ADVANCED PDE II - LECTURE 9

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Warning: This is a first draft of the lecture notes and should be used with care! Again, this lecture's notes very closely follow C. Sogge's "Nonlinear Wave Equations".

1. Symmetries and other higher norms

1.1. Symmetries.

Definition 1.1. *Let L be a linear differential operator. A* ***symmetry*** *for the linear PDE $Lu = 0$ is a differential operator S such that if $Lu = 0$ then $L(Su) = 0$.*

Theorem 1.2. *Let L and S be linear differential operators taking $\mathcal{S}$ to $\mathcal{S}$. If $[L, S] = 0$ or there is a (possibly order zero) differential operator P such that $[L, S] = P \circ L$, then S is a symmetry for the linear PDE $Lu = 0$.*

Proof. If $Lu = 0$ and $[L, S] = P \circ L$, then $0 = SLu = LSu - [L, Su] = LSu - PLu = L(Su)$. □

1.2. Higher norms.

Definition 1.3. *Let $\mathbb{S} = \{S_i\}_{i=1}^N$ be a collection of linear differential operators and $k \in \mathbb{N}$. The $\mathbb{S}$-k pointwise and spatial norms are defined by*

$$|f|^2_{\mathbb{S},k} = \sum_{j\le k} \sum_{i_1=1}^{N} \dots \sum_{i_j=1}^{N} |S_{i_j} \dots S_{i_1} f|^2,$$

$$\|f\|^2_{\mathbb{S},k} = \int_{\mathbb{R}^n} |f|^2_{\mathbb{S},k} \mathrm{d}^n x.$$

These are defined on functions for which the norms are finite, for example on the Schwartz class.

Given a vector field X, the $\mathbb{S} - k$ strengthened energy is

$$E_{X,\mathbb{S},k}[u](\Sigma) = \sum_{0\le j\le k} \sum_{i_1=1}^{N} \dots \sum_{i_j=1}^{N} E_x[S_{i_j} \dots S_{i_1} u](\Sigma).$$

1.3. Applications to the wave equation.

Definition 1.4. *In $\mathbb{R}^{1+n}$, let*

Vector fields		*Number*
$T = \partial_t,$		1
$X_i = \partial_i$	*for* $i \in \{1, \dots, n\}$	$n,$
$\Omega_{ij} = \sum_{k=0}^{n} (\eta_{jk} x^k \partial_i - \eta_{ik} x^k \partial_j)$	*for* $i, j \in \{0, \dots, n\}$	$\frac{n(n+1)}{2},$
$L = \sum_{k=0}^{n} x^k \partial_k$		$1.$

Let

$$\mathbb{X} = \{X_i\}_{i=1}^n,$$
$$\mathbb{\Gamma} = \{T, L\} \cup \mathbb{X} \cup \{\Omega_{ij}\}_{0 \le i < j \le n},$$
$$\mathbb{\Theta} = \{\Omega_{ij}\}_{1 \le i < j \le n}.$$

*$\mathbb{X}$ is a basis for translations of **translations**, $\mathbb{\Theta}$ generates **rotations**.*

Observe that the norms used in previous lectures can be expressed as

$$|u|_{1,k}^2 = \sum_{|\alpha| \le 1} \sum_{|\beta| \le k} |\partial^\alpha \vec{\partial}^\beta u|^2 = \sum_{|\alpha| \le 1} |\partial^\alpha u|_{\mathbb{X},k}^2.$$

Observe further that rotations can be applied to functions in $C^\infty(S^{n-1})$.

Theorem 1.5 (Commutators for $\mathbb{\Gamma}$)**.**

$$\forall \Gamma \in \mathbb{\Gamma} \backslash \{L\} : \qquad [\eta^{ij} \partial_i \partial_j, \Gamma] = 0,$$
$$[\eta^{ij} \partial_i \partial_j, L] = 2\eta^{ij} \partial_i \partial_j.$$

Furthermore,

$$\forall \Gamma_\alpha, \Gamma_\beta \in \mathbb{\Gamma} : \exists \{C^\gamma{}_{\alpha\beta}\}_\gamma : \qquad [\Gamma_\alpha, \Gamma_\beta] = C^\gamma{}_{\alpha\beta} \Gamma_\gamma.$$

Proof. The commutators can be computed by direct computation. The coefficients are rational of order 0, which shows they are of symbol type of order 0. □

Corollary 1.6 (Symmetries for the wave and Klein-Gordon equation)**.** *$\mathbb{\Gamma}$ is a collection of symmetries for the wave equation $\eta^{ij} \partial_i \partial_j u = 0$ and $\mathbb{\Gamma} \backslash \{L\}$ is a collection of symmetries for the Klein-Gordon equation $(\eta^{ij} \partial_i \partial_j - 1)u = 0$.*

Proof. The symmetry properties follow from the commutator properties involving $\eta^{ij} \partial_i \partial_j$. □

2. The Klainerman-Sobolev inequality

Theorem 2.1 (Klainerman-Sobolev inequality)**.** *Let $T > 0$. Let $u \in C^\infty([0, T] \times \mathbb{R}^n)$ vanish uniformly for $|x|$ sufficiently large. There is a C such that for $t \ge 0$,*

$$|u(t, x)| \le C(1 + t + |\vec{x}|)^{-\frac{n-1}{2}} (1 + |t - |\vec{x}||)^{-\frac{1}{2}} \|u(t)\|_{\mathbb{\Gamma}, \frac{n+2}{2}}.$$

The following results are useful for the proof

Definition 2.2. *A function $f : \mathbb{R}^{1+n} \to \mathbb{R}$ is of symbol class of order* 0 *if for all $x \in \mathbb{R} \times (\mathbb{R}^n \backslash \{\vec{0}\})$ and multiindices α there is a constant C_α such that*

$$|\partial^\alpha u(t,\vec{x})| \leq C_\alpha(|t| + |\vec{x}|)^{-|\alpha|}.$$

Lemma 2.3 (Formulae for t and r derivatives)**.** *Let $r = |\vec{x}|$ and $\partial_r = r^{-1}\sum_{i=1}^n x^i \partial_i$. In $\mathbb{R} \times (\mathbb{R}^n \backslash \{\vec{0}\})$,*

$$(t-r)\partial_r = -\frac{r}{r+t}L + \sum_{i=1}^n \frac{t}{r+t}\frac{x^i}{r}\Omega_{0i},$$
$$(t-r)\partial_t = \frac{t}{r+t}L - \sum_{i=1}^n \frac{1}{r+t}x^i\Omega_{0i}.$$

The coefficients of L and Ω_{0i} in the previous two formulae are of symbol class of order 0*.*

Furthermore,

$$(t-r)^2\sum_{i=0}^n |\partial_i u(t,\vec{x})|^2 \leq |Lu(t,x)|^2 + \sum_{0\leq j<k\leq n} |\Omega_{jk}u(t,\vec{x})|^2.$$

Proof. The identities can be verified by direct calculation. The symbol properties follow from the coefficients being homogeneous rational functions of order 0.

By rotational symmetry, we may assume $x^1 = r$, $x^2 = \ldots = x^n = 0$. At such a point,

$$(t^2+r^2)\sum_{i=2}^n |\partial_i u|^2 = \sum_{1\leq j<k\leq n} |\Omega_{jk}u|^2.$$

and, by the first two formulae in the the theorem

$$(t^2-r^2)\partial_t u = tLu - x^1\Omega_{01}u,$$
$$(t^2-r^2)\partial_1 u = -x^1Lu + t\Omega_{01}u.$$

Thus,

$$(t^2-r^2)^2(|\partial_t u|^2 + |\partial_r u|^2) \leq 2(t+r)^2(|Lu|^2 + |\Omega_{01}u|^2)$$
$$\leq (t^2+r^2)(|Lu|^2 + |\Omega_{01}u|^2),$$

which completes the proof. ∎

Lemma 2.4 (Radial, angular, and localised Sobolev estimate)**.** *Let $n \in \mathbb{Z}^+$ and $\delta > 0$. For a multiindex $\alpha \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, define $\Theta^\alpha = \Omega_{12}^{\alpha_1}\Omega_{13}^{\alpha_2}\ldots\Omega_{(n-1)n}^{\alpha_{\frac{n(n-1)}{2}}}$.*

There are constants $C_{n,\delta}$ and C_n such that:

- *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $\vec{x} \in \mathbb{R}^n$*

$$|f(\vec{x})|^2 \leq C_{n,\delta}\sum_{|\alpha|\leq\frac{n+2}{2}} \int_{|\vec{y}|\leq\delta} |(\vec{\partial}^\alpha f)(\vec{x}+\vec{y})|^2 \mathrm{d}^n y.$$

- *If $u \in C^\infty(S^{n-1})$ and $\omega \in S^{n-1}$, then*

$$|u(\omega)|^2 \leq C_n \sum_{|\alpha|<\frac{n+1}{2}} \int_{S^{n-1}} |\Theta^\alpha u|^2(\nu) \mathrm{d}^{n-1}\sigma(\nu).$$

- *If $v \in C^\infty((0,\infty)\times S^{n-1})$ and $(r,\omega) \in (0,\infty)\times S^{n-1}$, then*

$$|v(r,\omega)|^2 \leq C_{n,\delta} \sum_{j+|\alpha|} \int_{S^{n-1}} \int_{|q|\leq\delta} |(\partial_q^j \Theta^\alpha v)(r+q,\nu)|^2 \mathrm{d}q \mathrm{d}^{n-1}\sigma(\nu).$$

Proof. The first estimate follows from the standard Sobolev inequality and applying a smooth cut-off function. The second follows from working in local coordinates, applying smooth cut-off functions, and observing that the rotations span the tangent space. The third follows from applying the same ideas in $(0,\infty)\times S^{n-1}$. □

Lemma 2.5. *Let $T > 0$. Let $u \in C^\infty([0,T]\times\mathbb{R}^n)$ vanish uniformly for $|x|$ sufficiently large. There is a C such that for $t \geq 0$,*

$$|u(t,x)| \leq C(1+t+|\vec{x}|)^{-\frac{n-1}{2}} \|u(t)\|_{\mathbb{F},\frac{n+2}{2}}.$$

Remark 2.6. *This lemma is missing a factor of $(1+|t-|\vec{x}|)^{-\frac{1}{2}}$ relative to the Klainerman-Sobolev inequality.*

Proof. **Case 0:** $t+|vecx| \leq 1$ This follows from localised Sobolev estimate.
Case 1: $r \notin [t/2, 3t/2]$ here $|t-r| \sim |t+r|$. Let $|\vec{y}| \leq \frac{t+|\vec{x}|}{8}$, so $|t-|\vec{x}-\vec{y}|| \geq c|t+|\vec{x}||$. From the first localised Sobolev inequality to $f(\vec{z}) = u(t+\vec{x}+(t+|\vec{x}|)\vec{z})$, one finds

$$(t+|\vec{x}|)^n |u(t,\vec{x})|^2 \leq (t+|\vec{x}|)^n \sum_{|\alpha|\leq\frac{n+2}{2}} \int_{|\vec{y}|<1/8} |\partial_y^\alpha (u(t,\vec{x}+(t+|\vec{x}|)\vec{y}))|^2 \mathrm{d}^n y.$$

Now, applying a change of variables $\vec{y} \mapsto (t+|\vec{x}|)\vec{y}$

$$\begin{aligned}(t+|\vec{x}|)^n |u(t,\vec{x})|^2 &\leq \sum_{|\alpha|\leq\frac{n+2}{2}} \int_{|\vec{y}|<\frac{t+|\vec{y}|}{8}} |((t+|\vec{x})\partial_y)^\alpha u(t,\vec{x}+\vec{y})|^2 \mathrm{d}^n y \\ &\leq C\|u(t)\|^2_{\mathbb{F},\lfloor\frac{n+2}{2}\rfloor}.\end{aligned}$$

Case 2: $r \in [t/2, 3t/2]$ In this region $t \sim r \sim t+r$. Let $q = |\vec{x}| - t = r - t$, $\omega \in S^{n-1}$ be such that $\vec{x} = (t+q)\omega$, and $u(t,q,\omega)$ denote $u(t,(t+q)\omega)$. Let $di^{n-1}\sigma(\omega)$ denote integration in the ω variable wiht respect the standard volume form on the sphere. Thus,

$$\mathrm{d}^n x = (t+q)^{n-1}\mathrm{d}q\mathrm{d}^{n-1}\sigma(\omega), \qquad \partial_r = \partial_q, \qquad q\partial_q = (r-t)\partial_r.$$

From first applying the localised Sobolev estimate, then undoing the change of variables, and then recalling how ∂_r and $(t-r)\partial_r$ can be expanded in terms of the $\mathbb{\Gamma}$, one finds

$$\begin{aligned} t^{n-1}|u(t,\vec{x})|^2 &\leq t^{n-1} \sum_{j+k+|\alpha|\leq \frac{n+2}{2}} \int_{S^{n-1}} \int_{-\frac{3}{4}t\leq q\leq t} |(q\partial_q)^j \partial_q^k \Theta^\alpha u(t,q,\nu) \mathrm{d}q \mathrm{d}^{n-1}\sigma(\nu) \\ &\leq 4^{n-1} \sum_{j+k+|\alpha|<\frac{n+2}{2}} \int_{S^{n-1}} \int_{\frac{1}{4}t\leq r\leq t} |((t-r)\partial_r)^j \partial_r^k \Theta^\alpha u(t,r\nu)|^2 r^{n-1} \mathrm{d}r \mathrm{d}^{n-1}\sigma(\omega) \\ &\leq C\|u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}. \end{aligned}$$

□

Observe that in the previous proof, a decay rate of $(t+r)^{-n/2}$ was derived in the region where $r < t/2$ and $r > 3t/2$. The full Klainerman-Sobolev inequality, with decay of $\langle t+r\rangle^{-\frac{n-1}{2}}\langle t-r\rangle^{-1/2}$ can be found by dividing the region $r \in [t/2, 3t/2]$ into sectors where r/t varies in a narrow band, applying an argument like that used for $|r/t-1| > 1/2$, and tracking the constants carefully to find the dependence on the lower bound for $|r/t-1|$. The details can be found in many sources, including Sogge's "Nonlinear Wave Equations".

Corollary 2.7 (Proof of decay for solutions of the wave equation by vector fields)**.** *Let $n \geq 3$. Let $f, g \in C_c^\infty$. If u is a solution of*

$$\begin{aligned} \eta^{ij}\partial_i\partial_j u &= 0 \\ u(0,\vec{x} &= f(\vec{x}), \\ \partial_t u(0,\vec{x} &= g(\vec{x}), \end{aligned}$$

then $\forall (t,\vec{x}) \in \mathbb{R}^{1+n}$

$$\begin{aligned} |\partial u|(t,\vec{x}) &\leq C_n(1+t)^{-\frac{n-1}{2}}(1+|t-|\vec{x}||)^{-\frac{1}{2}}\|\partial u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}(0), \\ |u|(t,\vec{x}) &\leq C_n(1+t)^{-\frac{n-1}{2}}(1+|t-|\vec{x}||)^{\frac{1}{2}}\|\partial u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}(0). \end{aligned}$$

Remark 2.8. *The factor of $(1+|t-|\vec{x}||)^{\frac{1}{2}}$ can be removed if n is odd.*

Proof. Recall $\frac{1}{2}\|\partial u\|_{L^2}(t) = E[u](\{t\}\times\mathbb{R}^n)$. If $\Gamma \in \mathbb{\Gamma}$, then Γ is a symmetry, so Γu also has a conserved energy. Thus, for $t > 0$, using conservation of energy for $\Gamma_{i_k} \ldots \Gamma_{i_1}$ and the Klainerman-Sobolev inequality, one finds

$$\begin{aligned} \|\partial u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}(0) &= CE_{T,\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}[u](\{0\}\times\mathbb{R}^n) \\ &= CE_{T,\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}[u](\{t\}\times\mathbb{R}^n) \\ &= C\|\partial u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}(t) \\ &\geq C(1+t)^{\frac{n-1}{1}}(1+t-|\vec{x}|)^{1/2}|\partial u| \end{aligned}$$

Now consider u itself. For fixed t, conisder three regions of r, $r \geq 2t$, $r \in [t, 2t)$, and $r < t$. In the first region, we can integrate along a radial going from r to ∞. Using $N = \|\partial u\|_{\mathbb{\Gamma},\lfloor\frac{n+2}{2}\rfloor}$, this gives $|u(t,r,\omega)| \leq \int_r^\infty |\partial u(t,s,\omega)|\mathrm{d}s \leq CN\int_r^\infty (1+s+t)^{-\frac{n-1}{2}}(1+s-t)^{1/2}\mathrm{d}s$ $\leq CN\int_r^\infty (1+s-t)^{-(n+1)/2}\mathrm{d}s \leq CN(1+(r-t))^{-n/2+1} \leq CN(1+r+t)^{-\frac{n-1}{2}}(1+r-t)^{1/2}$. In the region, $r \in [t, 2t)$, we can integrate along a radial line from r to $2t$ and use the decay from the region $r \geq 2t$. Here, $|u(t, r, \omega)| \leq CN \int_r^{2t} (1+s+t)^{-\frac{n-1}{2}} (1+s-t)^{1/2} \mathrm{d}s + CN(1+t)^{-n/2+1}$ $\leq CN \int_r^{2t} (1+t)^{-\frac{n-1}{2}} (1+s-t)^{1/2} \mathrm{d}s + CN(1+t)^{-n/2+1} \leq CN \int (1+t)^{-n-1/2} (1+t-r)^{1/2}$.

Finally, consider the region $r < t$. Applying the same argument as in the region $r \in [t, 2t)$, and integrating along a radial line from r to t, one finds $|u(t, r, \omega)| \leq (1+t)^{-n/2+1}$ $\leq \int_r^t |\partial u| \mathrm{d}s + |u(t, t, \omega)|$. Using the same sort of argument as in the region $r \in [t, 2t)$, one can show that $\int_r^t |\partial u| \mathrm{d}s$ is bounded by the same sort of bound as $|u(t, t, \omega)|$, namely $CN(1+t)^{-n/2+1}$. Thus, for $n \geq 3$ and fixed (r, ω), $u(t, r, \omega)$ goes to zero as $t \to \infty$. Now, integrating in time, with r, ω fixed, one finds $|u(t, r, \omega| \leq \int_t^\infty |\partial u(s, r, \omega)| \mathrm{d}s \leq CN \int_t^\infty (1 + t + r)^{-(n-1)/2} (1+t-r)^{-1/2} \mathrm{d}s \leq CN(1+t+r)^{-(n-1)/2} (1+t-r)^{1/2}$. □

3. Global existence for quasilinear equations in high dimensions with small data

Theorem 3.1 (Principle of continuous induction/ Bootstrap argument.)**.** *Let $T_* > 0$, $C > 0$. If*

(1) *A is continuous on $[0, T_*)$,*
(2) *$A(0) \leq C$, and*
(3) *$\forall T \leq T_* : \sup_{t \leq T} A(t) \leq 2C \implies \sup_{t \leq T} A(t) \leq C$,*

then $\forall t < T_* : A(t) \leq C$.

Proof. By the continuity of A, the set of t such that $A(t) \leq C$ is closed. By the final condition, this set is open. Thus, it is the entire interval. □

Theorem 3.2. *Let $n \geq 4$. Assume condition $2Q(\mathbb{R}^{1+n}$, that $G^{00} = -1$, that G^{jk} and F depend only on ∂u, and that $F(0) = 0$ and $\delta F(0) = 0$.*

There is an $\epsilon > 0$ such that if $T > 0$ and u is a solution on $[0, T] \times \mathbb{R}^n$ of

$$\sum_{jk} G^{jk}(\partial u) \partial_i \partial_j u = F(\partial u)$$

and $\|u\|_{\Gamma, n+4}(0) < \epsilon$, then u can be extended to a solution of the PDE on $[0, \infty) \times \mathbb{R}^n$.

False with $n = 1, 3$ or no decay.

sketch. The main idea is to combine the continuation criterion involving $|u|_{1, \lceil \frac{n+3}{2} \rceil}$, the Klainerman-Sobolev inequality, and a bootstrap argument. □

Remark 3.3. *Fritz John's "Nonlinear Wave Equations: Formation of Singularities" shows that the previous theorem is false when $n = 1$ and $n = 3$, by giving examples of quasilinear wave equations that have solutions that cease to be C^∞ in finite time. Thus, in higher dimensions, some form of decay is necessary at infty, otherwise, one could take functions that are constant in $n-1$ (or $n-3$) directions, and reduce the nonlinear wave equation to one in $1+1$ (or $1+3$) dimensions, for which the existence of solutions that cease to be C^∞ in finite time.*