ADVANCED PDE II - LECTURE 8: ADDITIONAL RESULTS

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Warning: This is a first draft of the lecture notes and should be used with care![1]

1. Well-posedness for quasilinear wave equations

1.1. **Some further comments related to Burgers' equation and the lack of Lipschitz continuous dependence on initial data.**

The main lecture 8 notes prove existence and uniqueness for quasilinear wave equations. An earlier version claimed to show well-posedness using contraction mapping; such an approach cannot work. Recall that to be well-posed in the sense of Kato for first-order systems, the map from data in H^s to solutions in $C^0(H'^s)$ should be continuous. For second-order equations, this is replaced by the condition that the map from $H^s \times H^{s-1}$ to $C^0(H^s) \cap C^1(H^{s-1})$ is continuous. Recall further that a stronger condition than continuity is Lipschitz continuity. The method using energy and contraction mapping used to treat quasilinear wave equations in these notes can be applied equally well to first-order systems, including Burgers' equation. As shown in Home Work 2, contraction mapping, if it gives continuity, will give Lipschitz continuity. It can be shown that Burgers' equation is not Lipschitz continuous. Thus, a contraction mapping argument cannot hold.

1.2. **Well-posedness of quasilinear wave equations.**

Theorem 1.1 (Kato well-posedness for quasilinear wave equations)**.** *Assume condition 2Q($\mathbb{R}^{1+n}$) and $G^{00} = -1$.*

If $s \geq n + 4$, then for all $(f_0, g_0) \in H^s \times H^{s-1}$, there exists a $T > 0$ and a $\delta > 0$ such that the map from $B = \{(f, g) \in H^s \times H^{s-1} : \|f - f_0\|_{H^s} + \|g - g_0\|_{H^{s-1}} < \delta\}$ to $C^0([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ is continuous.

Remark 1.2. *Here, we take the"obvious" extension to second-order equations of Kato's definition of well-posedness, that the map $H^s \times H^{s-1} \to C^0(H^s) \cap C^1(H^{s-1})$ is continuous. For an inhomogeneous quasilinear wave equation, continuous dependence on the inhomogeneity, F, can also be shown.*

Proof. **Step 1: Uniform time** From step 2 of the existence and uniqueness proof, there is a time of existence which depends only on the norm of the initial data, which gives a uniform time of existence for solutions with inital data in a neighbourhood of (f_0, g_0). (This

[1] This lecture's notes very closely ideas from T. Tao's blog post "Quasilinear well-posedness" and N. Tzvetkov'z "Ill-posedness issues for nonlinear dispersive equations". Both of these build on J.L. Bona, R. Smith: "The initial-value problem for the Korteweg-de Vries equation", Philos. Trans. Roy. Soc. London Ser. A 278 (1975) 1287, 555-601.

time was the time on which energy estimates and Gronwall's inequality could be applied to prove a uniform bound on the energy. Frequently, solutions can be extended to much longer times of existence.) In particular, there is a $T > 0$ such that if (f_0, g_0) have $H^s \times H^{s-1}$ norm less than δ and (f, g) have norm less than 2δ, then the solution u arising from initial data (f, g) will have $C^0(H^s) \cap C^1(H^{s-1})$ norm not greater than 4δ.

Step 2: Lipschitz estimate with loss of regularity Let u and v be two solutions of QLWIVP with initial data (f_1, g_1) and (f_2, g_2) respectively. Let $w = u - v$. Observe that, since $L[u, \partial u]u = F[u, \partial u]$ and $L[v, \partial v]v = F[v, \partial v]$

$$\begin{aligned}
L[u, \partial u]u - L[v, \partial v]v &= F[u, \partial u] - F[v, \partial v], \\
L[u, \partial u]u - L[v, \partial v]v &= L[u, \partial u]u - L[u, \partial u]v + L[u, \partial u]v - L[v, \partial v]v \\
&= L[u, \partial u](u - v) + (L[u, \partial u] - L[v, \partial v])v \\
&= L[u, \partial u](u - v) - (L[u, \partial u] - L[v, \partial v])(u - v) + (L[u, \partial u] - L[v, \partial v])u.
\end{aligned}$$

Thus,

$$L[u, \partial u]w = (L[u, \partial u] - L[v, \partial v])(u - v) - (L[u, \partial u] - L[v, \partial v])u + F[u, \partial u] - F[v, \partial v].$$

Let $k \in \mathbb{N}$ and $|\beta| = k$. Applying $\vec{\partial}^\beta$, we find

$$L[u, \partial u]\vec{\partial}^\beta w = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

where

$$\begin{aligned}
I_1 &= -[\vec{\partial}^\beta, L[u, \partial u]]w \\
I_2 &= (L[u, \partial u] - L[v, \partial v])\vec{\partial}^\beta w \\
I_3 &= [\vec{\partial}^\beta, L[u, \partial u] - L[v, \partial v])w \\
I_4 &= -(L[u, \partial u] - L[v, \partial v])\vec{\partial}^\beta u \\
I_5 &= -[\vec{\partial}^\beta, L[u, \partial u] - L[v, \partial v]]u \\
I_6 &= \vec{\partial}^\beta(F[u, \partial u] - F[v, \partial v]).
\end{aligned}$$

We now derive an energy estimate by applying the energy estimate in divergence form to $\vec{\partial}^\beta w$ with $X = -\partial_t$. Using the energy based on $G = G[u, \partial u]$, one finds

$$\begin{aligned}
&E_k[w](t) - E_k[u](0) \\
&\quad = \int_0^t (-\partial_t \vec{\partial}^\beta w)(L[u, \partial u]\vec{\partial}^\beta w) \mathrm{d}^n x \mathrm{d}t' \\
&\qquad + \int_0^t \left(\left((\partial_i G^{ik})\partial_k \vec{\partial}^\beta w - B^i \partial_i \vec{\partial}^\beta w - (a+1)\vec{\partial}^\beta w \right) \partial_t \vec{\partial}^\beta w - \frac{1}{2}(-\partial_t G^{lm})\partial_l \vec{\partial}^\beta w \partial_m \vec{\partial}^\beta w \right) \mathrm{d}^n x \mathrm{d}t'.
\end{aligned}$$

The terms in the second integral can be estimated by the Cauchy-Schwarz inequality. All such terms can be estimated by $C(1 + \sum_{\|\alpha\| \le 1} |\partial^\alpha u|)|w|_{1,k}^2$.

It remains to estimate the terms from $(-\partial_t \vec{\partial}^\beta w)(L\vec{\partial}^\beta w)$. These can be treated using the Cauchy-Schwarz inequality when there is a good estimate on the corresponding terms in

$L\vec{\partial}w$. In particular, one finds

$$
\begin{aligned}
|I_1| &= |[\vec{\partial}^\beta, L[u,\partial u]]w| \le C\left((1+|u|_{1,\lceil\frac{k}{2}\rceil})^k|w|_{1,k} + (1+|u|_{1,\lceil\frac{k}{2}\rceil})^{k-1}|u|_{1,k}|w|_{1,\lceil\frac{k}{2}\rceil}\right),\\
|I_3| &= |[\vec{\partial}^\beta, L[u,\partial u]-L[v,\partial v])w| \le C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^k|w|_{1,k}\\
&\quad + C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^{k-1}(|u|_{1,k}+|v|_{1,k})|w|_{1,\lceil\frac{k}{2}\rceil},\\
|I_5| &= |[\vec{\partial}^\beta, L[u,\partial u]-L[v,\partial v]]u| \le C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^k|w|_{1,k}\\
&\quad + C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^{k-1}(|u|_{1,k}+|v|_{1,k})|w|_{1,\lceil\frac{k}{2}\rceil},\\
|I_6| &= |\vec{\partial}^\beta(F[u,\partial u]-F[v,\partial v])| \le C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^{k-1}|u-v|_{1,k}\\
&\quad + C(1+|u|_{1,\lceil\frac{k}{2}\rceil}+|v|_{1,\lceil\frac{k}{2}\rceil})^{k-2}(|u|_{1,k}+|v|_{1,k})|u-v|_{1,\lceil\frac{k}{2}\rceil}.
\end{aligned}
$$

The contribution to the change in the energy estimate from I_2 cannot be treated by the Cauchy-Schwarz inequality but can be treated using integration by parts. In particular,

$$
\begin{aligned}
&(\partial_t\vec{\partial}^\beta w)((L[u,\partial u]-L[v,\partial v])\vec{\partial}^\beta w\\
&\quad = (\partial_t\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_i\partial_j\vec{\partial}^\beta w)\\
&\qquad + (\partial_t\vec{\partial}^\beta w)(B[u,\partial u]^i\partial_i - B[v,\partial v]^i\partial_i + (A[u,\partial u]-A[v\partial v])(\vec{\partial}^\beta w)
\end{aligned}
$$

The terms on the last line can be estimated by $C(|u|_1+|v|_1)|w|_{1,k}^2$. The remaining term is

$$
\begin{aligned}
&(\partial_t\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_i\partial_j\vec{\partial}^\beta w)\\
&\quad = -(\partial_t\partial_i\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_j\vec{\partial}^\beta w)\\
&\qquad + \partial_i\left((\partial_t\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_j\vec{\partial}^\beta w)\right)\\
&\qquad + (\partial_t\vec{\partial}^\beta w)(\partial_i((G[u,\partial u]-G[v,\partial v])^{ij})\partial_j\vec{\partial}^\beta w)\\
&\quad = -\frac{1}{2}\partial_t\left((\partial_i\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_j\vec{\partial}^\beta w)\right)\\
&\qquad + (\partial_i\vec{\partial}^\beta w)(\partial_t((G[u,\partial u]-G[v,\partial v])^{ij})\partial_j\vec{\partial}^\beta w)\\
&\qquad + \partial_i\left((\partial_t\vec{\partial}^\beta w)((G[u,\partial u]-G[v,\partial v])^{ij}\partial_j\vec{\partial}^\beta w)\right)\\
&\qquad + (\partial_t\vec{\partial}^\beta w)(\partial_i((G[u,\partial u]-G[v,\partial v])^{ij})\partial_j\vec{\partial}^\beta w)
\end{aligned}
$$

The second and fourth terms on the right-hand side of the final equation can be estimated by $C(|u|_{1,1}+|v|_{1,1})|v|_{1,k}^2$, after applying the equation to eliminate terms involving $\partial_t^2 u$ or $\partial_t^2 v$ from $\partial_t G[u,\partial v]$ or $\partial_t G[v,\partial v]$. When the first and third term are integrated over $[0,t]\times B(\vec{0},R)$, where R is a sufficiently large radius in $\mathbb{R}^n$, the boundary terms at $|\vec{x}|=R$ vanish as $R\to\infty$ and there remain integrals over $\{0\}\times\mathbb{R}^n$ and $\{t\}\times\mathbb{R}^n$. The integrand on these surfaces is bounded by $|u-v|_1|w|_{1,k}^2$, so, as long as $u-v$ is small, they can be dominated by $1/10$ times the initial and final energy.

It remains to treat the contribution from I_4, which, from the perspective of regularity is far worse term. Since L is a second-order operator, but G^{00} is always one, regardless of the argument, $L[u,\partial u]-L[v,\partial v]$ is a second-order operator with at most one time derivative. Thus,

$$\begin{aligned}|(\partial_t\vec{\partial}^\beta w)I_4| &= |(\partial_t\vec{\partial}^\beta w)(L[u,\partial u]-L[v,\partial v])\vec{\partial}^\beta w|\\ &\leq C|w|_{1,k}|u-v|_1|u|_{1,k+1}.\end{aligned}$$

Summing the estimates on the growth of the k-energy for w, summing over $|\beta|\leq k$, using the equivalence of the k-energy and the $1,k$ norm squared, applying the Sobolev estimate, and integrating in space and then in time, we find

$$\begin{aligned}\|w\|_{1,k}(t)^2 \leq C\|w\|_{1,k}(0)^2 &+ C\int_0^t (1+\|u\|_{1,k}+\|v\|_{1,k})^k\|w\|_{1,k}^2\mathrm{d}t'\\ &+ C\int_0^t \|w\|_{1,k}\|w\|_1\|u\|_{1,k+1}\mathrm{d}t'.\end{aligned}$$

A further application of the Cauchy-Schwarz inequality yields the following energy estimate for differences

$$\|w\|_{1,k}(t)^2 \leq C\|w\|_{1,k}(0)^2 + C\int_0^t (1+\|u\|_{1,k}+\|v\|_{1,k})^k\|w\|_{1,k}^2\mathrm{d}t' + C\int_0^t \|w\|_1^2\|u\|_{1,k+1}^2\mathrm{d}t'. \tag{1}$$

When applying this formula, we will refer to u as the reference solution and v as the perturbation.

Step 3: Continuity by approximation by more regular reference solutions We now conclude the proof of continuity. Let $(f_0,g_0)\in H^s\times H^{s-1}$ as in step 1 and let $\epsilon>0$. Theorem A.1 provides a linear operator ρ_N from H^j to H^k for any $j<k$ such that for $i\leq j:\forall N\in[1,\infty):\forall u\in H^k$:

$$\begin{aligned}\|\rho_N(u)\|_{H^k} &\leq (1+N)^{k-j}\|u\|_{H^j},\\ \|\rho_N(u)-u\|_{H^i} &\leq (1+N)^{i-j}\|u\|_{H^j}.\end{aligned}$$

Furthermore, $\lim_{N\to\infty}\|\rho_N(u)-u\|\to 0$. Let N and δ be such that the following conditions hold:

$$\begin{aligned}N &\geq 1,\\ N^{s-3}\|(f_0,g_0)\|_{H^s\times H^{s-1}} &\leq \epsilon,\\ \|\rho_N(f_0)-f_0\|_{H^s}+\|\rho_N(g_0)-g_0\|_{H^{s-1}} &< \epsilon,\\ \delta &< \frac{\epsilon^2}{N(1+\|(f_0,g_0)\|_{H^s\times H^{s-1}})}.\end{aligned}$$

Suppose $\|(f_1,g_1)-(f_0,g_0)\|_{H^s\times H^{s-1}}<\delta$. Observe that

$$\begin{aligned}\|\rho_N(f_1)-f_1\|_{H^s} &\leq \|\rho_N(f_1)-\rho_n(f_0)\|_{H^s}+\|\rho_N(f_0)-f_0\|_{H^s}+\|f_0-f_1\|_{H^s}\\ &\leq \|f_1-f_0\|_{H^s}+\|\rho_N(f_0)-f_0\|_{H^s}+\|f_1-f_0\|_{H^s}\\ &\leq C\epsilon.\end{aligned}$$

Similarly, $\|\rho_N(f_1) - f_1\|_{H^{s-1}} \le C\epsilon$. In addition,

$$\begin{aligned}
&\|\rho_N(f_0)\|_{H^s} + \|\rho_N(g_0)\|_{H^{s-1}} \le \|f_0\|_{H^s} + \|g_0\|_{H^{s-1}},\\
&\|\rho_N(f_1)\|_{H^s} + \|\rho_N(g_1)\|_{H^{s-1}} \le \|f_1\|_{H^s} + \|g_1\|_{H^{s-1}} \le C(\|f_0\|_{H^s} + \|g_0\|_{H^{s-1}}),\\
&\|\rho_N(f_0)\|_{H^{s+1}} + \|\rho_N(g_0)\|_{H^s} \le CN(\|f_0\|_{H^s} + \|g_0\|_{H^{s-1}}),\\
&\|\rho_N(f_1)\|_{H^{s+1}} + \|\rho_N(g_1)\|_{H^s} \le CN(\|f_0\|_{H^s} + \|g_0\|_{H^{s-1}}).
\end{aligned}$$

Let u be the solution arising from (f_0, g_0), v be the solution arising from (f_1, g_1), u_N be the solution arising from $(\rho_N(f_0), \rho_N(g_0))$, and v_N be the solution arising from $(\rho_N(f_1), \rho_N(g_1))$. Observe that since these are all close to u, the solutions exist up to time T and all have a $C^0(H^s) \cap C^1(H^{s-1})$ norm that is bounded by $C\|(f_0, g_0)\|_{H^s \times H^{s-1}}$. Furthermore, from applying the continuation criterion, we find that u_N and v_N, which have initial data in the higher regularity $H^{s+1} \times H^s$, are in $C^0(H^{s+1}) \cap C^1(H^s)$ and have $\sup_{t\in[0,T]} \|u_N\|_{1,s} + \sup_{t\in[0,T]} \|v_N\|_{1,s} \le CN\|(f_0, g_0)\|_{H^s\times H^{s-1}}$.

Using the uniform bound on the solutions, for $k \le s - 2$, from the energy estimate for differences (1),

$$\begin{aligned}
\sup_{t\in[0,T]} \|u - u_N\|_{1,k} &\le CN^{(k+1)-(s-1)}(\|f_0 - \rho_N(f_0)\|_{H^s} + \|g_0 - \rho_N(g_0)\|_{H^{s-1}})\\
&\le CN^{k+2-s}\epsilon.
\end{aligned}$$

In particular,

$$\sup_{t\in[0,T]} \|u - u_N\|_1 \le CN^{2-s}\epsilon,$$

and similarly for v.

The smallness of $\|u - u_N\|_1$ can be used to compensate for the largeness of $\|u_N\|_{1,s}$ in the energy estimate for differences (1), when $k = s - 1$. Thus, taking u_N as the reference solution and u as the perturbation, we find

$$\begin{aligned}
\|u - u_N\|_{1,s-1}(t)^2 \le{}& C\|u - u_N\|_{1,s-1}(0)^2 + C\int_0^t (1 + \|u\|_{1,s-1} + \|u_N\|_{1,s})^k \|u - u_N\|^2_{1,s-1} \mathrm{d}t'\\
&+ C\int_0^t \|u - u_N\|_1^2 \|u_N\|^2_{1,s} \mathrm{d}t'\\
\le{}& C\epsilon^2 + C\int_0^t \|u - u_N\|^2_{1,s-1}\mathrm{d}t' + C\int_0^t N^{2-s}\epsilon N\|(f_0, g_0)\|_{H^s\times H^{s-1}}\mathrm{d}t'\\
\le{}& C\epsilon^2 + C\int_0^t \|u - u_N\|^2_{1,s-1}\mathrm{d}t'.
\end{aligned}$$

Thus, from Gronwall's inequality, for $t \in [0, T]$, $\|u - u_N\|_{1,s-1}(t)^2 \le \epsilon$. A similar estimate holds for $\|v - v_N\|_{1,s-1}$.

We now wish to take u_N as the reference solution and v_N as the perturbation, the energy estimate for differences. To do so, we note that $\|u_N - v_N\|_1 \leq \|u-v\|_1 \leq \delta = \epsilon^2/N$, so

$$\begin{aligned}\|u_N - v_N\|_{1,s-1}(t)^2 \leq \|u_N - v_N\|_{1,s-1}(0)^2 + \int_0^t (1 + \|u_N\|_{1,s-1} + \|v_N\|_{1,s-1})^2 \|u_n - v_N\|_{1,s-1}^2 \mathrm{d}t' \\ + C\int_0^t \frac{\epsilon^2}{N(1+\|(f_0,g_0)\|_{H^s\times H^{s-1}})} N\|(f_0,g_0)\|_{1,s-1}\mathrm{d}t' \\ \leq C\epsilon^2 + C\int_0^t \|u_n - v_N\|_{1,s-1}^2 \mathrm{d}t'.\end{aligned}$$

Gronwall's inequality again yields $\|u_N - v_N\|_{1,s-1}(t)^2 \leq C\epsilon^2$.

Summing gives, at each t, $\|u-v\|_{1,s-1} \leq \|u-u_N\|_{1,s} + \|u_N - v_N\|_{1,s-1} + \|v - v_N\|_{1,s-1} \leq C\epsilon$, which gives the continuity of the map from initial data in $H^s \times H^{s-1}$ to $C^0(H^s) \cap C^1(H^{s-1})$. □

Appendix A. Further results in Harmonic analysis: Approximation by more regular functions

Theorem A.1. *Let $i,j,k \in \mathbb{N}$ with $i \leq j \leq k$. There is a family of linear operators $\rho_N : H^i \to H^k$ such that $\forall N \in [1,\infty) : \forall u \in H^k$:*

$$\begin{aligned}\|\rho_N(u)\|_{H^k} &\leq C(1+N)^{k-j}\|u\|_{H^j},\\ \|\rho_N(u)\|_{H^j} &\leq \|u\|_{H^j},\\ \|\rho_N(u)-u\|_{H^i} &\leq C(1+N)^{i-j}\|u\|_{H^j}.\end{aligned}$$

Furthermore, $\lim_{N\to\infty}\|\rho_N(u)-u\| \to 0$.

Proof. Let $\mathcal{F}$ denote the Fourier transform. Let $\chi_N(\vec{\xi})$ be a smooth function that takes values in $[0,1]$ and is constantly 1 for $|\vec{\xi}| \leq N$ and constantly zero for $|\vec{\xi}| \geq N+1$. Let $\rho_N(u) = \mathcal{F}^{-1}(\chi_N \mathcal{F}(u))$. From Plancherel's theorem, $\|u\|_{H^k}^2 = \sum_{|\alpha|\leq k} \|\xi^\alpha \mathcal{F}(u)(\xi)\|_{L^2(\mathrm{d}^n\xi)}^2$. The three estimates then follow from the fact that $\chi_N \sum_{|\alpha|\leq k}(\xi^k)^2 \leq C(1+N)^{k-j}\sum_{|\alpha|<j}(\xi^\alpha)^2$, $\chi_N \leq 1$, and $(1-\chi_N)\sum_{|\alpha|\leq i}(\xi^\alpha)^2 \leq C(1+N)^{i-j}\sum_{|\alpha|\leq j}(\xi^\alpha)^2$. The convergence result follows from the the pointwise convergence of χ_N to 1 and from the Lebesgue dominated convergence theorem. □