Lecture 6

14/2/2018

Recall from Lecture 5:

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Thm 1 (GWP in $\dot{H}^1(\mathbb{R}^3)$ for the case of initial data of small energy)

There exists $\delta>0$ small such that $\forall\,(u_0,u_1)\in\dot{H}^1\times L^2$ with $\|(u_0,u_1)\|_{\dot{H}^1\times L^2}<\delta$, NLW_5 admits a unique global solution $(u,\partial_t u)\in C(\mathbb{R};\dot{H}^1\times L^2)$

Moreover, $\|(u,\partial_t u)\|_{L^\infty(\mathbb{R};\dot{H}^1\times L^2)}$, $\|u\|_{L^4(\mathbb{R},L^{12})}\leq 2\|(u_0,u_1)\|_{\dot{H}^1\times L^2}\leq 2\delta$.

Thm 2 (LWP in $\dot{H}^1$ of NLW_5 on $\mathbb{R}^3$)

Let $(u_0,u_1)\in\dot{H}^1\times L^2$ and let $t_0\in\mathbb{R}$, $I\subset\mathbb{R}$ interval s.t. $t_0\in I$.

There exists $\delta>0$ small s.t. if $\|S(\cdot-t_0)(u_0,u_1)\|_{L^4(I;L^{12})}<\delta$,

then there exists a unique solution u of NLW_5 on I:

$$\begin{cases} (u,\partial_t u)\in C(I;\dot{H}^1\times L^2) \\ \|(u,\partial_t u)\|_{L^\infty(I;\dot{H}^1\times L^2)}\leq 2\|(u_0,u_1)\|_{\dot{H}^1\times L^2} \\ \|u\|_{L^4(I;L^{12})}\leq 2\delta \end{cases}$$

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Global well-posedness of the defocusing energy-critical quintic NLW on $\mathbb{R}^3$ in $\dot{H}^1$

$$(NLW_5)\begin{cases} -\partial_t^2 u + \Delta u = u^5 \\ u(t_0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \\ \partial_t u(t_0) = u_1 \in L^2(\mathbb{R}^3) \end{cases}$$

Thm (GWP of NLW_5 in $\dot{H}^1(\mathbb{R}^3)$, Shatah-Struwe '93-'94)
For any $(u_0, u_1) \in \dot{H}^1 \times L^2$, NLW_5 admits a unique global solution in $C(\mathbb{R}, \dot{H}^1 \times L^2) \ni (u, \partial_t u)$.

T. Tao, 'Nonlinear dispersive equations': Prop. 5.3, 5.5, 5.6, 5.7
Shatah-Struwe, 'Geometric wave equations': Lemma 6.1

Strategy:
- Step 1: Finite time blowup implies concentration of the potential energy (Tao)
- Step 2: Non-concentration of the potential energy (Shatah-Struwe) 'energy flux machinery'

$\boxed{\text{Lemma 1}}$ Assume $\psi \in L^\infty(\mathbb{R}^3)$ s.t. $\nabla\psi \in L^3(\mathbb{R}^3)$. Then $\exists\, C = C(\psi) > 0$ s.t.

$$\left\| u(\cdot)\, \psi\!\left(\frac{\cdot - x}{R}\right) \right\|_{\dot H^1} \le C\, \|u\|_{\dot H^1\left(\operatorname{supp} \psi\left(\frac{\cdot - x}{R}\right)\right)}, \quad \forall u \in \dot H^1(\mathbb{R}^3) \quad \forall R > 0$$

$\boxed{\text{Lemma 2}}$ (Tao's book, Exe. 3.11, p. 128) (Gluing of strong solutions)

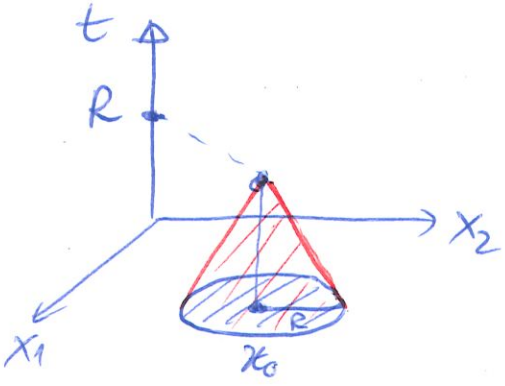
Def: A strong solution in H^s of NLW_5 is a sol $u \in C(I; H^s(\mathbb{R}^3))$ satisfying Duhamel's formula.

Let I, I' be time intervals s.t. $I \cap I' = \{t_0\}$.

Let u, u' be strong H^s solutions of NLW_5 on $I \times \mathbb{R}^3$, $I' \times \mathbb{R}^3$ respectively with data $\begin{cases} u(t_0) = u'(t_0) \\ \partial_t u(t_0) = \partial_t u'(t_0) \end{cases}$

Then $\begin{cases} u \text{ on } I \times \mathbb{R}^3 \\ u' \text{ on } I' \times \mathbb{R}^3 \end{cases}$ is a solution of NLW_5 on $(I \cup I') \times \mathbb{R}^3$.

$\boxed{\text{Lemma 3}}$ (Finite speed of propagation, Tao's book, Prop. 3.3, p.123)



Let I time interval, $0 \in I$.

Let $u, u' \in C^2_{t,x}(I \times \mathbb{R}^3 \to \mathbb{R})$ classical solutions of NLW s.t.

$$(u(0), \partial_t u(0))(x) = (u'(0), \partial_t u'(0))(x),$$

$$\forall x \in B(x_0, R) = \{x \in \mathbb{R}^3 ; |x - x_0| \leq R\}$$

Then $u(x,t) \equiv u'(x,t)$ for all (x,t) in:

$$\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} ; |x - x_0| \leq R - |t|\}$$

$\boxed{\text{Blowup criterion \#2}}$ (Blowup implies energy concentration) (Prop. 5.3, Tao)

If u is a $\dot{H}^1 \times L^2$ solution with maximal time of existence $0 < T_* < \infty$, then $\exists x \in \mathbb{R}^3$, $\exists \varepsilon_0 > 0$ absolute const. s.t.

$$\boxed{\limsup_{t \to T_*^-} E_{B(x, 3(T_* - t))}(u(t), \partial_t u(t)) \geq \varepsilon_0}$$

$$E(u(t),\partial_t u(t)) := \int_{\mathbb{R}^3}\Bigg(\underbrace{\frac{(\partial_t u)^2+|\nabla u|^2}{2}}_{\text{kinetic}} + \underbrace{\frac{|u|^6}{6}}_{\text{potential}}\Bigg)(t,x)\,dx$$ total energy

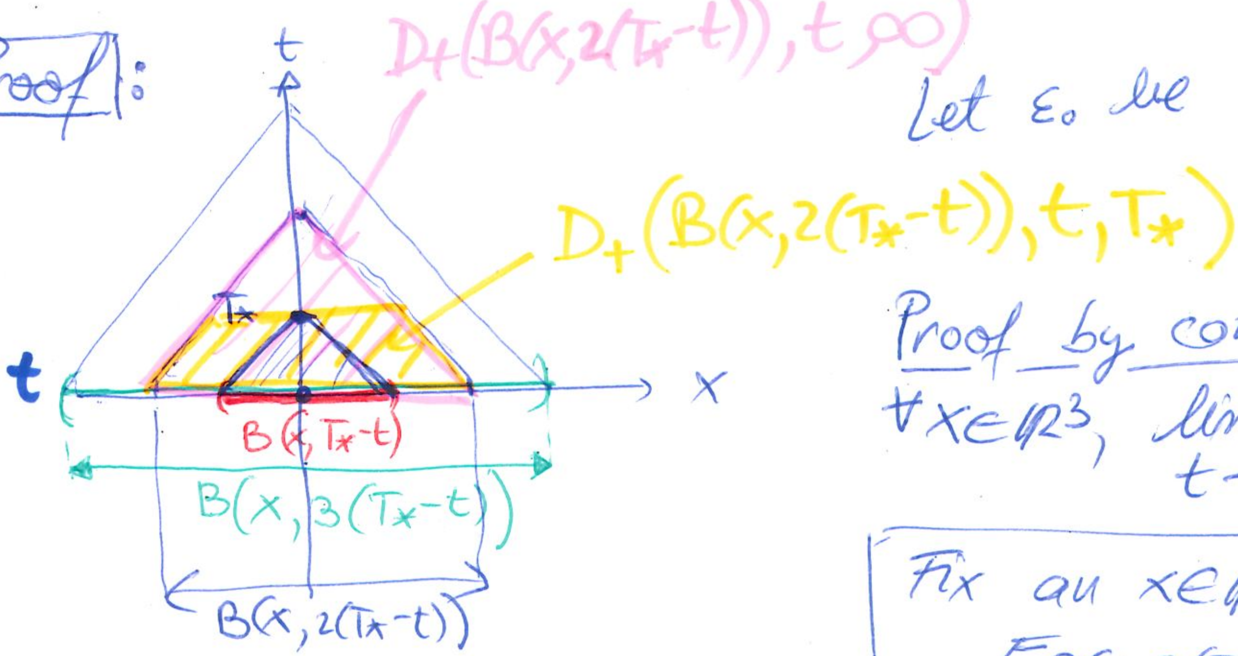
notations

- $E_\Omega(u,\partial_t u) = \int_\Omega$ — " —— dx for $\Omega \subseteq \mathbb{R}^3$
- $t_0 < t_1 \le \infty$, $\Omega \subset \mathbb{R}^3$

$$D_+(\Omega, t_0, t_1) :\overset{def}{=} \{(t,y)\in[t_0,t_1)\times\mathbb{R}^3 ;\ B(y,t-t_0)\subseteq\Omega\}$$

<u>truncated forward domain of dependence</u>

$\boxed{\text{Proof}}$:



Let ε_0 be s.t. $2C_0\varepsilon_0^{\frac{1}{2}} < \delta$, where δ is as in the LWP & C_0 is an absolute const. that will appear later in the proof.

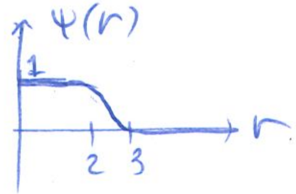
<u>Proof by contradiction</u>:

$\forall x\in\mathbb{R}^3$, $\limsup\limits_{t\to T_*^-} E_{B(x,3(T_*-t))}(u(t),\partial_t u(t)) < \varepsilon_0$

$$\boxed{\begin{array}{l}\text{Fix an } x\in\mathbb{R}^3 \Rightarrow \exists\, t=t(x) \text{ s.t.}\\ \quad E_{B(x,3(T_*-t))}(u(t),\partial_t u(t)) < \varepsilon_0 .\end{array}}$$

Localization of data on $B(x,3(T_*-t))$

Let $\psi_1(r) = \begin{cases} 1, & \text{if } 0<r<2 \\ 0, & \text{if } r>3 \end{cases}$ smooth



Let $(\tilde{u}_0(y), \tilde{u}_1(y)) := \left(u(t,y)\, \psi_1\left(\frac{|y-x|}{T_*-t}\right),\ \partial_t u(t,y)\, \psi_1\left(\frac{|y-x|}{T_*-t}\right)\right)$

By Lemma 1: $\begin{cases} \|\tilde{u}_0\|_{\dot{H}^1} \leq C \|u(t)\|_{\dot{H}^1(B(x,3(T_*-t)))} \leq C_0 \varepsilon_0^{\frac{1}{2}}. \\ \|\tilde{u}_1\|_{L^2} \leq \|\partial_t u\|_{L^2(B(x,3(T_*-t)))} \leq C_0 \varepsilon_0^{1/2} \end{cases}$

$$\Longrightarrow \|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2} \leq 2C_0 \cdot \varepsilon_0^{1/2} < \delta$$

Small energy data GWP

By small energy data GWP $\Longrightarrow \exists!$ global solution $\tilde{u}$ of NLW with $(\tilde{u}, \partial_t \tilde{u})(t) = (\tilde{u}_0, \tilde{u}_1)$.

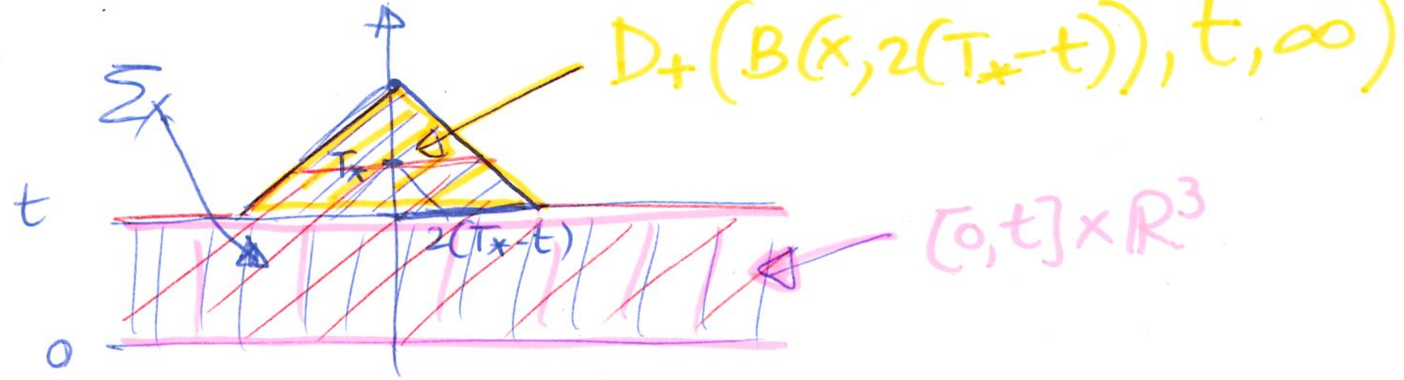
Finite speed of propagation

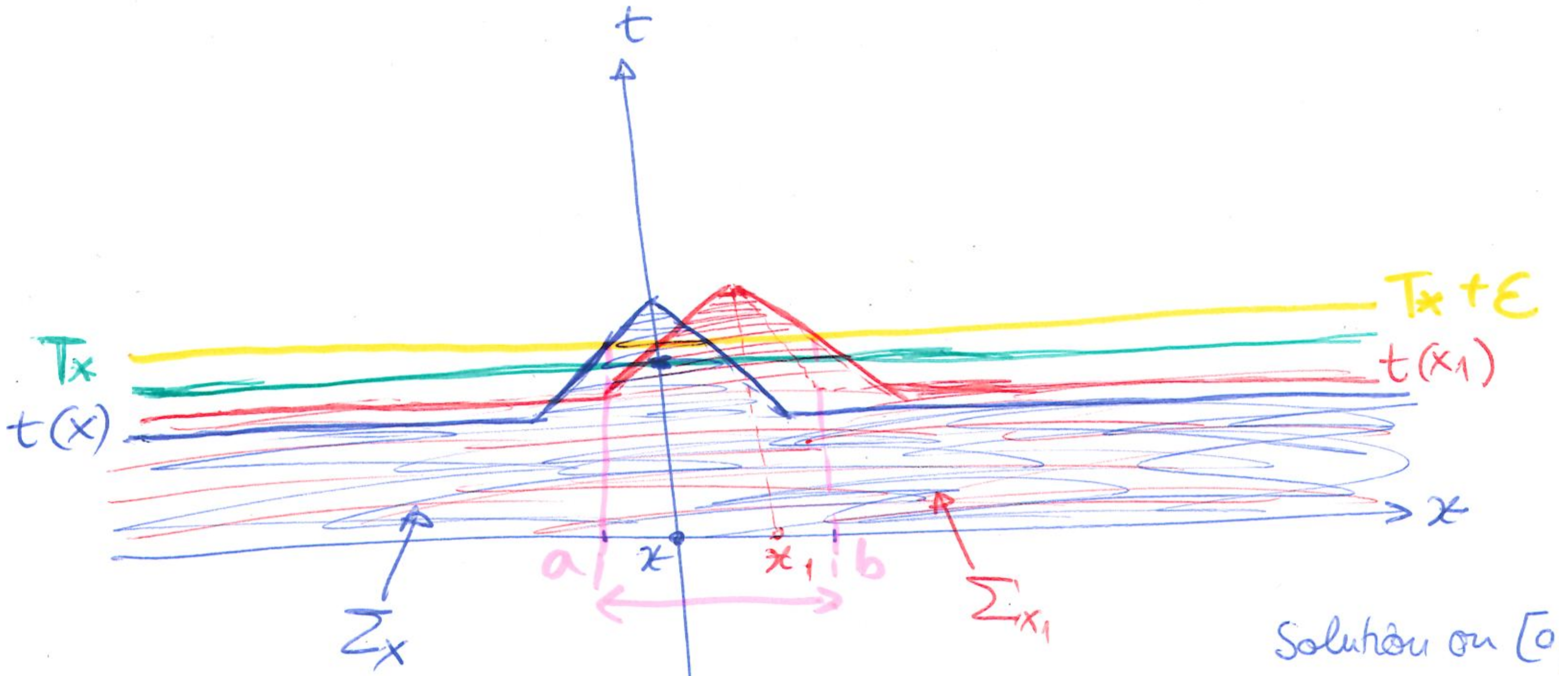
By finite speed of propagation, since $(\tilde{u}_0, \tilde{u}_1)(y) \equiv (u(t), \partial_t u(t))(y)$ on $y \in B(x, 2(T_*-t))$, it follows that

$$u(t,x) = \tilde{u}(t,x) \text{ for } (t,x) \in \Delta_+(B(x,2(T_*-t)), t, T_*)$$

We glue this with the solution u that already (6)
existed on $[0,t]\times\mathbb{R}^3$ and get a solution $u_{(x)}$ defined on:

$$\Sigma'_x := \left([0, t(x)]\times\mathbb{R}^3\right) \cup D_+\left(B(x, 2(T_*-t)), t(x), \infty\right)$$





Solution on $[0, T_* + \epsilon] \times \{a \le |x| \le b\}$

Σ_x contains a neighbourhood of (x, T_x)

$u_{(x)}$ coincides with $u_{(x_1)}$ (by finite speed of propagation) on $\Sigma_x \cap \Sigma_{x_1}$.

localization of data outside $B(0,R)$

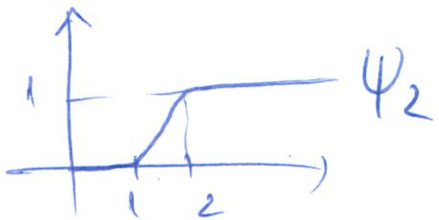
Fix $\tilde{\varepsilon}_0$ s.t. $C\tilde{\varepsilon}_0^{1/2} < \delta$. (You can take $\tilde{\varepsilon}_0 = \varepsilon_0$ for e.g.)

- $(u_0, u_1) \in \dot{H}^1 \times L^2 \iff E(u_0, u_1) < \infty \Rightarrow \exists\, R$ suff. large s.t. $R = R(\tilde{\varepsilon}_0)$

(8)

$$E_{\mathbb{R}^3 \setminus B(0,R)}(u_0, u_1) < \tilde{\varepsilon}_0.$$

- Let ψ_2 be a smooth funct. $\psi_2(r) = \begin{cases} 1, & \text{if } r > 2 \\ 0, & \text{if } 0 < r < 1 \end{cases}$



$$(u_0', u_1')(y) = \left(u_0(y)\psi_2\left(\frac{|y|}{R}\right), u_1(y)\psi_2\left(\frac{|y|}{R}\right)\right)$$

By <u>Lemma 1</u>: $\|(u_0', u_1')\|_{\dot{H}^1 \times L^2} \leq C\|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0,R))}$

$$\leq C_0 \tilde{\varepsilon}_0^{1/2} < \delta$$

small data GWP

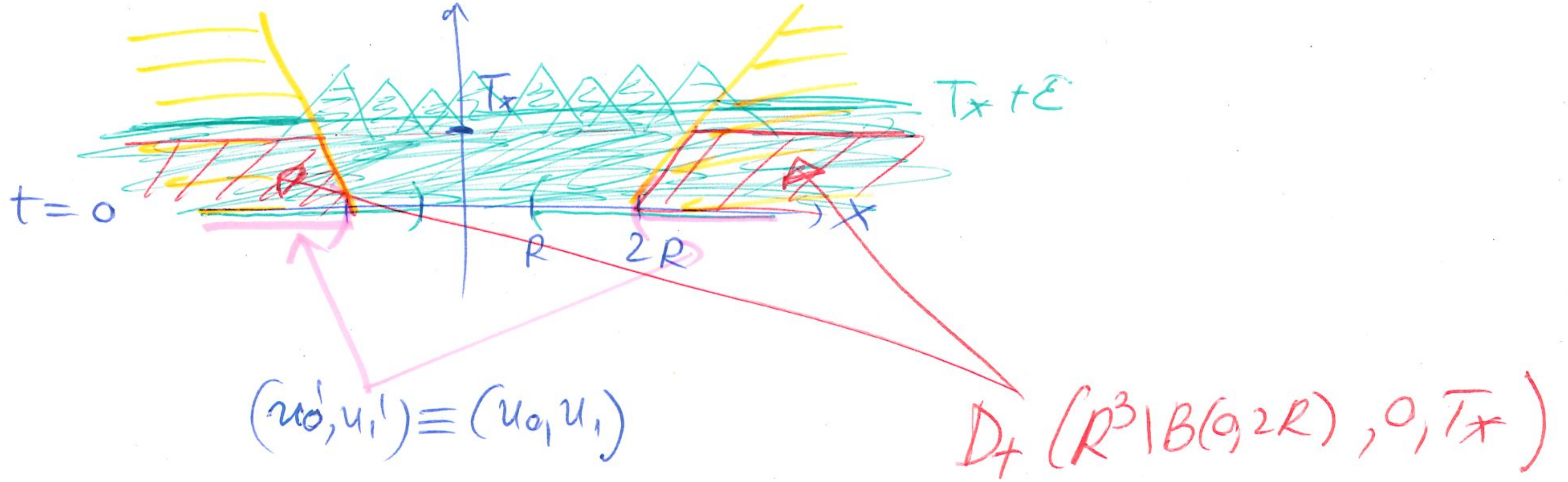
$\Rightarrow$ By small energy data GWP $\exists!$ global sol. $u_{>2R}$ with data $(u_{>2R}(0), \partial_t u_{>2R}(0)) = (u_0', u_1')$.

Since $(u_0', u_1')(x) = (u_0, u_1)(x)$ for $x \in \mathbb{R}^3 \setminus B(0, 2R)$

finite speed of propag.

$\Rightarrow$ by finite speed of propagation: ⑨

$$u(t,x) = u_{>2R}(t,x), \ \forall (t,x) \in D_+(\mathbb{R}^3 \setminus B(0,2R), 0, T_*)$$



We glue $u_{>2R}$ with finitely many $u_{(x)}$ and so we get an extension of u to:

$$[0, T_* + \varepsilon] \times \mathbb{R}^3 \subseteq D_+(\mathbb{R}^3 \setminus B(0,2R), 0, T_* + \varepsilon) \cup \Big(\bigcup_{\substack{\text{finitely many} \\ x}} \Sigma_x \Big)$$

This contradicts the fact that T_* is the maximal time of existence of u

$\Rightarrow$ initial assumption false

$\Rightarrow$ blowup criterion #2 holds.

Compactness argument to cover the compact $[0, T_* + \varepsilon] \times \mathbb{R}^3 \setminus D_+(\mathbb{R}^3 \setminus B(0,2R), 0, T_*)$ with finitely many Σ_x

Prop. 5.5, Tao (Exterior energy decay)

Let u be as in the previous prop. Then $\exists\, x \in \mathbb{R}^3$

$$\inf_{\sigma > 0} \limsup_{t \to T_*^-} E_{B(x, T_* - t + \sigma) \setminus B(x, T_* - t)}(u(t), \partial_t u(t)) = 0$$

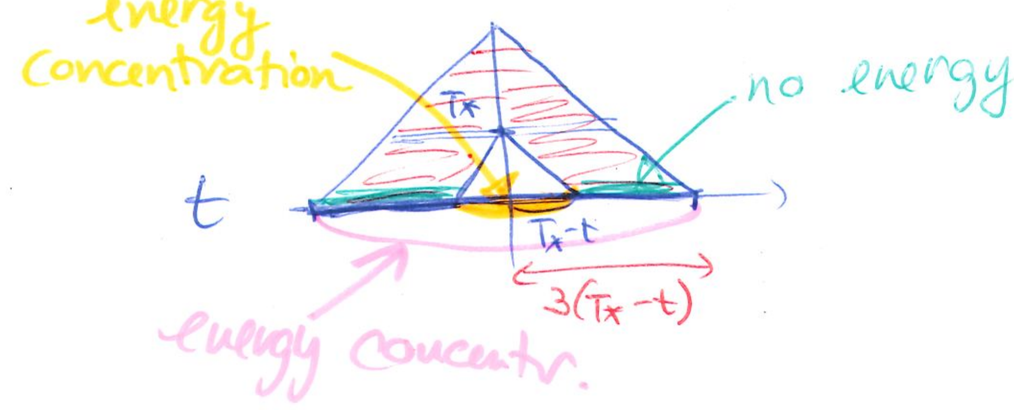
In particular, $\forall\, 1 < A < \infty$, we have

$$\boxed{\limsup_{t \to T_*^-} E_{B(x, A(T_* - t)) \setminus B(x, T_* - t)}(u(t), \partial_t u(t)) = 0}$$

Corollary of Blowup criterion #2 & Prop. 5.5. with $A = 3$:

Finite time blowup implies $\exists\, x \in \mathbb{R}^3$ s.t.

$$\boxed{\limsup_{t \to T_*^-} E_{B(x, T_* - t)}(u(t), \partial_t u(t)) \geq \varepsilon_0.}$$



$\boxed{\text{Blowup criterion \#3}}$ Blowup implies spacetime norm concentration, Prop. 5.6, Tao)

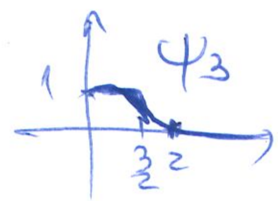
Let u be as in the blowup criterion #2 with $E(u, \partial_t u) \leq E_0$.
Then $\exists\ \varepsilon_1 = \varepsilon_1(E_0)$, $\exists\, x \in \mathbb{R}^3$ s.t.:

$$\boxed{\limsup_{t \to T_*^-} \| S(\cdot - t)(u(t), \partial_t u(t)) \|_{L^4 L^{12}(D_+(B(x, 2(T_* - t)), t, T_*))} \geq \varepsilon_1 .}$$

Proof: By contradiction, $\forall\, x \in \mathbb{R}^3$, $\exists\, t = t(x)$ s.t.

$$\| S(\cdot - t)(u(t), \partial_t u(t)) \|_{L^4 L^{12}(D_+(B(x, 2(T_* - t)), t, T_*))} < \varepsilon_1 .$$

ψ_3 smooth s.t. $\psi_3 = \begin{cases} 1, & \text{if } 0 < r < \frac{3}{2} \\ 0, & \text{if } r > 2 \end{cases}$



$$(\tilde{u}_0, \tilde{u}_1)(y) = \left(u(t,y)\, \psi_3\left(\frac{|y-x|}{T_* - t}\right), \partial_t u(t,y)\, \psi_3\left(\frac{|y-x|}{T_* - t}\right) \right)$$

It suffices to show: $\boxed{\| S(\cdot - t)(\tilde{u}_0, \tilde{u}_1) \|_{L^4 L^{12}([t, T_*] \times \mathbb{R}^3)} \leq c\,\varepsilon_1}$.

$$\|S(\cdot - t)(\tilde{u}_0, \tilde{u}_1)\|_{L^4 L^{12}(\mathbb{R}\times\mathbb{R}^3)} \underset{\text{Strichartz}}{\leq} \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \infty \quad (12)$$

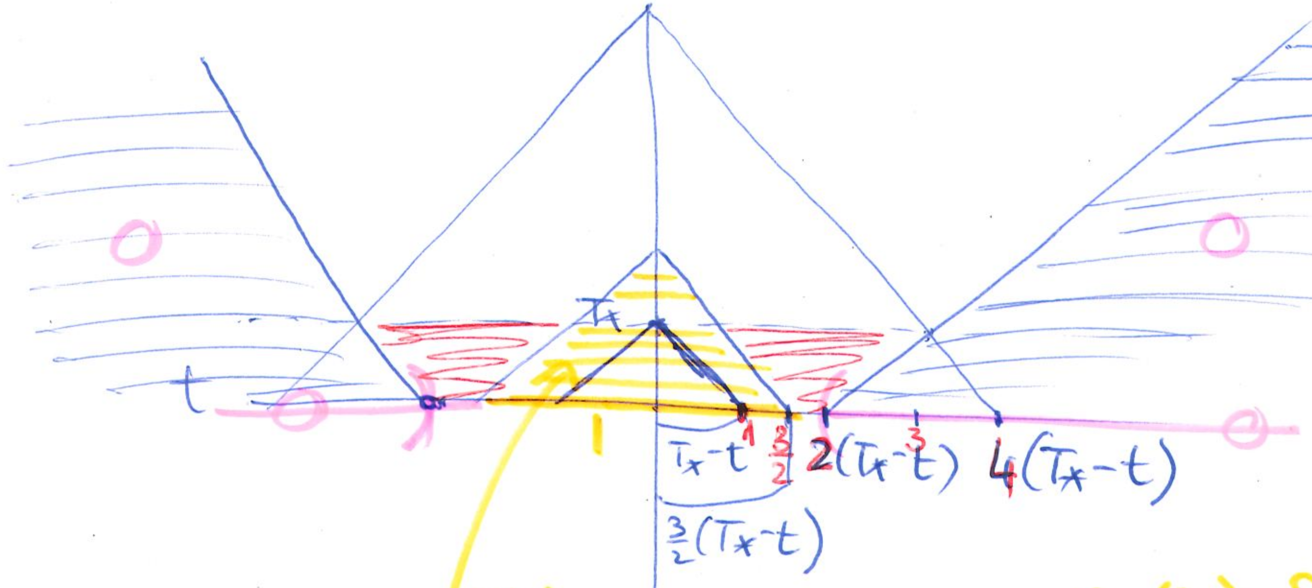
Then $\exists\ \eta > 0$ s.t.

$$\|S(\cdot - t)(\tilde{u}_0, \tilde{u}_1)\|_{L^4 L^{12}([t, T_+ + \eta]\times\mathbb{R}^3)} \leq 2C\varepsilon_1.$$

If $2C\varepsilon_1 < \delta$, with δ as in the LWP statement $\Rightarrow$

$\Rightarrow$ we build a solution $\tilde{u}$ to NLW_5 on $[t, T_+ + \eta]$ which coincides with our solution u on $D_+(B(x, \frac{3}{2}(T_+ - t)), t, T_+)$.

Then one proceeds as in the proof of the blowup criterion #2, and gets a contradiction to the maximality of T_+.



$S(\cdot - t)(\tilde{u}_0, \tilde{u}_1)$ coincides with $S(\cdot - t)(u(t), \partial_t u_t)$ hence is small

For the middle region: use localized version of Strichartz estimates

$$\|S(\cdot - t)(\tilde{u}_0, \tilde{u}_1)\|_{L^4 L^{12}(D_+(B(x, 4(T_* - t)) \setminus B(x, \frac{3}{2}(T_* - t)), t, T_*)}$$

$$\leq \|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2(B(x, 4(T_* - t)) \setminus B(x, \frac{3}{2}(T_* - t))}$$

$$\underset{\text{Lemma 1}}{\leq} C \left[E_{B(x, 4(T_* - t)) \setminus B(x, \frac{3}{2}(T_* - t))}(u(t), \partial_t u(t)) \right]^{1/2}$$

small from Prop. 5.5