 (Divergence theorem)**.** *Let Ω be a region with boundary $\partial\Omega$. Let Ω' be an open set such that $\Omega \subset \Omega'$. If $F : \Omega' \to \mathbb{R}^{1+n}$ is C^1, then*

$$\int_{\Omega} \left(\partial_i F^i\right) \mathrm{d}vol = \int_{\partial\Omega} F^i \mathrm{d}\nu_i.$$

1.3. **Tensors in the Minksowski spacetime.** Consider $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$. This is as in the general case of $\mathbb{R}^N$, except that we now index coordinates from 0 to n, instead of 1 to N. Furthermore, we use the notation

$$x = (x^0, \vec{x}) = (x^0, x^1, \ldots, x^n),$$
$$\partial = (\partial_0, \vec{\partial}) = (\partial_0, \partial_1, \ldots, \partial_n).$$

Two tensors that are frequently, but not always, useful are given by

$$\eta_{ij} = \eta^{ij} = \begin{cases} -1 & \text{if } i = j = 0, \\ 1 & \text{if } i = j \neq 0, \\ 0 & \text{if } i \neq j \end{cases}.$$

Both of these are called the Minkowski metric. Observe that these are symmetric and that

$$\eta_{ik}\eta^{kj} = \delta_i^j.$$

Minkowski space refers to $\mathbb{R}^{1+n}$ with η_{ij}.

If a hypersurface is given as the graph over the spatial coordinates, i.e. $f(\vec{y}) = (\phi(\vec{y}), vecy)$, then

$$\nu = (1, -\vec{\partial}\phi),$$
$$\nu_0 = 1,$$
$$\nu_i = -\partial_i \phi \qquad \text{if } i \geq 1.$$

We use the following norms

$$|\vec{\partial} u| = \left(\sum_{i=1}^{n} |\partial_i u|^2 \right)^{1/2},$$
$$|u|_1 = \left(\sum_{i=0}^{n} |\partial_i u|^2 + |u|^2 \right)^{1/2}.$$

2. Quasi-linear waves: An introduction

2.1. The basic form.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^{1+n}$.*

Condition $0(\Omega)$ is that Ω is connected with nonempty interior, that G^{ij}, B^i, A, and F are measurable tensor fields on Ω, that $G^{ij} = G^{ji}$, and that there is a second-order differential operator L on Ω given by

$$Lu = G^{ij}\partial_i\partial_j u + B^i\partial_i u + Au$$

For $r \in \mathbb{N} \cap \{\infty\}$, condition $1(s, \Omega)$ is that condition $0(\Omega)$ holds and that the tensor fields G^{ij}, B^i, A, and F are functions of $x \in \Omega$ that are C^r_b in the interior of Ω and have limits on the boundary of Ω for all their partial derivatives of orders up to k.

Condition $1Q(\Omega)$ is that condition $0(\Omega)$ holds and that G^{ij}, B^i, A, and F are C^∞ functions on $\Omega \times \mathbb{R} \times \mathbb{R}^{1+n}$ and are understood to be functions of $(x, u, \partial u)$.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^{1+n}$ and condition $0(\Omega)$ hold. For $\epsilon > 0$, G is ϵ close to η if*

$$\max_{l,m\in\{0,\dots,n\}} \left| G^{lm} - \eta^{lm} \right| \leq \frac{1}{(n+1) + \frac{(n+1)^2}{2}}\epsilon.$$

2.2. Energy and momentum.

Definition 2.3. *Assume condition $1(1, \Omega)$ from definition 2.1 holds. An energy-momentum-stress tensor is defined as a local[2] map from $C^1(\Omega)$ to tensor fields by*

$$\mathcal{T}[u]^i{}_j = G^{ik}\partial_k u \partial_j u - \frac{1}{2}\delta^i{}_j \left(G^{lm}\partial_l u \partial_m u + u^2 \right).$$

Given a vector-field $X \in C^1(\Omega; \mathbb{R}^{1+n})$, we define the associated momentum to be

$$\mathcal{P}_{(X)}[u]^i = \mathcal{T}[u]^i{}_j X^j.$$

Given a hypersurface Σ, we define the energy of u associated with X on the hypersurface Σ to be

$$E_X[u](\Sigma) = \int_\Sigma \mathcal{P}_{(X)}[u]^i \mathrm{d}\nu_i.$$

If some arguments are clear from context, they will be dropped. For example, we will often write $\mathcal{T}^i{}_j$ for $\mathcal{T}^i{}_j$. Frequently, we will use the notation

$$E_X(t) = E_X(\{t\} \times \mathbb{R}^n).$$

For other problems and in other cases, it may be useful to consider other energy-momentum tensors and to construct from them momenta and energies.

[2]A map on C^k is local if its value at a point depends only on the first k derivatives of its argument evaluated at the point.

Lemma 2.4 (Quasi-linear Klein-Gordon energy estimate in divergence-form)**.** *Assume condition* $1(1,\Omega)$ *from definition 2.1. If* $X \in C^1(\Omega;\mathbb{R}^{1+n})$ *and* $u \in C^2(\Omega)$*, then*

$$\partial_i \mathcal{T}^i{}_j = \Big((\partial_i G^{ik})\partial_k u + Lu - B^i\partial_i u - (a+1)u\Big)\,\partial_j u - \frac{1}{2}(\partial_j G^{lm})\partial_l u \partial_m u,$$
$$\partial_i {\mathcal{P}_{(X)}}^i = \partial_i \mathcal{T}^i{}_j X^j + \mathcal{T}^i{}_j \partial_i X^j.$$

Proof. By direct computation,

$$\begin{aligned}\partial_i \mathcal{T}^i{}_j &= (\partial_i G^{ik})\partial_k u \partial_j u + G^{ik}\partial_i\partial_k u \partial_j u + G^{ik}\partial_k u \partial_i \partial_j u \\ &\quad - \frac{1}{2}(\partial_i G^{lm})\partial_l u \partial_m u - G^{lm}\partial_l u \partial_i \partial_m u + u\partial_j u.\end{aligned}$$

The third and fifth terms cancel. Substituting the definition of L gives the remaining result.

The second result follows from the product rule. □

Lemma 2.5 (Positivity of energy density)**.** *Assume condition* $1(1,\Omega)$ *from definition 2.1,* $\epsilon > 0$*, and* G *is* ϵ *close to* η *as in definition 2.2.*

If $u \in C^1(\Omega)$ *and* $X \in C^1(\Omega;\mathbb{R}^{1+n})$

$$\left|-\mathcal{T}^0{}_0 - \frac{1}{2}|u|_1^2\right| \le \epsilon |u|_1^2,$$

If $i \neq 0$

$$\left|\mathcal{T}^i{}_0\right| \le (1+\epsilon)\,\frac{1}{2}|u|_1^2.$$

Remark 2.6. *Observe that the first estimate implies that* $\mathcal{T}_0^0$ *and* $|u|_1^2$ *are not only uniformly equivalent, but that the constants relating the quantities are very close to* $1/2$.

Proof. First observe that

$$\eta^{lm}\partial_l u \partial_m u = -(\partial_t u)^2 + |\vec{\partial} u|^2,$$

so

$$-\eta^{00}\partial_t u \partial_t u + \frac{1}{2}\left(\eta^{lm}\partial_l u \partial_m u + u^2\right) = \frac{1}{2}\left(|\partial_t u|^2 + |\vec{\partial} u|^2 + |u|^2\right) = \frac{1}{2}|u|_1^2.$$

Furthermore, for $i \neq 0$,

$$\begin{aligned}&\eta^{ik}\partial_k u \partial_t u = \partial_i u \partial_t u,\\ &\left|\eta^{ik}\partial_k u \partial_t u\right| = |\partial_i u||\partial_t u| \le \frac{1}{2}\left(|\vec{\partial} u|^2 + |\partial_t u|^2\right) \le \frac{1}{2}|u|_1^2.\end{aligned}$$

Turning to the terms to be estimated in the current lemma, one finds

$$\begin{aligned}-\mathcal{T}^0{}_0 &= -G^{0k}\partial_k u \partial_t u + \frac{1}{2}G^{lm}\partial_l u \partial_m u \\ &= -\eta^{0k}\partial_k u \partial_t u + \frac{1}{2}\eta^{lm}\partial_l u \partial_m u \\ &\qquad + (-G^{0k} + \eta^{0k})\partial_k u \partial_t u + \frac{1}{2}\left(G^{lm} - \eta^{lm}\right)\partial_l u \partial_m u,\\ \left|-\mathcal{T}^0{}_0 - \frac{1}{2}|u|_1^2\right| &\le \max_k |-G^{0k} + \eta^{0k}|(n+1)|\partial u|^2 + \frac{1}{2}\max_{lm}|-G^{lm} + \eta^{lm}|(n+1)^2|\partial u|^2 \\ &\le \epsilon |u|_1^2.\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{T}^j{}_0 &= G^{jk}\partial_k u \partial_t u, \\
&= \eta^{jk}\partial_k u\partial_t u + (G^{jk}-\eta^{jk})\partial_k u\partial_t u, \\
\left|\mathcal{T}^j{}_0\right| &\leq \frac{1}{2}(1+\epsilon)|u|_1^2.
\end{aligned}$$

□

3. Uniqueness of solutions and finite speed of propogation

Theorem 3.1 (C^2 uniqueness of the zero solution)**.** *Let $\epsilon \in [0,1)$, $T > 0$ and $\vec{x}_0 \in \mathbb{R}^n$ and let*

$$\begin{aligned}
c &= \frac{1+\epsilon}{1-\epsilon}, \\
B &= \left\{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{x_0}| \leq cT\right\}, \\
\Lambda &= \left\{(t,\vec{x}) \in \mathbb{R}\times\mathbb{R}^n : t \in [0,T], |\vec{x}-\vec{x_0}| \leq c(T-t)\right\}.
\end{aligned}$$

Assume either

- *that condition $1(1,\Lambda)$ holds*[3] *and G is ϵ close to η, or*
- *that condition $1Q(\Lambda)$ holds and that $G(x,0,0)$ is ϵ close to η for $x \in \Lambda$.*

Furthermore, suppose there is a constant C such that everywhere in Λ

$$|F| \leq C|u|_1.$$

If $u \in C^2$ is a solution of

$$Lu = F, \qquad \forall \vec{x} \in B : u(0,\vec{x}) = 0, \partial_t u(0,\vec{x}) = 0,$$

then $u = 0$ in Λ.

Proof. To begin assume condition $1(\infty,\Lambda)$ and G is ϵ close to η.

Step 1: Foliate the cone. Define, for $\tau \in [0,T]$, $\phi_\tau : B \to \mathbb{R}$ by

$$\phi_\tau(\vec{x}) = \tau - \frac{\tau}{cT}|\vec{x}-\vec{x_0}|,$$

and let

$$\begin{aligned}
\Sigma_\tau &= \left\{(t,\vec{x}) : \vec{x} \in B, t = \phi_\tau(\vec{x})\right\}, \\
\Lambda_\tau &= \bigcup_{\tau' \in [0,\tau]} \Sigma_{\tau'}.
\end{aligned}$$

Observe that

$$\Lambda = \Lambda_T.$$

Also, Σ_τ is the graph of $t = \phi_\tau$, so its normal is given by

$$\nu_0 = 1, \qquad \nu_i = -\partial_i\phi_\tau = \frac{\tau}{cT}\frac{(x^i - x^i_0)}{|\vec{x}-\vec{x}_0|}.$$

[3] In lecture, I mistakenly said condition $1(\infty,\Lambda)$. We only need G^{ij}, B^i, A, F to be C^1 to hold. Furthermore, in step 5, when we pass to the quasilinear case, we need to allow G to be merely C^1.

From now on, if no argument is given, quantities are evaluated at $(\phi_\tau(\vec{x}), \vec{x})$. We will only consider $\tau \in [0, T]$ and $\vec{x} \in B$.

Step 2: Estimate the energy on slices. Consider the energy associated with $T = -\partial_t$, which has $T^0 = -1$ and $T^i = 0$ for $i \neq 0$. Observe that for $i \in \{0, 1, \ldots, n\}$,

$$\mathcal{P}_{(X)}{}^i = -\mathcal{T}^i{}_j \partial_t^j = -\mathcal{T}^i{}_0.$$

Thus,

$$\begin{aligned} E(\Sigma_\tau) &= \int_{\Sigma_\tau} \mathcal{P}^i \mathrm{d}\nu_i \\ &= \int_B \mathcal{P}^i \nu_i \mathrm{d}^n\vec{x} \\ &\geq \int_B \mathcal{P}^0 \cdot 1 \mathrm{d}^n\vec{x} - \int_B |\vec{\mathcal{P}}||\vec{\partial}\phi_\tau| \mathrm{d}^n\vec{x} \\ &\geq (1-\epsilon)\frac{1}{2}\int_B \frac{1}{2}|u|_1^2 \mathrm{d}^n\vec{x} - (1+\epsilon)\frac{1}{2}\frac{\tau}{cT}\int_B |u|_1^2 \mathrm{d}^n\vec{x} \\ &\geq (1-\epsilon)\left(1 - \frac{\tau}{T}\right)\frac{1}{2}\int_B |u|_1^2 \mathrm{d}^n\vec{x}. \end{aligned}$$

Step 3: Estimate the change in energy. On the other hand, from the energy estimate in divergence form, we find there is a constant C_1 such that

$$|\partial_i \mathcal{P}^i| \leq C_1 |u|_1^2.$$

We now wish to integrate this over $\Lambda_\tau = \bigcup_{\tau' \in [0,\tau]} \Sigma_{\tau'}$. In doing so, it is convenient to introduce new coordinates $(\tau, \vec{x})$. Observe that

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}\tau'} &= 1 - \frac{1}{cT}|\vec{x} - \vec{x}_0|, \\ \left|\frac{\mathrm{d}t}{\mathrm{d}\tau'}\right| &\leq 1. \end{aligned}$$

This leads to the following

$$\begin{aligned} E(\Sigma_\tau) - E(\Sigma_0) &\leq \int_{\Lambda_\tau} |\partial_i \mathcal{P}^i| \mathrm{d}t \mathrm{d}^n\vec{x} \\ &\leq C_1 \int_{\vec{x} \in B} \int_0^\tau |u|_1^2 \left|\frac{\mathrm{d}t}{\mathrm{d}\tau'}\right| \mathrm{d}\tau' \mathrm{d}^n\vec{x} \\ &\leq \int_0^\tau C_1 \int_{\vec{x} \in B} |u|_1^2 \mathrm{d}^n\vec{x} \mathrm{d}\tau' \\ &\leq \int_0^\tau C_1 \int_{\vec{x} \in B} |u|_1^2 \mathrm{d}^n\vec{x} \mathrm{d}\tau'. \end{aligned}$$

Step 4: Finish with Gronwall's inequality. Let $T_0 < T$. Thus, there is a constant $C_2 > 0$ such that for $\tau \in [0, T_0]$

$$(1-\epsilon)\left(1 - \frac{\tau}{T}\right) \geq (1-\epsilon)\left(1 - \frac{T_0}{T}\right) \geq C_2.$$

Let $f(\tau) = \frac{1}{2}\int_B |u|_1^2 \mathrm{d}^n\vec{x}$. Combining the previous two steps, we find there is a constant C_3 such that for $\tau \in [0, T_0]$,

$$f(\tau) \leq f(0) + \int_0^\tau C_3 f(\tau')\mathrm{d}\tau'.$$

Since $f(0) = 0$, by Gronwall's inequality, we find $f(\tau) = 0$ for $\tau \in [0, T_0]$. This implies that $u = 0$ in Λ_{T_0} for all $T_0 < T$. Taking the limit as $T_0 \to T$ and using the continuity of u, u must vanish in $\Lambda = \Lambda_T$.

Step 5: Treat the quasi-linear case. Since $\epsilon < 1$, we can choose $\delta > 0$ be such that $\epsilon + \delta \in (0, 1)$. Let T_0 be the supremum value of t such that G is ϵ close to η on Λ_t. Since $G(x, 0, 0)$ is ϵ close to η, $T_0 \geq 0$. Furthermore, since $u \in C^2$, it means that G^{ij} is at least C^1, so $T_0 > 0$. However, if $T_0 < T$ by applying the theorem in the $1(1, \Lambda_{T_0})$, we find u is zero on Λ_{T_0}, which, by continuity of G implies, that G is $\epsilon + \delta$ close to η on a slightly larger set, contradicting the definition of T_0. Taking the limit as $\delta \searrow 0$, we find $T_0 \geq T$. Applying the theorem in the $1(1, \Lambda_T)$ shows $u = 0$ on Λ. □

4. Existence and uniqueness for linear equations

Theorem 4.1. *Let $s \in \mathbb{Z}$,*