ADVANCED PDE II - LECTURE 8

PIETER BLUE AND OANA POCOVNICU

Warning: This is a first draft of the lecture notes and should be used with care! This lecture's notes very closely follow C. Sogge's "Nonlinear Wave Equations".

1. Existence and uniqueness for quasilinear wave equations

Definition 1.1. *Condition 2Q(Ω) is condition 1Q(Ω) and that for all multiindices $\alpha \in \mathbb{Z}^{(1+n)+1+(1+n)}$, there are constants C_α such that $|\delta^\alpha G^{ij}| < C_\alpha$, $|\delta^\alpha B^i| < C_\alpha$, and $|\delta^\alpha A| < C_\alpha$.*

(Recall that G, B, and A are functions of $(x, u, \partial u) \in \mathbb{R}^{1+n} \times \mathbb{R} \times \mathbb{R}^{1+n}$. δ denotes the partial derivative operator in $\mathbb{R}^{1+n} \times \mathbb{R} \times \mathbb{R}^{1+n}$, as opposed to ∂ which denotes partial differentiation in $\mathbb{R}^{1+n}$.)

Definition 1.2.

$$\begin{aligned}
|w|_{0,s}(t,\vec{x}) &= \sum_{|\beta|\le s} |\vec{\partial}^\beta w(t,x)|, \\
|w|_{1,s}(t,\vec{x}) &= \sum_{|\alpha|\le 1}\sum_{|\beta|\le s} |\partial^\alpha \vec{\partial}^\beta w(t,x)|, \\
\|w\|_{1,s}^2(t) &= \int_{\mathbb{R}^n} |w|_{1,s}(t,\vec{x})^2 \mathrm{d}^n\vec{x}.
\end{aligned}$$

Theorem 1.3 (s-chain rule)**.** *Assume condition 2Q(Ω). Assume further G^{00} is constant. Let $s \in \mathbb{N}$. If $|\alpha| \le s$, then $\exists C : \forall v, w \in C^\infty(\Omega)$, $\forall (t, \vec{x}) \in \Omega$*

$$\begin{aligned}
|F(x,v,\partial v)|_{0,s} &\le C\left(1 + |v|_{1,\lceil\frac{s+1}{2}\rceil}\right)^{s-1} |v|_{1,s}, \\
\left|[\vec{\partial}^\alpha, L(\vec{x},v,\partial v)]w\right| &\le C\left(1 + |v|_{1,\lceil\frac{s+1}{2}\rceil}\right)^{s} |w|_{1,s} \\
&\quad + C\left(1 + |v|_{1,\lceil\frac{s+1}{2}\rceil}\right)^{s-1} |w|_{1,\lceil\frac{s+1}{2}\rceil} |v|_{1,s}.
\end{aligned}$$

Proof. See lecture 7 notes. ∎

Corollary 1.4 (s-chain rule with differences)**.** *Assume condition 2Q(Ω). Assume further G^{00} is constant. Let $s \in \mathbb{N}$. If $|\alpha| \le s$, then $\exists C : \forall v_1, v_2, w \in C^\infty(\Omega)$, $\forall (t, \vec{x}) \in \Omega$*

$$\begin{aligned}
&|F(x, v_1, \partial v_1) - F(x, v_2, \partial v_2)|_{0,s} \\
&\quad \le C|v_1 - v_2|_{1,\lceil \frac{s+1}{2} \rceil} \left(1 + |v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}\right)^{s-1} (|v_1|_{1,s} + |v_2|_{1,s}), \\
&\qquad + C|v_1 - v_2|_{1,s} \left(1 + |v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}\right)^{s-1} (|v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}), \\
&\left|[\vec{\partial}^\alpha, L(\vec{x}, v_1, \partial v_1) - L(\vec{x}, v_2, \partial v_2)]w\right| \\
&\quad \le C|v_1 - v_2|_{1,\lceil \frac{s+1}{2} \rceil} \left(1 + |v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}\right)^{s} |w|_{1,s} \\
&\qquad + C|v_1 - v_2|_{1,s} \left(1 + |v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}\right)^{s} |w|_{1,\lceil \frac{s+1}{2} \rceil} \\
&\qquad + C|v_1 - v_2|_{1,\lceil \frac{s+1}{2} \rceil} \left(1 + |v_1|_{1,\lceil \frac{s+1}{2} \rceil} + |v_2|_{1,\lceil \frac{s+1}{2} \rceil}\right)^{s-1} (|v_1|_{1,s} + |v_2|_{1,s}) |w|_{1,\lceil \frac{s+1}{2} \rceil}.
\end{aligned}$$

Proof. This largely follows the proof for the previous theorem. For $s = 0$, by condition 2Q, it follows that $|F(x, v_1, \partial v_1) - F(x, v_2, \partial v_2)|_{0,0} \le C|v_1 - v_2|_1$. For the induction step, define a difference term to be of the form $h(x, v_1, \partial v_1) - h(x, v_2, \partial v_2)$ for some smooth function h or of the form $\vec{\partial}^\alpha(v_1 - v_2)$ or $\vec{\partial}^\alpha \partial(v_1 - v_2)$. Consider $h(x, v, \partial v) = h(v)$ for simplicity. In this case,

$$\begin{aligned}
\vec{\partial}\,(h(v_1) - h(v_2)) &= h'(v_1)\vec{\partial} v_1 - h'(v_2)\vec{\partial} v_2 \\
&= h'(v_1)\vec{\partial} v_1 - h'(v_2)\vec{\partial} v_1 \\
&\quad + h'(v_2)\vec{\partial} v_1 - h'(v_2)\vec{\partial} v_2 \\
&= (h'(v_1) - h'(v_2))\vec{\partial} v_1 + h'(v_2)\vec{\partial}(v_1 - v_2).
\end{aligned}$$

A similar argument for general $h(x, v, \partial v)$ shows that the derivative of a difference term is a sum of products in which at least one factor is a difference term. By induction, $|F(x, v_1, \partial v_1) - F(x, v_2, \partial v_2)|_{0,s}$ is a sum of at most $s + 1$ products, where the first term in the product is a derivative of $F(x, v_1, \partial v_1) - F(x, v_2, \partial v_2)$ and the remaining s terms are derivatives of v or ∂v with at most s spatial derivatives distributed between them. By induction, each such sum can be written as a sum of products in which at least one factor is a difference term and in which the first term involves either F, its derivatives of such terms. By condition 2Q, difference terms involving differences of F or its derivatives can be $|v_1 - v_2|_{1,0}$. Difference terms involving $\vec{\partial}^\alpha \partial(v_1 - v_2)$ can be estimated by $|v_1 - v_2|_{1,|\alpha|}$.

The argument for $[\vec{\partial}^\alpha, L(x, v_1, \partial v_1] - L(x, v_2, \partial v_2)]w$ is similar. ☐

Corollary 1.5 (H^s-chain rule)**.** *Assume condition 2Q($\mathbb{R}^{1+n}$). Assume further G^{00} is constant. Let $s \ge n + 3$.*

If $|\alpha| \le s$, then $\exists C : \forall v, w, v_1, v_2 \in C^\infty(\Omega)$, $\forall t \in \mathbb{R}$:

$$\|F(x, v, \partial v)\|_{0,s} \le C(1 + \|v\|_{1,s})^{s-1} \|v\|_{1,s},$$
$$\|[\vec{\partial}^\alpha, L(x, v, \partial v)]w\|_{L^2} \le C(1 + \|v\|_{1,s})^{s} \|w\|_{1,s}.$$

Furthermore,

$$\|F(x,v_1,\partial v_1) - F(x,v_2,\partial v_2)\|_{0,s} \leq C\|v_1 - v_2\|_{1,0}\Big(\|v_1\|_{1,s} + \|v_2\|_{1,s}\Big)^s,$$

$$\|(L(x,v_1,\partial v_1) - L(x,v_2,\partial v_2))w\|_{0,s} \leq C\|v_1 - v_2\|_{1,s}(1 + \|v_1\|_{1,s} + \|v_2\|_{1,s})^{s-1}\|w\|_{1,s+1},$$

$$\Big\|[\vec{\partial}^\alpha, L(\vec{x},v_1,\partial v_1) - L(\vec{x},v_2,\partial v_2)]w\Big\|_{L^2} \leq C\|v_1 - v_2\|_{1,s}\Big(\|v_1\|_{1,s} + \|v_2\|_{1,s}\Big)^s\|w\|_{1,s}.$$

Proof. From the definition of the L^2 norm and the s-chain rule,

$$\begin{aligned}\|F(x,v,\partial v)\|_{0,s}^2 &= \int |F(x,v,\partial v)|_{0,s}^2 \mathrm{d}^n x \\ &\leq C\int \Big(1 + |v|_{1,\lceil \frac{s+1}{2}\rceil}\Big)^{2s-2} |v|_{1,s}^2 \mathrm{d}^n x \\ &\leq C\|1 + |v|_{1,\lceil \frac{s+1}{2}\rceil}\|_{L^\infty}^{2s-2}\int |v|_{1,s}^2 \mathrm{d}^n x.\end{aligned}$$

Now observe that since $s \geq n+3$, one has

$$\left\lceil \frac{s+1}{2}\right\rceil + \frac{n}{2} \leq \frac{s+2}{2} + \frac{s-3}{2} < s.$$

Thus, the previous computation and the Sobolev embedding theorem give the first result. (Recall $|\partial|u|| \leq |\partial u|$ a.e.) The remaining results follow similarly. □

Definition 1.6. *Assume condition 2Q($\mathbb{R}^{1+n}$). Consider*

$$L(x,u,\partial u)u = F(x,u,\partial u), \tag{1a}$$

$$u(0,\vec{x}) = f(\vec{x}), \tag{1b}$$

$$\partial_t u(0,\vec{x}) = g(\vec{x}). \tag{1c}$$

This system is called the ***quasilinear wave initial value problem, QLWIVP****.*

Theorem 1.7 (Quasilinear existence and uniqueness)**.** *Assume condition 2Q($\mathbb{R}^{1+n}$) and $G^{00} = -1$.*

If $s \geq n+3$, then for all $(f,g) \in H^s \times H^{s-1}$, there exists $T > 0$ and a unique $u \in C^0([0,T];H^s) \cap C^1([0,T],H^{s-1})$ that solves QLWIVP. [1]

Proof. Initially, assume $f,g \in C_0^\infty(\mathbb{R}^n)$. Let $k = s-1$.

[1] Please note that, contrary to what was claimed in lecture and an earlier draft of these notes, this argument does not prove that solutions depend continuously on the initial data. (This error was mine. In particular, it was not present in the notes of C. Sogge.) Thanks to Leonardo Tolomeo and Hiro Oh for useful discussions on this point. I hope to post an updated version which will cover the proof of continuity of the map from initial data to solutions. In step 5, one cannot easily obtain an estimate on the energy of the difference of two solutions, $u-v$, because in estimating $L[u]u - L[v]v$, one must estimate both $(L[u]+L[v])(u-v)$, which is fine, and $(L[u]-L[v])(u+v)$, which must be estimate as an inhomogeneity, F, and which involves 2 derivatives on $u+v$. In particular, if k derivatives are applied, then $\vec{\partial}^k((L[u]-L[v])(u+v))$ involves a term with $\partial^2\vec{\partial}^k(u+v)$, which cannot be controlled by $\|u+v\|_{1,k}$. Even if the direction of derivatives could be controlled, $\vec{\partial}^{k+2}(u+v)$ could not be controlled by $\|u+v\|_{1,k}$.

Step 1: Define the Picard iterates Let $u_{-1}=0$. For $m\in\mathbb{N}$, let u_m solve

$$\begin{aligned} L(x,u_{m-1},\partial u_{m-1})u_m &= F(x,u_{m-1},\partial u_{m-1}),\\ u(0,\vec{x}) &= f(\vec{x}),\\ \partial_t u(0,\vec{x}) &= g(\vec{x}). \end{aligned}$$

By induction and the existence of solutions to linear equations, each u_m exists and is smooth. By finite speed of propagation (from uniqueness), for each t and m, $u_m(t,\vec{x})$ vanishes for sufficiently large $|\vec{x}|$.

For simplicity, let $L_m = L(x,u_m,\partial u_m)$ and $F_m = F(x,u_m,\partial u_m)$.

Step 2: Find T

Recall, $\exists C$ (depending on n, s) such that for all u and t,

$$C^{-1}E_k[u](t) \leq \|u\|_{1,k}(t)^2 \leq CE_k[u](t).$$

By the alternative energy bound

$$\begin{aligned} E_k[u_m](t) \leq E_m[u](0)& \\ &+ C\int_0^t \|F_{m-1}\|_k^2 \mathrm{d}t' + \int_0^t\int\sum_{|\alpha|\leq k}|[\vec{\partial}^\alpha, L_{m-1}]u_m|^2\mathrm{d}^n x\mathrm{d}t'\\ &+ C\int_0^t E_k[u_m](t')\mathrm{d}t'. \end{aligned}$$

By the s-chain rule,

$$\begin{aligned} E_k[u_m](t)^2 \leq{}& E_k[u_m](0)\\ &+ C\int_0^t (1+\|u_{m-1}\|_{1,k})^{2k}\mathrm{d}t'\\ &+ C\int_0^t (1+\|u_{m-1}\|_{1,k})^{2k}E_k[u_m]\mathrm{d}t'\\ &+ C\int_0^t E_k[u_m](t')\mathrm{d}t'\\ \leq{}& E_k[u_m](0) + C\int_0^t (1+\|u_{m-1}\|_{1,k})^{2k}E_k[u_m](t')\mathrm{d}t'. \end{aligned}$$

By Gronwall's inequality

$$E_k[u_m](t)^2 \leq 2E_k[u_m](0),$$

if

$$C\int_0^t \left(1+E_k[u_{m-1}](t')\right)^k\mathrm{d}t' \leq \ln 2.$$

This holds by induction if T is chosen sufficiently small relative to $\|f\|_{H^s}^2 + \|g\|_{H^{s-1}}^2 = E_k[u_m](0)$. Similarly integral restrictions on T will be imposed later, with different constants C.

Step 3: Show the Picard iterates are Cauchy We will prove by induction that there is a C such that $\forall m \in \mathbb{N}$.

$$\|u_m(t) - u_{m-1}(t)\|_{1,k}^2 \leq C2^{-2m}.$$

Since $u_{-1} = 0$, the base case follows from choosing C based on $\|f\|_{H^s}^2 + \|g\|_{H^{s-1}}^2$.

Observe that

$$L_{m-1}(u_{m-1} - u_m) = (L_{m-1} - L_{m-2})u_{m-1} - F_{m-2} - F_{m-1}.$$

Since $u_{m-1} - u_{m-2} = 0$ at $t = 0$, by the alternative s-energy estimate

$$\begin{aligned} E_k[u_{m-1} - u_m](t) \leq C \int_0^t & \left(\|(L_{m-1} - L_{m-2})u_{m-1}\|_k^2 + \|F_{m-2} - F_{m-1}\|_k^2\right) \mathrm{d}t' \\ & + C \int_0^t \sum_{|\beta| \leq k} \|[\vec{\partial}^\beta, L_{m-1}](u_{m-1} - u_m)\|_{L^2}^2 \mathrm{d}t'. \end{aligned}$$

From the equivalence of the square of the k-energy and the $1, k$ norm, and from the H^s-chain rule, one finds

$$\begin{aligned} \|u_{m-1} - u_m\|_{1,k}(t)^2 \leq C \int_0^t & \|u_{m-1} - u_{m-2}\|_{1,k}^2 \left(1 + \|u_{m-1}\|_{1,k} + \|u_{m-2}\|_{1,k}\right)^{2k} \|u_{m-1}\|_{1,k}^2 \mathrm{d}t' \\ & + C \int_0^t \|u_{m-1} - u_{m-2}\|_{1,k}^2 \left(1 + \|u_{m-1}\|_{1,k} + \|u_{m-2}\|_{1,k}\right)^{2k} \mathrm{d}t' \\ & + C \int_0^t (1 + \|u_{m-1}\|_{1,k})^{2k} \|u_{m-1} - u_m\|_{1,k}^2 \mathrm{d}t'. \end{aligned}$$

Since the norms $\|u_j\|_{1,k}$ are uniformly bounded, by restricting T to be sufficiently small depending only on the uniform bound, we find

$$\|u_{m-1} - u_m\|_{1,k}(t)^2 \leq \frac{1}{4} \left(\sup_{t'} \|u_{m-1} - u_{m-2}\|_{1,k}(t)^2 + \sup_{t'} \|u_{m-1} - u_m\|_{1,k}(t)^2 \right),$$

$$\sup_{t'} \|u_{m-1} - u_m\|_{1,k}(t)^2 \leq \frac{1}{2} \sup_{t'} \|u_{m-1} - u_{m-2}\|_{1,k}(t)^2.$$

This implies the u_j are Cauchy in $C^0(H^s) \cap C^1(H^{s-1})$. Let u denote the limit.

Step 4: The limit for smooth data is a solution Solving the definition of the Picard iterates for $\partial_t^2 u_m$ and using the fact that $u \in C^0(H^s) \cap C^1(H^{s-1})$, one finds $u_m \in C^2(H^{s-2})$. Furthermore, the convergence of u_m in $C^0(H^s) \cap C^1(H^{s-1})$ implies the convergence in $C^2(H^{s-2})$. Thus, $L_{m-1}u_m - F_{m-1}$ is well defined in $C^0(H^{s-2})$ and identically zero. Taking the limit, $L(x, u, \partial u)u - F(x, u, \partial u)$ is zero as an element of $C^0(H^{s-2}) \subset L^\infty(H^{s-2})$. Since every test function is in $L^1(H^{-s+2})$, it follows that $L(x, u, \partial u)u - F(x, u, \partial u)$ is defined and vanishes as a distribution.

Step 5: completing the argument Since the sequence u_m was Cauchy with respect to the norm $\max_{t \in [0,T]} \|u\|_{1,k}$ and consisted of smooth functions, we find that the limit is in $C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$. Furthermore, as a solution of the equations, we find $u \in C^2([0, T], H^{s-2})$.

Since $s > n+3 > n/2+2$, we find that $C^0([0,T],H^s) \cap C^1([0,T],H^{s-1}) \cap C^2([0,T],H^{s-2}) \subset C^2([0,T]\times\mathbb{R}^n)$. C^2 uniqueness of solutions was proved in a previous theorem, which gives the uniqueness in this space. □

Definition 1.8. *Consider a locally well-posed initial value problem. A continuation criterion involving t and a solution u is a criterion that is sufficient to guarantee that the solution u exists until time t.*

Theorem 1.9 (Continuation criterion)**.** *Assume condition 2Q($\mathbb{R}^{1+n}$) and $G^{00}=-1$. Assume further that G^{ij}, B^i, A, and F do not depend explicitly on $t=x^0$.*

Let $s \geq n+3$. Let $(f,g)\in H^s\times H^{s-1}$, and let T_* be the supremum of times T such that QLWIVP has a solution $u\in C^0([0,T];H^s)\cap C^1([0,T];H^{s-1})$.

The following are equivalent

1. $T_* < \infty$,
2. $\|u\|_{1,s-1} \notin L^\infty([0,T_*))$.
3. $|u|_{1,\lceil \frac{s}{2}\rceil} \notin L^\infty([0,T_*)\times\mathbb{R}^n)$.

Proof. First, observe that since the problem does not depend explicitly on t, we can consider any time t as the initial time and extend to time $t+T$ with T as in the proof of the previous theorem.

Second, although the statement of the previous theorem gave T as depending on (f,g), in the proof, T depended only on the norm $\|f\|_{H^s}+\|g\|_{H^{s-1}}$, in particular, being sufficiently small that $C(1+\|f\|_{H^s}+\|g\|_{H^{s-1}})^sT \leq 1/2$.

Now, consider the statements in the theorem. Since $C^0\subset L^\infty$, (1) implies (2). If $T_*<\infty$ but $\|u\|_{1,s-1}$ is bounded, then at time $T_*-T/2$, we can construct a solution existing until $(T_*-T/2)+T$, contradicting the definition of T_*. Thus, (2) implies (1). By Sobolev embedding, (2) implies (3).

Finally, consider (3) implies (2). In steps 2 and 3 of the proof, we used the energy estimate to estimate the growth of the $1,s-1$ norm using the H^s-chain rule. However, if we had used the s-chain rule, we could, instead have proved estimates of the form

$$\begin{aligned}\int \|F(x,u,\partial u)\|_{0,k}^2 \mathrm{d}t' &= \int\int |F(x,u,\partial u)|_{0,k}^2 \mathrm{d}^n x \mathrm{d}t' \\ &\leq C\int\int (1+|u|_{1,\lceil\frac{k+1}{2}\rceil})^{2k-2}|u|_{1,k}^2 \mathrm{d}^n x\mathrm{d}t' \\ &\leq C\sup_{t,\vec{x}}(1+|u|_{1,\lceil\frac{k+1}{2}\rceil})^{2k-2}\int |u|_{1,k}^2\mathrm{d}^n x \\ &\leq C(1+\sup_{t,\vec{x}}|u|_{1,\lceil\frac{k+1}{2}\rceil})^{2k-2}\|u\|_{1,k}^2,\end{aligned}$$

and similarly for the remaining terms. Substituting $k=s-1$ as in the well-posedness proof gives the desired result. □

Theorem 1.10 (Propagation of regularity)**.** *Assume condition 2Q($\mathbb{R}^{1+n}$) and $G^{00}=-1$. Assume further that G^{ij}, B^i, A, and F do not depend explicitly on $t=x^0$.*

Let $s\geq n+4$.

If $(f,g) \in H^{s+1} \times H^s$, $T > 0$, *and* u *is a solution of QLWIVP in* $C^0([0,T);H^s) \cap C^1([0,T);H^{s-1}$ *is a solution of QLWIVP, then* $u \in C^0([0,T);H^{s+1}) \cap C^1([0,T);H^s)$.

Proof. Let $T' \leq T$. Since $u \in C^0([0,T'];H^s) \cap C^1([0,T'];H^{s-1})$, the $\|u\|_{1,s-1}$ norm remains bounded on $[0,T']$. Since

$$s - 1 \geq \frac{s+1}{2} + \frac{n}{2} + 1 > \left\lceil \frac{s+1}{2} \right\rceil + \frac{n}{2},$$

by Sobolev embedding, $|u|_{1,\lceil \frac{s+1}{2} \rceil}$ remains bounded on $[0,T'] \times \mathbb{R}^n$. By the continuation criterion in $H^{s+1} \times H^s$, this means that u exists in $C^0([0,T'];H^{s+1}) \cap C^1([0,T'];H^s)$. $\square$

Remark 1.11. *From the continuation criterion and propagation of regularity, if* $f,g \in C^\infty$ *and* $|u|_{1,\lceil \frac{n}{2}+2 \rceil}$ *remains bounded, then there are global, smooth solutions.*