Lecture 7

Recall from Lecture 6:

Blowup criterion #3 (Blowup implies spacetime norm concentration)

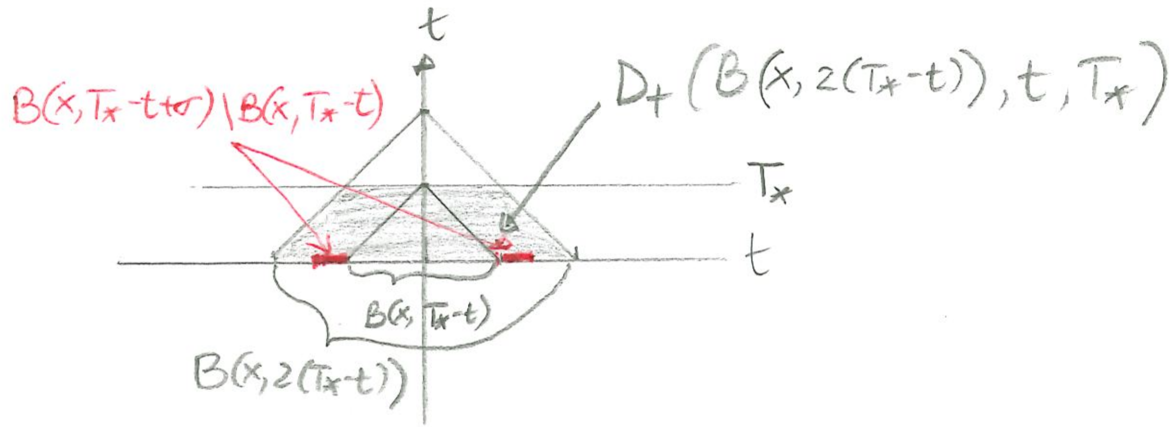
If u is a $\dot{H}^1 \times L^2$-solution of NLW with $E(u, \partial_t u) \leq E_0$, with a maximal time of existence $0 < T_* < \infty$, then $\exists\, \varepsilon_1 = \varepsilon_1(E_0) > 0$, $\exists\, x \in \mathbb{R}^3$:

$$\boxed{\limsup_{t \to T_*^-} \| S(\cdot - t)(u(t), \partial_t u(t)) \|_{L^4 L^{12}(D_+(B(x, 2(T_* - t)), t, T_*))} \geq \varepsilon_1 .}$$

Exterior energy decay

Let u be as above. Then $\forall x \in \mathbb{R}^3$:

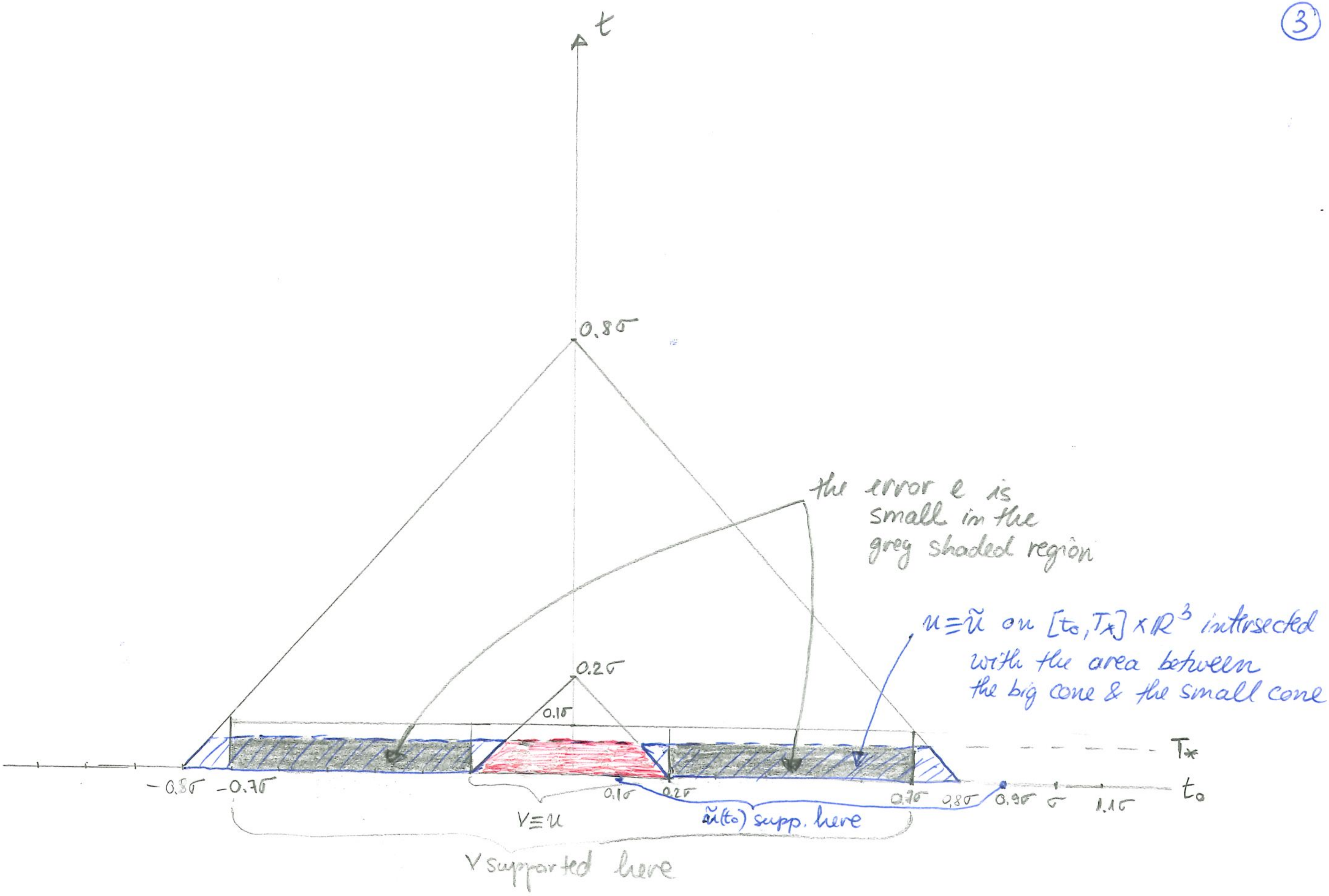
$$\boxed{\inf_{\sigma > 0} \limsup_{t \to T_*^-} E_{B(x, T_* - t + \sigma) \setminus B(x, T_* - t)} (u(t), \partial_t u(t)) = 0 .}$$



Blowup criterion #4 (Blowup implies potential energy concentration)
(Prop 5.7, Tao's book)

Let $E_0 > 0$. If u is a $\dot{H}^1 \times L^2$ solution of NLW_5 with $E(u) \le E_0$ and a maximal time of existence $0 < T_* < \infty$, then $\exists\, x \in \mathbb{R}^3$, $\exists\, \varepsilon_2(E_0) > 0$ s.t.

$$\boxed{\limsup_{t \to T_*^-} \int_{B(x, T_* - t)} |u(t,y)|^6 \, dy \ge \varepsilon_2}$$



t
0.8σ
the error e is small in the grey shaded region
u ≡ ũ on [t₀, T*] × ℝ³ intersected with the area between the big cone & the small cone
0.2σ
0.1σ
T*
t₀
−0.8σ
−0.7σ
0.1σ
0.2σ
0.7σ
0.8σ
0.9σ
σ
1.1σ
V ≡ u
ũ(t₀) supp. here
V supported here

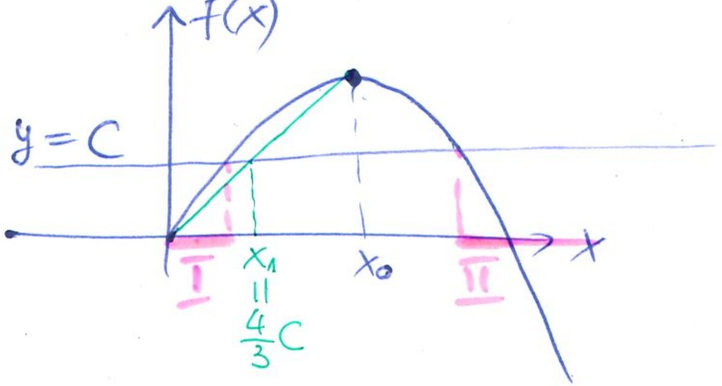
Trap Lemma (version). Let $0 < \varepsilon << 1$, $T > 0$

If $w : [0,T] \to \mathbb{R}_+$ s.t. $w(0) = 0$, w is continuous and $\exists\, C > 0$ s.t.

$$w(t) - \varepsilon . w(t)^4 \le C \quad , \forall t \in [0,T].$$

Then, $w(t) \le \frac{4}{3} C \quad , \forall t \in [0,T]$.

"Proof": $f(x) = x - \varepsilon x^4$



Proof of blowup criterion #4: Let $\varepsilon_2 > 0$ s.t. $\begin{cases} C \varepsilon_2^{\frac{1}{2}} < \delta & (\delta \text{ as in the small energy data GWP}) \\ \varepsilon_2^{\frac{1}{6}} E_0^{\frac{1}{2}} \le 1 \\ C \varepsilon_2^{\frac{1}{24}} E_0^{\frac{1}{2}} < \varepsilon_{\frac{1}{2}} \end{cases}$

$$\forall x \in \mathbb{R}^3, \quad \limsup_{t \to T_*^-} \int_{B(x, T_* - t)} |u(t,y)|^6 \, dy < \varepsilon_2$$

Fix $x \in \mathbb{R}^3$, $\exists \delta > 0$, $\exists\, 0 < t_0 < T_*$ s.t. $\forall\, t_0 \le t < T_*$:

(5)

$$(0)\quad \boxed{\int_{B(x,T_*-t+\sigma)} |u(t,y)|^6\,dy + \underbrace{\int_{B(x,T_*-t+\sigma)\setminus B(x,T_*-t)} |\nabla_{t,x} u(t,y)|^2\,dy}_{\text{exterior energy decay}} < \varepsilon_2}$$

By moving t_0 closer to T_*, we can guarantee that $T_* - t_0 < 0.1\sigma$

Strategy:

(IV) Get a contradiction by showing:

$$(1)\quad \boxed{\|S(\cdot - t_0)\,u(t_0)\|_{L^4L^{12}(D_+(x,0.2\sigma),t,T_*)} < \varepsilon_1}.$$

This contradicts blowup criterion #3.

Take $v(y,t) = u(y,t)\chi(y)$, $\chi \begin{cases} \text{supp on } B(x,0.7\sigma) \\ \equiv 1 \text{ on } B(x,0.2\sigma) \end{cases}$

By finite speed of propag. for the linear wave eqn.:

$$S(\cdot - t_0)u(t_0) \equiv S(\cdot - t_0)v(t_0) \text{ on } D_+(B(x,0.2\sigma),t,T_*)$$

So, enough to show:

$$(2)\quad \boxed{\|S(\cdot - t_0)\,v(t_0)\|_{L^4L^{12}(D_+(x,0.2\sigma),\,t,T_*)} < \varepsilon_1}.$$

(III) By Duhamel's formula, (2) follows if we show:

(3) $\boxed{\|v\|_{L^4L^{12}(D_+(B(x,0.2\sigma),t,T^*))} < \frac{\varepsilon_1}{2}}$

(II) v solves a perturbed NLW:

$$(-\partial_t^2+\Delta)v + v^5 = \underbrace{u^5\chi(1-\chi^4)+\dots}_{=e}$$

error supported on $B(x,0.7\sigma)\setminus B(x,0.2\sigma)$

Nonlin. estimates, using Strichartz, $\|u\|_{L^\infty L^6([t_0,T_*)\times B(x,0.7\sigma))} < \varepsilon_2$

$\Rightarrow \|v\|_{L^4L^{12}([t_0,T_*)\times \mathbb{R}^3)} < \frac{\varepsilon_1}{2}$. and so, (3) follows.

For the above nonlin. estimates, it's crucial to have:

(4) $\boxed{\|e\|_{L^1L^2([t_0,T_*)\times(B(x,0.7\sigma)\setminus B(x,0.2\sigma)))} < \varepsilon_2.}$

For (4), it's enough to show:

(5) $\boxed{\|u\|_{L^4L^{12}([t_0,T_*)\times(B(x,0.7\sigma)\setminus B(x,0.2\sigma)))} \lesssim \varepsilon^{\frac{1}{2}}}$

(I) Take $\tilde{u}(t_0, y) = u(t_0, y)\, \Psi_1\left(\frac{|y-x|}{\sigma}\right) \rightarrow \begin{cases} \text{supp. on } B(x, 0.9\sigma) \setminus B(x, 0.1\sigma) \\ \equiv 1 \text{ on } B(x, 0.8\sigma) \setminus B(x, 0.2\sigma) \end{cases}$

By finite speed of propagation of NLW$_5$, we have

$$\tilde{u} \equiv u \quad \text{on } D_+\left(\left(B(x, 0.8\sigma) \setminus B(x, 0.2\sigma)\right), t_0, T_*\right)$$

$$\cup\!\!/$$

$$[t_0, T_*) \times \left[B(x, 0.7\sigma) \setminus B(x, 0.2\sigma)\right].$$

Sufficient to show

$$(6) \quad \boxed{\|\tilde{u}\|_{L^4 L^{12}\left(D_+\left(B(x, 0.8\sigma) \setminus B(x, 0.2\sigma)\right), t_0, T_*\right)} \leq 2C\varepsilon_2^{\frac{1}{2}}.}$$

This will be true because $\tilde{u}(t_0)$ has small energy due to exterior energy decay in (0).

Detailed proof: First, we need to prove (6). ⑧

By a previous lemma, since $\tilde{u}(t_0)$ is localized on $B(x, 0.9\sigma) \setminus B(x, 0.1\sigma)$:

$$\partial_t \tilde{u}(t_0) \overset{\text{def}}{=} \partial_t u(t_0) \cdot \tilde{\chi}$$
$$\|\partial_t \tilde{u}(t_0)\|_{L^2} \leq c \|\partial_t u(t_0)\|_{L^2}$$

$$\|\tilde{u}(t_0)\|_{\dot{H}^1} \leq C \|u(t_0)\|_{\dot{H}^1(B(x,0.9\sigma) \setminus B(x,0.1\sigma))}$$
$$\overset{(0)}{\leq} C \varepsilon_2^{\frac{1}{2}} < \delta$$

(δ is as in the small energy data GWP)

By small energy data GWP $\exists!$ solution of NLW_5 $\tilde{u}$ on $[t_0, \infty)$ with data $(\tilde{u}(t_0), \partial_t \tilde{u}(t_0))$ and, moreover,

$$\|\tilde{u}\|_{L^4 L^{12}([t_0,\infty) \times \mathbb{R}^3)} \leq 2 \|(\tilde{u}(t_0), \partial_t \tilde{u}(t_0))\|_{\dot{H}^1 \times L^2} \leq 2C \varepsilon_2^{\frac{1}{2}}$$

This shows (6), and thus (5).

Using (5), we will now prove (4):
?

Recall $v(y,t) = u(y,t)\,\chi(y)$ with $\chi \begin{cases} \text{supp. on } B(x, 0.7\sigma) \\ \equiv 1 \text{ on } B(x, 0.2\sigma) \end{cases}$. Then,

$$(-\partial_t^2 v + \Delta)v + v^5 = \left[(-\partial_t^2 + \Delta)u\right]\cdot\chi + 2\nabla u\cdot\nabla\chi + u\,\Delta\chi + v^5$$

$$= -u^5\cdot\chi + u^5\cdot\chi^5 + 2\nabla u\cdot\nabla\chi + u\cdot\Delta\chi$$

$$= \underbrace{\underbrace{u^5\chi(\chi^4-1)}_{e_1} + \underbrace{2\nabla u\cdot\nabla\chi}_{e_2} + \underbrace{u\cdot\Delta\chi}_{e_3}}_{e''}$$

supp on $B(x,0.7\sigma)\setminus B(x,0.2\sigma)$

$$\|e_1\|_{L^1L^2([t_0,T_*)\times\mathbb{R}^3)} \le \|u\|_{L^\infty L^6(B(x,0.7\sigma)\setminus B(x,0.2\sigma))}$$

$$\cdot\ \|u\|^4_{L^4L^{12}([t_0,T_*)\times B(x,0.7\sigma)\setminus B(x,0.2\sigma))}$$

$$\overset{(0)\&(5)}{\le} \varepsilon_2^{\frac{1}{6}}\left(C\varepsilon_2^{\frac{1}{2}}\right)^4 \le \frac{\varepsilon_2}{3}.$$

Similarly, $\|e_2\|_{L^1L^2([t_0,T_*)\times\mathbb{R}^3)}$, $\|e_3\|_{-''-} \le \frac{\varepsilon_2}{3}$.

Using (4), we will now prove (3):

$$\|v\|_{L^4L^{12}([t_0,T_*)\times\mathbb{R}^3)} \overset{\text{Strich.}}{\leq} \underbrace{\|(u(t_0),\partial_t u(t_0))\|_{\dot H^1\times L^2}}_{\leq E_0^{\frac12}}$$

$$+ \underbrace{C\,\|u\|_{L^\infty L^6([t_0,T_*)\times B(x,0.7\sigma))}}_{\leq C\varepsilon_2^{\frac16}} \|v\|^4_{L^4L^{12}([t_0,T_*)\times\mathbb{R}^3)}$$

the smallness of the potential energy is crucial here!

$$+ \underbrace{\|e\|_{L^1L^2([t_0,T_*)\times\mathbb{R}^3)}}_{\leq \varepsilon_2 \text{ by } (4)}$$

$$\leq \underbrace{E_0^{\frac12}+\varepsilon_2}+C\varepsilon_2^{\frac16}\|v\|^4_{L^4L^{12}([t_0,T_*)\times\mathbb{R}^3)}$$

Then, by the trap Lemma for $w(t)=\|v\|_{L^4L^{12}([t_0,t]\times\mathbb{R}^3)}$, $t_0\leq t<T_*$:

$$\|v\|_{L^4L^{12}([t_0,T_*)\times\mathbb{R}^3)} \leq \underbrace{C E_0^{\frac12}}_{\text{not small!!}}.$$

Similarly, $(3,18)$ is a wave-admissible pair and:

$$\|v\|_{L^3L^{18}([t_0,T_*)\times\mathbb{R}^3)} \leq C E_0^{\frac12}.$$

Interpolate $L^4 L^{12}$ between $L^3 L^{18}$ & $L^\infty L^6$: (11)

$$\|v\|_{L^4 L^{12}(D_+(B(x,0.2\sigma),t,T_*))} \le$$

$$\le \|v\|^{\frac{1}{4}}_{L^\infty L^6(-"-)} \|v\|^{\frac{3}{4}}_{L^3 L^{18}(-"-)}$$

$$\le C \|u\|^{\frac{1}{4}}_{L^\infty L^6(D_+(B(x,0.2\sigma),t,T_*))} \cdot E_0^{\frac{1}{2}}$$

$$\le C\, \varepsilon_2^{\frac{1}{24}} E_0^{\frac{1}{2}} < \frac{\varepsilon_1}{2}. \quad \text{(from the choice of } \varepsilon_2\text{)}.$$

So, (3) is proved.

Finally, we use (3), to prove (2):

By Duhamel's formula:

$$S(t-t_0)V(t_0) = V(t) + \int_{t_0}^{t} \frac{\sin((t-s)|D|)}{|D|}(V^5+e)(s)\,ds$$

$\xrightarrow[\text{estim.}]{\text{nonlin.}}$ $\|S(t-t_0)V(t_0)\|_{L^4L^{12}(D_+(B(x,0.2\sigma),t,T_*))} \le \varepsilon_1$.

Prop (Local energy identity)

Let u be a sol. of NLW_5 on $\mathbb{R}^3$. Then $\forall (t_0, x_0) \in \mathbb{R}\times\mathbb{R}^3$, $\forall S < T < t_0$, we have

The flux measures how much energy escapes the backward light cone at (t_0, x_0) from time $t=S$ to time $t=T$

$$\boxed{E_{B(x_0, t_0-T)}(u(T)) + \underbrace{\mathrm{Flux}(u; M_S^T(t_0, x_0))}_{\geq 0} = E_{B(x_0, t_0-S)}[u(S)]}$$

where

$$E_{B(x_0, t_0-T)}(u(T)) := \int_{B(x_0, t_0-T)} \left[\frac{(\partial_t u)^2 + |Du|^2}{2} + \frac{u^6}{6}\right](T, y)\, dy$$

$M_S^T :=$ lateral boundary of the truncated cone $D_+(B(x_0, t_0-S), S, T)$

$$\mathrm{Flux}(u; M_S^T(t_0, x_0)) := \frac{1}{\sqrt{2}} \int_{M_S^T(t_0, x_0)} \frac{\left|Du - \frac{x-x_0}{|x-x_0|}\partial_t u\right|^2}{2} + \frac{u^6}{6}\, d\sigma$$

Rem: $\lim_{S\to t_0^-} \lim_{T\to t_0^-} \text{Flux}(u, M_S^T(t_0,x_0)) = 0$.

Proof: Since $E_{B(x_0, t_0-T)}(u(T)) \geq 0$, by the local energy identity, we have:

$$0 \leq \text{Flux}(u; M_S^T(t_0,x_0)) \leq E_{B(x_0,t_0-S)}(u(S)) \leq E(u(S)) = E(u_0,u_1)$$

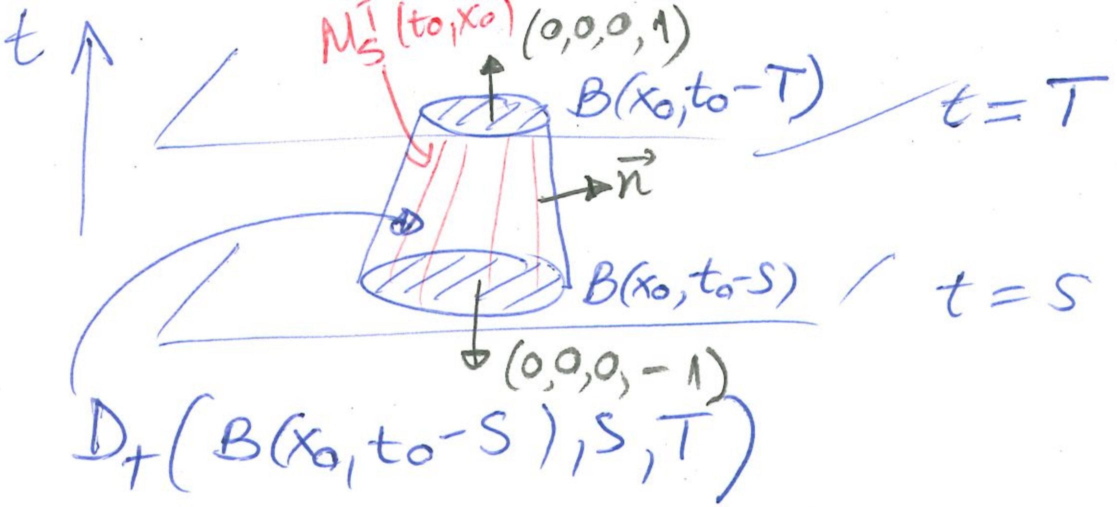
So, for all $S<T<t_0$, we have

$$\text{Flux}(u; M_S^T(t_0,x_0)) \in [0, E(u_0,u_1)]$$

Since, $\text{Flux}(u; M_S^T(t_0,x_0))$ is non-decreasing as a function of $T \Rightarrow \lim_{T\to t_0^-} \text{Flux}(u; M_S^T(t_0,x_0))$ exists.

By noting that $\lim_{T\to t_0^-} E_{B(x_0,t_0-T)}(u(T)) = 0$ via the absolute continuity of the integral, we have from local energy id., by taking $T\to t_0^-$ first and $S\to t_0$ secondly:

$$\lim_{S\to t_0^-} \lim_{T\to t_0^-} \text{Flux}(u, M_S^T(t_0,x_0)) = \lim_{S\to t_0^-} E_{B(x_0,t_0-S)}[u(S)] = 0.$$



M_S^T is given by

$$F(x,t)=0$$

with $F(x,t) = t - t_0 + |x-x_0|$

$$\Rightarrow \vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|}\left(\nabla_x(|x-x_0|),\, 1\right) = \frac{1}{\sqrt{2}}\left(\frac{x-x_0}{|x-x_0|},\, 1\right)$$

Thm (Non-concentration of the potential energy, Shatah-Struwe book, Lemma 6.1)

Let $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^3$, let u be a classical solution of NLW_5 on $D_+(B(x,t_0),0,t_0)$. Then

$$\lim_{S\to t_0^-} \int_{B(x_0,t_0-S)} |u(S)|^6\, dx \le C \lim_{S\to t_0} \left[Flux(u, M_S^{t_0}(t_0,x_0))\right]^{\frac{1}{3}} = 0.$$

Conclusion: the blowup criterion #4 together with the above thm. show that any data $(u_0,u_1) \in \dot{H}^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ lead to global-in-time solutions in $C(\mathbb{R}, \dot{H}^1 \times L^2)$.

<u>Proof of Local energy identity</u>: From a previous lecture (Lecture 4), we have:

$$\partial_t\left[\frac{(\partial_t u)^2+|Du|^2}{2}+\frac{u^6}{6}\right]-\operatorname{div}(Du\,\partial_t u)=0$$

Integrate this over $D_+(B(x_0,t_0-S),S,T)$:

$$\int_{D_+(B(x_0,t_0-S),S,T)}\operatorname{div}_{x,t}\left(0,0,0,\frac{(\partial_t u)^2+|Du|^2}{2}+\frac{u^6}{6}\right)dx\,dt$$

$$-\int_{\text{---}''\text{---}}\operatorname{div}_{x,t}(Du\cdot\partial_t u,\,0)\,dx\,dt=0$$

By the divergence theorem:

$E_{B(x_0,t_0-T)}(u(T))$ $\qquad$ $E_{B(x_0,t_0-S)}(u(S))$

$$\left\{\int_{B(x_0,t_0-T)}\frac{(\partial_t u)^2+|Du|^2}{2}+\frac{u^6}{6}\,dx-\int_{B(x_0,t_0-S)}\frac{(\partial_t u)^2+|Du|^2}{2}+\frac{u^6}{6}\,dx\right.$$

Flux $\longrightarrow$

$$\left.+\frac{1}{\sqrt2}\int_{M_S^T(t_0,x_0)}\frac{(\partial_t u)^2+|Du|^2}{2}+\frac{u^6}{6}\,d\sigma\right\}-\frac{1}{\sqrt2}\int_{M_S^T(t_0,x_0)}\partial_t u\,Du\cdot\frac{x-x_0}{|x-x_0|}\,d\sigma=0$$