Existence & uniqueness for linear equations

Def'n Schwartz class $\mathcal{S} = \{ f : \mathbb{R}^n \Rightarrow \mathbb{R} \mid \forall \alpha, \beta \ \exists C_{\alpha,\beta} : |\vec{x}^\alpha \vec{\partial}^\beta f(\vec{x})| < C_{\alpha,\beta} \}$

Th'm Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$.
$\mathcal{S}$ is dense in H^s and L^p for $s \in \mathbb{R}$, $p \in [1, \infty)$.

Th'm Let $s \in \mathbb{Z}$, $0 < T < \infty$. Assume cond'n $1(\infty, \mathbb{R}^{1+n})$. Assume $\varepsilon \in [0, \frac{1}{4})$
and G is ε close to η.
If $u \in C([0,T]; H^{s+1}) \cap C^1([0,T]; H^s)$
and $Lu \in L^1([0,T]; H^s)$,
then $\exists C_{s,T} : \forall t \in [0,T]$

$$\sum_{|\alpha| \le 1} \|\partial^\alpha u^{(t)}\|_{H^s} \le C_{s,T} \left(\sum_{|\alpha| \le 1} \|\partial^\alpha u^{(0)}\|_{H^s} + \int_0^t \|Lu(\tau, \cdot)\|_{H^s} \, d\tau \right)$$

s-Energy estimate)

pf C will denote a constant depending on C^{ij}, B^i, A, F and varying from line to line.

By density, it is sufficient to show this for $u \in \mathcal{S}$.

<u>Case 1</u> $s=0$

Consider $\Sigma_t = \{t\} \times \mathbb{R}^n$. Let $T = -\partial_t$

$$E[u](t) = \int_{\Sigma_t} T^0{}_0 \, d^n\vec{x}$$

$$\geq (1-\varepsilon) \frac{1}{2} \int_{\Sigma_t} |u|_1^2 \, d^n\vec{x}$$

Similarly

$$E[u](0) \leq (1+\varepsilon) \frac{1}{2} \int_{\Sigma_0} |u|_1^2 \, d^n\vec{x}$$

By the energy estimate in divergence-form

$$E[u](t) - E[u](0)$$

$$\leq \int \Big((\partial_i C^{ik}) \partial_k u - B^i \partial_i u - (A+1) u \Big) \partial_0 u - \frac{1}{2} (\partial_0 C^{lm}) \partial_l u \, \partial_m u + (Lu) \partial_0 u \Big) \, d^n x \, dt'$$

$$\leq C \int |u|_1^2 + |Lu|^2 \, d^n x \, dt'$$

$$E[u](t) \leq \underbrace{E[u](0) + C \int_0^T |Lu|^2 \, d^n x \, dt'} + \int_0^t C E[u](t') \, dt'$$

By Gronwall's inequality

$$(1) \quad E[u](t) \leq \left(E[u](0) + C\int_0^T \|Lu\|^2 d^n x\, dt' \right) e^{\int_0^t C\, dt'}$$

$$\leq C \left(E[u](0) + C\int_0^T \|Lu\|^2 d^n x\, dt' \right).$$

<u>Case 2</u> $s \in \mathbb{Z}^+$

Since $|G^{00} - -1| \leq \frac{1}{4}$ and G^{00} is smooth,

$$\tilde{G} = \frac{G}{G^{00}}, \tilde{B} = \frac{B}{G^{00}}, \tilde{A} = \frac{A}{G^{00}}, \text{ and } \tilde{L} \text{ from these,}$$

they also satisfy cond'n $1(\infty, R^{1+n})$ and $\tilde{G}$ is 2ε close to η with $2\varepsilon \in [0, 1/2)$

Thus, we may ~~continue~~ now assume $G^{00} = -1$.

Given k, let

$$|u|_{1,k} = \sum_{|\alpha| \leq 1} \sum_{|\beta| \leq k} |\partial^\alpha \vec{\partial}^\beta u|$$

$$E_k[u] = \sum_{|\beta| \leq k} E[\vec{\partial}^\beta u]$$

Observe

$$(2)\quad C^{-1}\sum_{|\alpha|\le 1}\|\partial^\alpha u\|_{H^k}^2 \le E_k[u] \le C\sum_{|\alpha|\le 1}\|\partial^\alpha u\|_{H^k}^2$$

For $|\beta|\le s$, $[L,\vec{\partial}^\beta] = L\vec{\partial}^\beta - \vec{\partial}^\beta L$ is an order $s+1$ differential operator with at most 1 time derivative.

$$\therefore \quad |[L,\vec{\partial}^\beta]u| \le C|u|_{1,s}$$

Since $L\vec{\partial}^\beta u = \vec{\partial}^\beta Lu + [L,\vec{\partial}^\beta]u$, by (I) applied to $\vec{\partial}^\beta u$

$$E[\vec{\partial}^\beta u](t) \le E[\vec{\partial}^\beta u](0) + C\int_0^t\int |\vec{\partial}^\beta Lu|^2 + |[L,\vec{\partial}^\beta]u|^2\, d^n x\, dt' + C\int_0^t E[\vec{\partial}^\beta u](t')\,dt'$$

(alternative s-energy estimate)

Summing in β

$$E_s[u](t) \le E_s[u](0) + C\int_0^t \|Lu\|_{H^s}^2\,dt' + C\int_0^t\int |u|_{1,s}^2\,d^n x\,dt' + C\int_0^t E_s[u](t')\,dt'$$

$$\le E_s[u](0) + C\int_0^t \|Lu\|_{H^s}^2\,dt' + C\int_0^t E_s[u](t')\,dt'$$

By Gronwall's: $E_s[u](t) \le C\left(E_s[u](0) + \int_0^t \|Lu\|_{H^s}^2\,dt'\right)$ (3)

Case 3 $s \in \mathbb{Z}^-$. Let $k = -s \in \mathbb{Z}^+$

Since $u(t,x) \in \mathcal{S}_x$, $\hat{u}(t,\xi) \in \mathcal{S}_\xi$, $(1+|\xi|^2)^{-k}\hat{u}(\xi) \in \mathcal{S}_\xi$.

Define $v = (1-\Delta)^{-k} u$ to be $\mathcal{F}_\xi^{-1}\left((1+\xi^2)^{-k}\hat{u}(t,\xi)\right) \in \mathcal{S}_x$

Observe

$$\|v(t)\|_{H^k} = \|(1-\Delta)^{k/2} v(t)\|_{L^2} = \|(1-\Delta)^{-\frac{k}{2}} u(t)\|_{L^2}$$
$$= \|u(t)\|_{H^{-k}} = \|u(t)\|_{H^s}$$

Applying (3) with v,k replacing u,s

(4) $$\sum_{|\alpha|\le 1}\sum_{|\beta|\le k}\int |\partial^\alpha \vec{\partial}^\beta v(t)|^2\, d^n\vec{x}$$
$$\le \sum_{|\alpha|\le 1}\sum_{|\beta|\le k}\int |\partial^\alpha\vec{\partial}^\beta v(0)|^2 d^n\vec{x} + \int_0^t \int\sum_{|\beta|\le k} |\vec{\partial}^\beta Lv|^2 d^n x\, dt'$$

Note

$$(1-\Delta)^k Lv = Lu + [(1-\Delta)^k, L]v$$ and $[(1-\Delta)^k, L]$ is of order $2k+1$ with at most 1 t derivative.

$$\int\sum_{|\beta|\le k} |\vec{\partial}^\beta Lv|^2 d^n x = \|Lv\|_{H^k}^2$$
$$= \|(1-\Delta)^k Lv\|_{H^{-k}}^2$$

$$\leq C\left(\|Lu\|^2_{H^{-k}} + \|[(1-\Delta)^k, L]\, v\|^2_{H^{-k}}\right)$$

$$\leq C\left(\|Lu\|^2_{H^{-k}} + \sum_{\substack{|\alpha|\leq 1\\ |\gamma|\leq 2k}} \|\partial^\alpha \overrightarrow{\partial}^\gamma v\|^2_{H^{-k}}\right)$$

$$\leq C\left(\|Lu\|^2_{H^{-k}} + \sum_{|\alpha|\leq 1} \|\partial^\alpha v\|^2_{H^k}\right)$$

Substituting in (4)

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha u(t)\|_{H^{-k}} \leq \sum_{|\alpha|\leq 1} \|\partial^\alpha u(0)\|_{H^{-k}} + C\int_0^t \left(\|Lu\|^2_{H^{-k}} + \sum_{|\alpha|\leq 1} \|\partial^\alpha u(t')\|^2\right) dt'$$

Conclude as before using Gronwall's inequality. $\square$

Theorem

Assume cond'n $1(\infty, \mathbb{R}^{1+n})$, $\varepsilon \in [0, \frac{1}{4})$, and G is ε close to η.

Let $s \in \mathbb{Z}$. $T \in (0, \infty)$

For $f \in H^s(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$, and $F \in L^1([0,T]; H^{s-1}(\mathbb{R}^n))$

then there is a unique

$$u \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$ (Unique here).

such that

$$\text{(IVP)} \quad \begin{cases} Lu = F & \text{for } t \in (0,T) \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \end{cases}$$

pf let $L^p(H^s)$ and $C^k(H^s)$ denotes $L^p([0,T]; H^s(\mathbb{R}^n))$ and $C^k([0,T], H^s(\mathbb{R}^n))$.

Step 1 Uniqueness

If u_1 and u_2 solve (IVP), then $L(u_1 - u_2) = 0$ and $(u_1 - u_2)|_{t=0} = 0$ and $\partial_t (u_1 - u_2)|_{t=0}$. By the s-energy estimate, $E_s[u_1 - u_2](t) = 0$ $\forall t \geq 0$, so $(u_1 - u_2)(t,x) = 0$ $\forall (t,x)$.

Step 2 <u>L^∞ existence with $f=0=g$ in the wrong space</u>

Let $L^*u = \partial_i\partial_j(G^{ij}u) - \partial_i(B^i u) + Au$, so L^* is the formal adjoint of L. Note L^* also satisfies cond'n $1(\infty, \mathbb{R}^n)$ with G ε close to η.

For $\psi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$, by the s-energy estimate for L^* with $T-t$ replacing t and $-s-1$ replacing s.

$$\|\psi(t)\|_{H^{-s}} \leq 0 + C\int_t^T \|L^*\psi\|_{H^{-s-1}}\,dt'$$

Thus, for $F \in L^1([0,T]; H^s)$ extend F as 0 for $t<0$

$$\begin{aligned}|\langle F,\psi\rangle_{t,x}| &= \left|\int_0^T \langle F(t'), \psi(t')\rangle_x\,dt'\right| \\ &\leq \|F\|_{L^1(H^s)}\,\|\psi\|_{L^\infty(H^{-s})} \\ &\leq C\int_0^T \|L^*\psi\|_{H^{-s-1}}\,dt'\end{aligned}$$

By Hahn-Banach & HW2.3 $\exists\, W \in (L^1([0,T]; H^{-s-1}))^*$ st $\forall \psi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$

$$W(L^*\psi) = \int_0^T\int_{\mathbb{R}^n} F\,\psi\,d^nx\,dt'$$

Recall for $p \in [1,\infty)$ with $\frac{1}{p'} = 1 - \frac{1}{p}$ for $p > 1$ and $p' = \infty$ if $p = 1$ and for X a Banach space.

$$\left(L^p(I,X)\right)^* = L^{p'}(I,X^*)$$

and $(H^s)^* = H^{-s}$

Thus, $\exists\, u \in L^\infty([0,T]; H^{s+1})$ (i.e. $u = W$) such that

$$\langle u, L^*\psi\rangle = \langle F, \psi\rangle$$

By def'n, u is a weak solution of (IVP) with $f = g = 0$.

If $\operatorname{supp} \psi \subset (-\infty, 0) \times \mathbb{R}^n$, then

$\langle u, L^*\psi\rangle = \langle F, \psi\rangle = 0$, so $u(t) = 0$ for $t < 0$.

<u>Step 3</u> <u>C^∞ existence for $f = g = 0$ & $F \in C_0^\infty$</u>

Again pass to $\tilde{G} = \frac{G}{G^{00}}$.

Let $v = \partial_t u$. From $Lu = F$

$$(5)\quad \partial_t v + 2\sum_{j=1}^{n} \tilde{G}^{j0}\partial_j v + \tilde{B}^0 v = -\sum_{j,k=1}^{n} \tilde{G}^{jk}\partial_j\partial_k u - \sum \tilde{B}^j \partial_j u - \hat{A}u + \tilde{F}$$

The RHS is in $L^\infty(H^{s-1})$ since $u \in L^\infty(H^{s+1})$ & $F \in C_0^\infty$

Applying $\vec{\partial}^\beta$ with $|\beta| \leq s-1$, multiplying by $\vec{\partial}^\beta v$ & integrating in $\mathbb{R}^{1+n}$,

$$\frac{d}{dt}\|v(t)\|_{H^{s-1}}^2 \leq C\left(\max_j |\partial_j G^{j0}|\, \|v\|_{H^{s-1}}^2 + \|v\|_{H^{s-1}}^2 + \|v\|_{H^{s-1}}\, \|u\|_{H^{s+1}}\right)$$

By Grenwall's inequality, $\partial_t u = v \in L^\infty(H^{s-1})$, $\partial_t^2 u = \partial_t v \in L^\infty(H^{s-2})$.

By ~~the Fund~~ integration, $u \in C^0(H^{s-1}) \cap C^1(H^{s-2})$.

Since $F \in C_0^\infty$, we can simply replace $s \longmapsto s+2$, to get $u \in C^0(H^s) \cap C^1(H^{s-1})$.

<u>Step 4</u> <u>Existence for $f=g=0$ in the right space</u>

Let $F_m \in C_0^\infty$ converge to F in $L^1(H^{s-1})$

For each F_m, there is a solution $u_m \in C^0(H^s) \cap C^1(H^{s-1})$

From the s-energy estimate (with s replaced by s-1), the u_m are Cauchy in $C^0(H^s) \cap C^1(H^{s-1})$ and hence converge there.

Step 5 **General f, g**

Let $u_0(t,x) = f(x) + t g(x)$, which has the correct data and is in $L^1(H^{s-1})$. Solve

$$L v = F - L u_0$$
$$v|_{t=0} = 0$$
$$\partial_t v|_{t=0} = 0$$

Now, $u = v + u_0 \in C^0(H^s) \cap C^1(H^{s-1})$ solves IVP. $\square$

Rmk $u \in C^0(H^s) \cap C^1(H^{s-1}) \cap C^2(H^{s-2})$

Thus, $\partial_j \partial_k u \in C^0(H^{s-2})$

and $Lu = F$ holds in $C^0(H^{s-2})$ if $F \in C^0(H^{s-2})$

not merely in the weak sense.

Local Existence for Quasilinear Equations

Def'n Condition $2Q(\Omega)$ is condition $1Q(\Omega)$ and that for all multiindices α, there are constants C_α such that $|\partial^\alpha G^{ij}| < C_\alpha$, $|\partial^\alpha B^i| < C_\alpha$, $|\partial^\alpha A| < C_\alpha$, $|\partial^\alpha F| < C_\alpha$ (Here α is in $\mathbb{Z}^{(1+n)+1+(1+n)}$.)

Rmk: u^2 fails to satisfy this, but $u^2 \tilde{\chi}_{[-N,N]}(u)$ with $\tilde{\chi}_{[-N,N]}$ smooth, 1 on $[-N,N]$, and 0 outside $[-N-1,N+1]$, can be used to approximate solutions & does satisfy the cond'n.

Lemma [s-chain rule] Assume cond'n $2Q(\Omega)$.

Let $|w|_s(t,x) = \sum_{|\alpha|\le s} |\partial^\alpha w|(t,x)$

Let $s \in \mathbb{N}$. If $|\alpha| \le s$, then $\exists C : \forall v,w \in C^\infty(\Omega)\ \forall (t,x) \in \Omega$

$$|F(v,\partial v)|_{0,s} \le C\left(1 + |v|_{1,\lceil \frac{s+1}{2}\rceil}\right)^{s-1} |v|_{1,s}$$

$1+s \mapsto 1,s$

$\frac{s+3}{2} \mapsto 1, \frac{s+1}{2}$

$$|[\partial^\alpha, G^{\ell m}(v,\partial v)]\partial_\ell \partial_m w| \le C\left(1 + |v|_{\frac{s+3}{2}}\right)^s |w|_{1,s} + C\left(1 + |v|_{\frac{s+3}{2}}\right)^{s-1} |w|_{\frac{s+3}{2}} |v|_{1,s}$$

$$|[\vec{\partial}^\alpha, B^\ell(v,\partial v)]\partial_\ell w| \le C(1+|v|_{\frac{s+3}{2}})^s |w|_s + C(1+|v|_{\frac{s+3}{2}})^{s-1} |w|_{\frac{s+3}{2}} |v|_{s+1}$$

$$|[\vec{\partial}^\alpha, A(v,\partial v)]w| \le C(1+|v|_{\frac{s+3}{2}})|w|_{s-1} + C(1+|v|_{\frac{s+3}{2}})^{s-1} |w|_{\frac{s+3}{2}} |v|_{s+1}.$$

Pf By induction on s, for $|\alpha| \le s$, $\vec{\partial}^\alpha F(v,\partial v)$ can be written as a sum of terms, each of which has at most s factors and the total number of derivatives on factors of v or ∂v is s. Thus, the most derivatives on any factor of v or ∂v is s, all remaining factors have at most $\lceil \frac{s}{2} \rceil$ derivatives. With the uniform bound on derivatives of F, this proves the first part.

$[\partial^\alpha, G^{\ell m}]$ can be written as a sum of order $k \le s-1$ differential operators with coefficients involving $s-k$ derivatives of $G^{\ell m}$. If $k \ge \frac{s-1}{2}$, then estimating $\partial^\beta G^{\ell m}$ with $|\beta| \le s-k$

as in the previous paragraph gives terms on the ~~first~~ first line. If $k \le \frac{s-1}{2}$, then at most $\frac{s+3}{2}$ derivatives are applied to w, and the terms are controlled by the second line.

There's only ever one ∂_t in each factor.

The B & A terms are similar. $\square$