GLOBAL WELL POSEDNESS FOR ENERGY CRITICAL NLS

LEONARDO TOLOMEO

Contents

1. Introduction

Consider the defocusing quintic nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^4 u \tag{NLS}$$

describing the evolution of a complex - valued function $u : (I \subseteq \mathbb{R}_t) \times \mathbb{R}^3_x \to \mathbb{C}$. The solution to this equation conserves both the energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6 \mathrm{d}x \tag{1}$$

and the mass

$$M(u(t)) = \int_{\mathbb{R}^3} |u(x,t)|^2 \mathrm{d}x. \tag{2}$$

By Sobolev embedding, $u(0)$ has a finte energy if and only if $u(0) \in \dot{H}^1$, which is the space in which we will consider the initial data u_0. This is also a scale - invariant space; if we consider the family of transformations

$$u(t,x) \mapsto u^{\lambda,x_0}(t,x) = \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda(x + x(t)), \tag{3}$$

they preserve both the space of solutions to (NLS) and the $\dot{H}^1$ norm. For this reason, the equation is called *energy critical*.

A function $u : I \times \mathbb{R}^3 \to \mathbb{C}$ on a nonempty interval $I \ni 0$ is called a *strong solution* to NLS if $u \in C_t \dot{H}_1 \cap L^{10}_{t,x}(K \times \mathbb{R}^3)$ for all compact intervals $K \subseteq I$, and it obeys the Duhamel formula

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')|u(t')|^4 u(t') \mathrm{d}t', \tag{Duhamel}$$

for all $t \in I$, where $S(t) = e^{it\Delta}$ is the Scrödinger propagator. Since this is the only kind of solution we are interested in this report, we will drop the term *strong* and we will just talk about solutions in this sense.
We say that u is a *maximal* solution if it cannot be extended to any strictly larger interval (in this class).
We say that u scatters at $\pm\infty$ if there exists $u_\pm \in \dot{H}^1$ such that $\|u - S(t)u_\pm\| \to 0$ as $t \to \pm\infty$.
The main result is the following:

Theorem 1.1. *Let $u_0 \in \dot{H}^1$. Recall that $E(u) = \frac{1}{2}\|u\|^2_{\dot{H}^1} + \frac{1}{6}\|u\|^6_{L^6}$, and let $E(u_0) = E$. Then there exists a unique global solution u to (NLS) which satisfies*

$$\|u\|_{L^{10}_{t,x}(I_{\max}\times\mathbb{R}^3)} \le L(E) < +\infty.$$

In particular, the solution scatters, that is, there exists $u_\pm$ such that

$$\|u(t) - S(t)u_\pm\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

Theorem 1.1 was proven in the radial setting by Bourgain[1] and in its full generality by Colliander, Keel, Staffilani, Takaoka and Tao in [3]. In this report, we will follow a different kind of approach, introduced by a discovery of Keraani [5] of the existence of a *minimal* blowup solution, which was first used by Kenig and Merle in [4] for the focusing equation in the radial setting in dimension $3 \leq d \leq 5$. This approach for (NLS) is due to Killip and Vişan ([6]), from which we have taken sections 4 and 5, up to very slight tweaks and modifications. In sections 2 and 3, we follow mainly [8], which proves a similar result in dimension 4.

2. Preliminary results and notations

We will use the notation $X \lesssim Y$ to denote that there exists some constant C for which $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$. We will denote eventual dependencies on the constant C using subscripts, for instance $X \lesssim_u Y$ means $X \leq C(u)Y$.

We will use frequently the operator $|\nabla|^s$, defined on the Fourier side by having the multiplier $|\xi|^2$, togheter with the corresponding homogeneous Sobolev norms

$$\|f\|_{\dot{W}^{s,p}} := \||\nabla|^s f\|_{L^p_x} ,$$

and $\dot{H}^1 = \dot{W}^{s,p}$.

We will also need some Littlewood-Paley theory. Let $\varphi \in C^\infty_c(\mathbb{R}^+)$, $\operatorname{supp}(\varphi) \subset [-\frac{1}{2}, 2]$, and $\sum_{N \in 2^{\mathbb{Z}}} \varphi(\frac{r}{N}) \equiv 1$. For $N \in 2^{\mathbb{Z}}$, let $\varphi_N(\xi) = \varphi(\frac{|\xi|}{N})$, and define the operators (Littlewood-Paley projections)

$$\widehat{P_{\leq N} u} = \sum_{M \leq N} \varphi_M \hat{u}, \quad \widehat{P_{>N} u} = \sum_{M > N} \varphi_M \hat{u}, \quad \widehat{P_N u} = \varphi_N \hat{u}. \tag{4}$$

Other operators like $P_{\geq N}$, $P_{<N}$, $P_{M \leq \cdot \leq N}$ may appear in this report and are defined similarly. We will often use the notation $u_N := P_N u$, $u_{\leq N} := P_{\leq N} u$, etc.. The Littlewood - Paley projections commute with everything that commutes with translations, like derivatives, $S(t)$, convolution operators. Moreover, they are self-adjoint for every $\dot{H}^s$ space and bounded for every $\dot{W}^{s,p}$ for any $1 \leq p \leq \infty$. They also obey the following estimates

$$\left\||\nabla|^{\pm s} f_N\right\|_{L^p_x} \sim N^{\pm s} \|f_N\|_{L^p}, \quad \|f_N\|_{L^q_x} \lesssim_s N^{\frac{3}{p} - \frac{3}{q}} \|f_N\|_{L^p} . \tag{Bernstein}$$

whenever $s \geq 0$ and $1 \leq p \leq q \leq \infty$, and

$$\|f\|_{L^p_x} \sim \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |f_N(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x} \tag{5}$$

. for $1 < p < \infty$. We will often write $F(u) := |u|^4 u$ and, following [6], we will use the notation $Y = \text{Ø}(X)$ to denote a quantity that resembles X, in the sense that in order to estimate Y in whichever norm we need, it is enough to estimate X. Namely, this will often mean that Y is a linear combination of pieces with the same factors of X, up to complex conjugation, like

$$F(u+v)=\sum_{j=0}^{5}\varnothing(u^j v^{5-j}),$$

or up to Littlewood-Paley projections, like

$$F(u)=F(u_{>N})+\varnothing(u_{\leq N}u^4).$$

Another notation we will use in this report is that a set $\{u(t)\}_{t\in I}$ is *precompact modulo symmetries* in $\dot{H}^1$: by this we mean that there exists $N(t)\in\mathbb{R}^+$, $x(t)\in\mathbb{R}^3$ such that $\{v(t)|v(t,x)=N(t)^{\frac{1}{2}}u(t,N(t)(x+x(t)))\}_{t\in I}$ is precompact in $\dot{H}^1$.

As basic results go, we first need a couple of results on the linear theory of the Schrödinger propagator $S(t)$:

Proposition 2.1 (Strichartz estimates)**.** *We call a couple $q,r\geq 2$* admissibile *if $\frac{2}{q}+\frac{3}{r}=\frac{3}{2}$. Then for any admissible couples (q,r), $(\tilde{q},\tilde{r})$ the following inequalities hold:*

$$\|S(t)\psi\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^3}\lesssim\|\psi\|_{L^2_x(\mathbb{R}^3)} \tag{6}$$

$$\left\|\int_R S(-t)F(t)\mathrm{d}t\right\|_{L^2_x(\mathbb{R}^3)}\lesssim\|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^3)} \tag{7}$$

$$\left\|\int_0^t S(t-t')F(t')\mathrm{d}t'\right\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^3}\lesssim\|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^3)}. \tag{8}$$

Proof. See Theorem 2.3 in [7]. □

Lemma 2.2 (Fraunhofer formula)**.** *Let $\psi\in L^2(\mathbb{R}^d)$. Then*

$$\left\|(S(t)\psi)(x)-(2it)^{-\frac{d}{2}}e^{i\frac{|x|^2}{4t}}\hat{\psi}\left(\frac{x}{2t}\right)\right\|_{L^2}\to 0 \tag{9}$$

as $|t|\to\infty$.

Proof. See Lemma 8.8 of [8]. □

The next result is still a linear result on the Schrödinger propagator $S(t)$, but it is more technical, and it is a precise formulation of the heuristic statement

"Proposition". *The embedding $\dot{H}^1\hookrightarrow L^{10}_{t,x}$ given by $u\mapsto S(t)u$, which is composition of the embedding $u\mapsto S(t)u$ granted by the Strichartz inequality $\dot{H}^1\hookrightarrow L^{10}_t\dot{W}^{1,\frac{30}{13}x}_x$ and the Sobolev embedding $L^{10}_t\dot{W}^{1,\frac{30}{13}}_x\hookrightarrow L^{10}_{t,x}$, is compact modulo scaling and translations.*

Theorem 2.3 (Profile decomposition)**.** *Let f_n be a sequence of functions bounded in $\dot{H}^1$. Up to subsequences, there exists a $J^* \in \mathbb{N} \cup \{\infty\}$, functions $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1$, $\{\lambda_n\} \subset (0, \infty)$ and $\{t_n^j, x_n^j\} \subset \mathbb{R} \times \mathbb{R}^3$ such that for each finite $0 \le J \le J^*$, we have the decomposition*

$$f_n = \sum_{j=1}^{J} (\lambda_n^j)^{-\frac{1}{2}} \left(S(t_n^j)\phi^j\right) \left(\frac{x - x_n^j}{\lambda_n^j}\right) + w_n^J \tag{10}$$

with the following properties:

$$\lim_{J \to J^*} \limsup_{n \to \infty} \left\|S(t) w_n^J\right\|_{L^{10}_{t,x}} = 0 \tag{11}$$

$$\lim_{n\to\infty} \left(\|f_n\|_{\dot{H}^1}^2 - \sum_{j=1}^{J} \left\|\phi^j\right\|_{\dot{H}^1}^2 - \left\|w_n^J\right\|_{\dot{H}^1}^2 \right) = 0 \tag{12}$$

$$\lim_{n\to\infty} \left(\|f_n\|_{L^6}^6 - \sum_{j=1}^{J} \left\|S(t_n^j)\phi^j\right\|_{L^6}^6 - \left\|w_n^J\right\|_{L^6}^6 \right) = 0 \tag{13}$$

$$S(-t_n^j) \left(\lambda_n^j\right)^{\frac{1}{2}} w_n^J \left(\lambda_n^J x + x_n^J\right) \rightharpoonup 0 \tag{14}$$

$$\text{If } j \neq k, \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{\left|t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2\right|}{\lambda_n^j \lambda_n^k} \to \infty \qquad \text{as } n \to \infty \tag{15}$$

Moreover, we can assume that $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$.

Proof. See Theorem 4.1 in [8] □

Now we are ready to talk about our equation. The basic result, which gives local existence and uniqueness for any initial data in $\dot{H}^1$ and global existence and uniqueness for small initial data is the following, due to Cazenave and Weissler [2]:

Theorem 2.4. *Let $u_0 \in \dot{H}^1$. There exists a $\delta_0 > 0$ such that, if $0 \in I$ is an interval in which $\|S(t)u_0\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} < \delta_0$, then there exists a unique solution to (NLS) in $I \times \mathbb{R}^3$. This solutions satisfies $\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^1}$.*

With a very similar proof (arguably, the same proof), the following holds:

Theorem 2.5. *Let $u_+ \in \dot{H}^1$. There exists a $\delta_0 > 0$ such that, if $I = (t, +\infty)$ is an interval in which $\|S(t)u_+\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} < \delta_0$, then there exists a unique solution to (NLS) in $I \times \mathbb{R}^3$ which scatters to u_+ as $t \to \infty$. This solutions satisfies $\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|_{L_t^{10}\dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^1}$.*

In order to control the $\dot{W}_x^{1,\frac{30}{13}}$ norm of a solution, it is actually sufficient to control the weaker $L^{10}_{t,x}$ norm:

Lemma 2.6. *Let u be a solution to (NLS) on $I\times\mathbb{R}^3$ such that $\|u\|_{L^{10}_{t.x}} \le L < +\infty$ and $u_{t_0}\in \dot{H}^1$. Then $\|u(t)\|_{\dot{H}^1} + \|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \le C(E,L) < +\infty$, where E is the energy.*

Proof. By Duhamel, Strichartz and Hölder, we have, for every interval $[t_0,t_1]\subset [0,T_{\max})$

$$\|u\|_{L^\infty_t \dot{H}^1(I\times\mathbb{R}^3)} \lesssim \|u_{t_0}\|_{\dot{H}^1} + \|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \|u\|^4_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \tag{16}$$

$$\|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \lesssim \|u_{t_0}\|_{\dot{H}^1} + \|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \|u\|^4_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \tag{17}$$

Therefore, chopping down $[0,T_{\max}]$ in a finite number $\left(\lesssim \left(\frac{L}{\delta}\right)^{10}\right)$ of intervals $J_k = [t_k,t_{k+1}]$ where $\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)}\le\delta$, we get for δ small enough (depending only on the implicit constants in the previous inequalities):

$$\|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}([t_k,t_k+1]\times\mathbb{R}^3)} \lesssim \|u\|_{L^\infty_t \dot{H}^1_x([t_{k-1},t_k]\times\mathbb{R}^3)}, \tag{18}$$

$$\|u\|_{L^\infty_t \dot{H}^1_x([t_k,t_{k+1}]\times\mathbb{R}^3)} \lesssim \|u\|_{L^\infty_t \dot{H}^1_x([0,t_k]\times\mathbb{R}^3)} + \delta^4 \|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}([t_k,t_{k+1}]\times\mathbb{R}^3)}, \tag{19}$$

from which, proceeding inductively, we get $\|u_{t_0}\|_{\dot{H}^1} + \|u\|_{L^{10}_t \dot{W}_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} < C(E,L) < +\infty$. $\square$

This $L^{10}_{t,x}$ norm of the solution it is a good indicator for the eventual blowup. Indeed we have that in case $\|u\|_{L^{10}_{t,x}} = \infty$, we have blowup:

Lemma 2.7. *Let u a solution to NLS such that $\|u\|_{L^{10}_{t,x}((0,T_{\max})\times\mathbb{R}^3} = +\infty$. Then u cannot be extended to a solution $v\in C_t\dot{H}^1((0,T)\times\mathbb{R}^3)$ for any $T > T_{\max}$.*

Proof. Suppose by contradiction that there exists $v:(0,T)\to\mathbb{R}^3$ solution to NLS which extends u. Let $0<\overline{T}\le T_{\max}$ the minimum time at which $\|u\|_{L^{10}_{t,x}((0,\overline{T})\times\mathbb{R}^3} = +\infty$ (it exists because of Theorem 2.4). Let $\delta>0$ such that

$$\|u\|_{L^{10}_{t,x}([\overline{T}-\delta,\overline{T}]\times\mathbb{R}^3)} = \|v\|_{L^{10}_{t,x}([\overline{T}-\delta,\overline{T}]\times\mathbb{R}^3)} < +\infty$$

(it exists because of Theorem 2.4 again). Then we have

$$\|u\|_{L^{10}_{t,x}} \le \|u\|_{L^{10}_{t,x}([0,\overline{T}-\delta]\times\mathbb{R}^3)} + \|u\|_{L^{10}_{t,x}([\overline{T}-\delta,\overline{T}]\times\mathbb{R}^3)} < +\infty,$$

contradiction. $\square$

On the other hand, in case $\|u\|_{L^{10}_{t,x}((0,T_{\max})\times\mathbb{R}^3} < +\infty$, then we cannot have any blowup in finite time:

Proposition 2.8. *Let u a maximal solution to (NLS) in $[0,T_{\max})\times\mathbb{R}^3$ with $\|u\|_{L^{10}_{t,x}} < +\infty$. Then $T_{\max} = +\infty$.*

Proof. Now let us suppose by contradiction that $T_{\max} < +\infty$, and let $t_n \uparrow T_{\max}$. We have, by Duhamel with $t_0 = t_n$, Strichartz inequality and Hölder:

$$\|S(t-t_n)u(t_n)\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x((t_n,T_{\max})\times\mathbb{R}^3)} \lesssim \|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x(t_n,T_{\max})} + \|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x((t_n,T_{\max})\times\mathbb{R}^3)} \|u\|^4_{L^{10}_{t,x}((t_n,T_{\max})\times\mathbb{R}^3)}. \tag{20}$$

The RHS in this last inequality is approaching 0 when $n \to \infty$, so we have that for n big enough,

$$\|S(t-t_n)u(t_n)\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x((t_n,T_{\max})\times\mathbb{R}^3)} < \frac{\delta_0}{2}.$$

Moreover, since $\|S(t-t_n)u(t_n)\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u(t_n)\|_{\dot{H}^1} < +\infty$, we have that there exists $\eta > 0$ such that $\|S(t-t_n)u(t_n)\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x((t_n,T_{\max}+\eta)\times\mathbb{R}^3)} < \delta$. Therefore, by Theorem 2.4, one can extend u up to $T_{\max}+\eta$, contradiction. $\square$

Moreover, if $\|u\|_{L^{10}_{t,x}((0,+\infty)\times\mathbb{R}^3} < +\infty$, we have a good behavior at $+\infty$, namely scattering:

Proposition 2.9. *Let u be a solution to (NLS) on $[0,+\infty)\times\mathbb{R}^3$ such that $\|u\|_{L^{10}_{t,x}} < +\infty$. Then u* scatters *at $+\infty$, that is, there exists $u_+ \in \dot{H}^1$ such that $\|u - S(t)u_+\|_{\dot{H}^1} \to 0$ for $t \to +\infty$.*

Proof. Let $u_+ = u_0 - i\int_0^{+\infty} S(-t')|u|^4u(t')\mathrm{d}t'$. This is well defined in $\dot{H}^1$, since by Strichartz and Hölder,

$$\begin{aligned}\left\|\int_{T_0}^{T_1} S(-t')|u|^4u(t')\mathrm{d}t'\right\|_{\dot{H}^1} &= \left\|\int_0^{T_1-T_0} S(-t')|u|^4u(T_0+t')\mathrm{d}t'\right\|_{\dot{H}^1} \\ &\lesssim \|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x((T_0,+\infty)\times\mathbb{R}^3)} \|u\|^4_{L^{10}_{t,x}((T_0,+\infty)\times\mathbb{R}^3)}\end{aligned}$$

which is going to 0 as $T_0 \to \infty$ since $\|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x(\mathbb{R}^+\times\mathbb{R}^3)}$, $\|u\|_{L^{10}_{t,x}(\mathbb{R}^+\times\mathbb{R}^3)} < +\infty$. Therefore we have, by Duhamel,

$$\|u(t) - S(t)u_+\|_{\dot{H}^1} = \|S(-t)u(t) - u_+\|_{\dot{H}^1} = \left\|\int_t^\infty S(-t')|u|^4u(t')\mathrm{d}t'\right\|_{\dot{H}^1},$$

which is converging to 0 as $t \to \infty$ (as we just proved). $\square$

In order to complete this list of basic results on NLS, we need some kind of stability result, which united with Theorem 2.4 will give LWP and GWP for small data for NLS, other then being a fundamental technical tool that we will need in the further sections. The next statement is what we need. Stronger versions (with weaker hypotheses) are available, but this is the easiest (to prove) that the writer knows, and it is enough for our purposes.

Theorem 2.10 (Perturbation theory)**.** *Let I a compact time interval and let $\tilde{u}$ satisfy a perturbed NLS*

$$i\tilde{u}_t = -\Delta\tilde{u} \pm |\tilde{u}|^4\tilde{u} + e \tag{PNLS}$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^3)} \le E \tag{21}$$

$$\|\tilde{u}\|_{L_{t,x}^{10}(I\times\mathbb{R}^3)} \le L \tag{22}$$

for some positve constants E and L. Let $t_0 \in I$ and $u_0 \in \dot{H}_x^1$ and assume the smallness conditions

$$\|u_0 - \tilde{u}_0\|_{\dot{H}^1} \le \varepsilon \tag{23}$$

$$\|\nabla e\|_{N^0(I)} \le \varepsilon \tag{24}$$

for some $0 < \varepsilon < \varepsilon_1(E, L)$. (Here $\|e\|_{N^0}$ is the dual norm of the norm of $L_t^2 L_x^6 \cap L_t^\infty L_x^2$). Then there exists a unique solution $u : I \times \mathbb{R}^3 \to \mathbb{C}$ to (NLS) with initial data u_0 at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L_{t,x}^{10}} \le C(E, L)\varepsilon \tag{25}$$

$$\|\nabla(u - \tilde{u})\|_{L_t^{10} L^{\frac{30}{13}}(I\times\mathbb{R}^3)} \le C(E, L) \tag{26}$$

$$\|\nabla u\|_{L_t^{10} L^{\frac{30}{13}}(I\times\mathbb{R}^3)} \le C(E, L). \tag{27}$$

Proof. See Theorem 5.3 in [8]. □

The rest of this section will be dedicated to a few technical lemmas we will need in the following sections,

Lemma 2.11. *Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a solution to the forced Schröndinger equation*

$$iu_t + \Delta u = G$$

for some function G, and let (q, r) and $(\tilde{q}, \tilde{r})$ be Schröndinger admissible ($\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$ and $q, r \ge 2$). Then we have

$$\left(\sum_{N\in 2^{\mathbb{Z}}} \|\nabla u_N\|^2_{L_t^q L_x^r(I\times\mathbb{R}^3)} \right)^{\frac{1}{2}} \lesssim \|u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} + \|\nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}$$

Proof. By Duhamel and Strichartz,

$$\|\nabla u_N\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim \|u(t_0)_N\|_{\dot{H}^1} + \|P_N \nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}.$$

By Littlewood - Paley theory $\|\nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)} \sim \|P_N \nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} l_N^2(I\times\mathbb{R}^3\times\mathbb{Z})}$, so by Minkowski inequality,

$$LHS = \|\nabla u_N\|_{l_N^2 L_t^q L_x^r(\mathbb{Z}\times I\times\mathbb{R}^3)} \lesssim \|u(t_0)_N\|_{l_N^2 \dot{H}_x^1} + \|P_N \nabla G\|_{l_N^2 L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{Z}\times I\times\mathbb{R}^3)}$$

$$\overset{\tilde{q}',\tilde{r}'\leq 1}{\lesssim} \|u(t_0)\|_{\dot{H}^1} + \|P_N \nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} l_N^2(I\times\mathbb{R}^3\times\mathbb{Z})} \sim \|u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} + \|\nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}$$

□

Lemma 2.12. *For any $u : I\times\mathbb{R}^3 \to \mathbb{C}$ we have*

$$\|u\|_{L_t^4 L_x^\infty(I\times\mathbb{R}^3)} \lesssim \|\nabla u\|_{L_t^\infty L_t^2}^{\frac{1}{2}} \left(\sum_{N\in 2^{\mathbb{Z}}} \|\nabla u_N\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}^2\right)^{\frac{1}{4}}.$$

In particular,

$$\|u_{\leq N}\|_{L_t^4 L_x^\infty(I\times\mathbb{R}^3)} \lesssim \|\nabla u_{\leq N}\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \left(\sum_{M\leq N} \|\nabla u_M\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}^2\right)^{\frac{1}{4}}.$$

Proof. From Minkowski and Littlewood Paley theory, we have that

$$\|\nabla u\|_{L_t^2 L_x^6} \sim \|\nabla u_N\|_{L_t^2 L_x^6 l_N^2(I\times\mathbb{R}^3\times\mathbb{Z})} \lesssim \|\nabla u_N\|_{l_N^2 L_t^2 L_x^6(\mathbb{Z}\times I\times\mathbb{R}^3)}.$$

Then we have, by Hölder,

$$\|u\|_{L_t^4 L_x^\infty} \leq \|\nabla u\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla u\|_{L_t^2 L_x^6}^{\frac{1}{2}} \lesssim \|\nabla u\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla u_N\|_{l_N^2 L_t^2 L_x^6(\mathbb{Z}\times I\times\mathbb{R}^3)}^{\frac{1}{2}}.$$

□

Proposition 2.13 (Maximal Strichartz Estimate)**.** *Let $v : I\times\mathbb{R}^3 \to \mathbb{C}$ be a solution to the forced Schröndinger equation*

$$iv_t + \Delta v = F + G.$$

Then for each $6 < q \leq \infty$,

$$\left\| M(t)^{\frac{3}{q}-1} \left\|P_{M(t)} v(t)\right\|_{L_x^q} \right\|_{L_t^2} \lesssim \left\| |\nabla|^{-\frac{1}{2}} v \right\|_{L_t^\infty L_x^2} + \left\| |\nabla|^{-\frac{1}{2}} G \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \|F\|_{L_t^2 L_x^1}$$

uniformly for all functions $M : [0,T] \to 2^{\mathbb{Z}}$. All spacetime norms are over $[0,T]\times\mathbb{R}^3$.

Proof. See Theorem 3.1 in [6]. □

3. Minimal blowup solution for the defocusing case

Suppose by contradiction that Theorem 1.1 fails. Therefore, defining

$$L(E) := \sup\{\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \,|E(u_0)\le E\},$$

$L(E)=\infty$ for some $E<\infty$. By Theorem 2.4, $L(E)<+\infty$ for E small enough. Therefore, by monotonicity of L, there exists a critical level $0<E_c<+\infty$ such that $E_c=\sup\{E|L(E)<+\infty\}=\inf\{E|L(E)=+\infty\}$.

Lemma 3.1. *$L(E_c)=+\infty$.*

Proof. Suppose by contradiction that $L(E_c)=L_c<+\infty$, and let $0<\delta\ll 1$. Let u_0 a generic initial data with $E(u_0)=E_c+\delta$, and let u be the maximal solution starting from u_0. Let $\tilde{u}_0$ such that $E(\tilde{u}_0)\le E_c$ and $\|u_0-\tilde{u}_0\|_{\dot{H}^1}\lesssim\delta$. Let $\tilde{u}$ be the solution starting from $\tilde{u}_0$.
We have that $\|\tilde{u_0}\|^2_{\dot{H}^1}\le 2E_c$ and $\|\tilde{u_0}\|_{L^{10}_{t,x}}\le L_c$, so we can apply Theorem 2.10 to this $\tilde{u}$, getting that $\|u\|_{L^{10}_{t,x}}\lesssim C(\sqrt{2E_c},L_c)\delta+L_c$. Therefore $L(E_c+\delta)\lesssim\delta+L_c<+\infty$, contradiction. □

Since $L(E_c)=+\infty$, by definition it means that there exists a sequence of initial data $u_n(0)$ with $E(u_n)\le E_c$ such that $\|u_n\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)}\to+\infty$. Without loss of generality, by translating u_n in time, we can have that $\|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)}=\|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^-)\times\mathbb{R}^3)}\to+\infty$. The goal is to extract a blowup solution from this sequence. The main tool is the following:

Proposition 3.2 (Palais - Smale condition)**.** *Let $u_n: I_n\times\mathbb{R}^3\to\mathbb{C}$ be a sequence of solutions to NLS with $E(u_n)\to E_c$ such that*

$$\lim_{n\to\infty}\|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)}=\lim_{n\to\infty}\|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^-)\times\mathbb{R}^3)}\to\infty.$$

Then the sequence $u_n(0)$ has a converging subsequence in $\dot{H}^1$ modulo scaling and spatial translations.

Proof. Using Theorem 2.3, we write

$$u_n=\sum_{j=1}^{J}(\lambda_n^j)^{-\frac{1}{2}}\left(S(t_n^j)\phi^j\right)\left(\frac{x-x_n^j}{\lambda_n^j}\right)+w_n^J=\sum_{j=1}^{J}S(t_n^j)T_n^j\phi^j+w_n^J.$$

By 12 and 13, we get that

$$\lim_{n\to\infty}E(u_n)-\sum_{j=1}^{J}E(S(t_n^j)\phi^j)-E(r_n^J)=0. \tag{28}$$

Scenario 1: $\sup_j \limsup_{n\to\infty} E(S(t_n^j)\phi^j) = E_c$.
By 12, if $J^* = \infty$, we have that $\lim_{j\to\infty} \|\phi_n^j\|_{\dot H^1} = 0$, which implies that there exists $J_0 \in \mathbb{N}$ such that $E(S(t_n^j)\phi^j) < \frac{1}{2}E_c$ for $j > J_0$. The same holds trivially if $J^* < \infty$, just by taking $J_0 = J^* + 1$.
Therefore, up to passing to a subsequence in n, we have that for a certain $j_0 \le J_0$, $\lim_n E(S(t_n^{j_0}\phi^{j_0})) = E_c$. In this case, by (28), we get that for every $j \neq j_0$,

$$0 = \limsup_n E(S(t_n^j)\phi^j) \gtrsim \limsup_n \left\|\phi^j\right\|_{\dot H^1}^2,$$

so we have that $J^* = 1$. Again by (28), this implies $\|r_n^1\|_{\dot H^1} \to 0$. If $t_n^1 \equiv 0$ we obtain our desired compactness, so we just need to prove that $t_n^1 \not\to \pm\infty$. In the case $t_n^1 \to \infty$, we have that

$$\|\nabla S(t)u_n\|_{L_t^{10}L_x^{\frac{30}{13}}(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim \left\|\nabla S(t)\phi^1\right\|_{L_t^{10}L_x^{\frac{30}{13}}([t_n^1,+\infty)\times\mathbb{R}^3)} + \left\|\nabla S(t)w_n^1\right\|_{L_t^{10}L_x^{\frac{30}{13}}(\mathbb{R}^+\times\mathbb{R}^3)} \to 0$$

as $n \to \infty$. Therefore, $\tilde u = S(t)u_n$ satisfies (PNLS) on $\mathbb{R}^+ \times \mathbb{R}^3$ with $e = -|\tilde u|^4\tilde u$, and

$$\|\nabla e\|_{N^0} \lesssim \|\nabla e\|_{L^2L^{\frac{6}{5}}} \lesssim \|\tilde u\|_{L_{t,x}^{10}}^4 \|\nabla \tilde u\|_{L_t^{10}L_x^{\frac{30}{13}}} \lesssim \|\nabla \tilde u\|_{L_t^{10}L_x^{\frac{30}{13}}}^5 \to 0$$

as $n \to 0$. Therefore, by Theorem 2.10 applied to u_n and $\tilde u$, we have that, for n big enough,

$$\|u_n\|_{L_{t,x}^{10}(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim 1 + \|\tilde u\|_{L_{t,x}^{10}(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim 1 + \|u_n(0)\|_{\dot H^1} \lesssim 1, \tag{29}$$

which contradicts our hypothesis. An analogous argument holds for $t_n \to -\infty$.

Scenario 2: $\sup_j \limsup_{n\to\infty} E(S(t_n^j)\phi^j) \le E_c - 2\delta$.
Fix $J < +\infty$. Then, for n sufficiently large, $E(S(t_n^j)\phi^j) \le E_c - \delta \ \forall j \le J$. Define v_j in the following way:

- If $t_n^j \equiv 0$, then v^j is the maximal solution to NLS with initial data ϕ^j. Since $E(\phi_j) < E_c$, this solution is defined on the whole real line,
- If $t_n^j \to +\infty$, then v^j is the solution to NLS that scatters to ϕ^j when $t \to +\infty$ (which existence is guaranteed by Theorem 2.5),
- If $t_n^j \to -\infty$, then v^j is the solution to NLS that scatters to ϕ^j when $t \to -\infty$ (which existence is guaranteed by Theorem 2.5).

Let $v_n^j(t) = T_n^j v^j\left(\frac{t}{(\lambda_n^j)^2} + t_n^j\right)$. This is still a solution to NLS, and satisfies

$$\lim_{n\to\infty} \left\|v_n^j(0) - S\left(t_n^j\right)T_n^j\phi^j\right\|_{\dot H^1} = 0,$$

therefore for n big enough, $E(v_n^j) \le (\limsup E(S(t)\phi^j) + \frac{\delta}{2}) \le E_c - \frac{\delta}{2}$, so v_n^j is defined for all times and from Proposition 2.4 we get $\|v_n^j\|_{L_{t,x}^{10}} \lesssim_{E_c,\delta,L(E_c-\frac{\delta}{2})} \|\phi\|_{\dot H^1}$.

Now consider

$$\tilde{u}_n^J := \sum_{j=1}^{J} v_n^j + S(t)w_n^J.$$

Because of Theorem 2.3, points 11,15 and 12, we have that

(30)

$$\left\|\tilde{u}_n^J\right\|_{L^{10}} \lesssim \left\|\left\|v_n^j\right\|_{L^{10}}\right\|_{l_j^{10}} + \left\|S(t)w_n^J\right\|_{L^{10}} \lesssim \left\|\|\phi\|_{\dot{H}^1}\right\|_{l^2} + \varepsilon_J \lesssim \|u_n\|_{\dot{H}^1} + \varepsilon_J \lesssim \sqrt{E}_c + \varepsilon_J.$$

Claim I: $\lim_{n\to\infty}\left\|\tilde{u}_n^J(0) - u_n(0)\right\| = 0.$

$$\left\|\tilde{u}_n^J(0) - u_n(0)\right\|_{\dot{H}^1}^2 = \left\|\sum_{j=1}^{J} v_n^j(0) - S(t_n^j)T_n^j\phi^j\right\|_{\dot{H}^1}^2 \to 0.$$

Claim II: For J big enough (depending on ε),

$$\limsup_{n\to\infty}\left\||\tilde{u}_n^J|^4\tilde{u}_n^J - \sum_{j=1}^{J}|v_n^j|^4 v_n^j\right\|_{L_t^2\dot{W}_x^{1,\frac{6}{5}}} \le \varepsilon$$

$$\left\||\tilde{u}_n^J|^4\tilde{u}_n^J - \sum_{j=1}^{J}|v_n^j|^4 v_n^j\right\|_{L_t^2\dot{W}_x^{1,\frac{6}{5}}}$$

$$\text{(I)} \qquad \lesssim \sum_{j\neq k}\sum_{l+m+p+q+1=5}\left\|(\nabla v_n^j)\overline{(v_n^j)}^l(v_n^j)^m\overline{(v_n^k)^p}(v_n^k)^q\right\|_{L_t^2L_x^{\frac{6}{5}}}$$

$$\text{(II)} \qquad +\sum_{j,k}\sum_{l+m+p+q+1=5}\left\|(S(t)\nabla w_n^J)\overline{(v_n^j)}^l(v_n^j)^m\overline{(v_n^k)^p}(v_n^k)^q\right\|_{L_t^2L_x^{\frac{6}{5}}}$$

$$\text{(III)} \qquad +\sum_{j,k}\sum_{l+m+p+q+2=5}\left\|(\nabla v_n^j)S(t)w_n^J\overline{(v_n^j)}^l(v_n^j)^m\overline{(v_n^k)^p}(v_n^k)^q\right\|_{L_t^2L_x^{\frac{6}{5}}}$$

$$\text{(IV)} \qquad +\sum_{j,k}\sum_{l+m+p+q+2=5}\left\|(\nabla v_n^j)\overline{S(t)w_n^J}\overline{(v_n^j)}^l(v_n^j)^m\overline{(v_n^k)^p}(v_n^k)^q\right\|_{L_t^2L_x^{\frac{6}{5}}}$$

We have that:

I: Because of Theorem 2.3, point 15, I $\to 0$.

II: II $\lesssim \sum_{j=1}^{J}\left\|(\nabla S(t)w_n^J)|v_n^j|^4\right\|_{L^2L^{\frac{6}{5}}} \to 0$ because of Theorem 2.3, point 14 (by approximating $|v^j|$ in L^{10} with functions with space-time compact support).

III + IV:

$$\text{III} + \text{IV} \lesssim \sum_{j=1}^{J}\left\||\nabla v_n^j||v_n^j|^3 S(t)w_n^J\right\|_{L^2L^{\frac{6}{5}}} \lesssim \left\|v_n^j\right\|_{L_t^{10}L_x^{\frac{30}{13}}}\left\|v_n^j\right\|_{L_{t,x}^{10}}^3\left\|w_n^J\right\| \lesssim \varepsilon_J$$

because of Lemma 2.6 and Theorem 2.3, point 11.

Therefore, taking J such that $\varepsilon \leq \varepsilon_1(2\sqrt{E_c}, C\sqrt{E_c})$ and n big enough, we can apply Theorem 2.10 with $\tilde{u} = \tilde{u}_n$ and $u = u_n$, getting that $\|u_n\|_{L^{10}} \lesssim \sqrt{E_c}$. But we know that $\|u_n\|_{L^{10}} \to +\infty$, contradiction. ☐

Theorem 3.3 (Existence of minimal counterexample)**.** *Suppose Theorem 1.1 fails to be true. Then there exists a maximal solution $u : I \times \mathbb{R}^3 \to \mathbb{C}$ to the defocusing energy-critical NLS with $E(u) = E_c$, which blows up in both time directions in the sense that*

$$\|u\|_{L^{10}_{t,x}((I\cap\mathbb{R}^-)\times\mathbb{R}^3)} = \|u\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)} = \infty,$$

and whose orbit $\{u(t)|t \in I\}$ is precompact in $\dot{H}^1$ modulo scaling and spacial translations

Proof. Let u_n with $E(u_n) \uparrow E_c$, $\|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)} = \|u_n\|_{L^{10}_{t,x}((I\cap\mathbb{R}^-)\times\mathbb{R}^3)} \to +\infty$. Because of Proposition 3.2, up to rescaling and translating u_n, we can assume $u_n(0) \to \phi$ in $\dot{H}^1$. In particular, $E(\phi) = E_c$.

Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ the maximal solution with initial data ϕ. Applying Theorem 2.10 with $\tilde{u} = u$ and $u_0 = u_n(0)$, we get that $\|u\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)} = \|u\|_{L^{10}_{t,x}((I\cap\mathbb{R}^+)\times\mathbb{R}^3)} = +\infty$. Finally, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in I. Because of Lemma 2.7, $\|u\|_{L^{10}_{t,x}(K\times\mathbb{R}^3)} < +\infty$, so we have

$$\|u\|_{L^{10}_{t,x}((I\cap\{t<t_n\})\times\mathbb{R}^3)} = \|u\|_{L^{10}_{t,x}((I\cap\{t>t_n\})\times\mathbb{R}^3)} = +\infty.$$

Hence we can apply Proposition 3.2 to the sequence $u(t_n)$, obtaining that $u(t_n)$ has a converging subsequence modulo scaling and translations, from which we get the precompactness. ☐

What we just proved to exist in Theorem 3.3 can be already considered a minimal blowup solution, and of course proving Theorem 1.1 is equivalent to proving that this solution cannot exist. However, this solution does not have all the properties we need yet, and a further study os needed to discover the general behavior of almost periodic solutions and to build a blowup solution with slightly stronger features.

Proposition 3.4. *Let $\{u(t)\} \subset \dot{H}^1$ be precompact modulo dilations and translations. Then we have*

(1) *For every $\eta > 0$, there exists $C(\eta)$ such that*

$$\int_{|x-x(t)|\geq\frac{C(\eta)}{N(t)}} |\nabla u(t)(x)|^2 \,\mathrm{d}x \leq \eta, \qquad \int_{|\xi|\geq C(\eta)N(t)} |\xi\hat{u}(t)|^2 \leq \eta$$

(2) *For every $\eta > 0$, there exists $c(\eta)$ such that*

$$\int_{|x-x(t)|\leq\frac{c(\eta)}{N(t)}} |\nabla u(t)(x)|^2 \,\mathrm{d}x \leq \eta, \qquad \int_{|\xi|\leq c(\eta)N(t)} |\xi\hat{u}(t)|^2 \leq \eta$$

Proof.

(1) Let $\tilde{u}(t)$ be the rescaling and translation of $u(t)$, in such a way that $\{\tilde{u}(t)\}$ is precompact in $\dot{H}^1$. Suppose by contradiction that there exists $\eta > 0$ such that for every $n \in \mathbb{N}$, there exists t_n for which

$$\int_{|x-x(t)|\geq \frac{n}{N(t_n)}} |\nabla u(t_n)(x)|^2 \,\mathrm{d}x > \eta \Leftrightarrow \int_{|x|>n} |\nabla \tilde{u}(t_n)|^2 > \eta.$$

Because of precompactness, up to subsequences we have that $\tilde{u}(t_n) \to \tilde{u}$ in $\dot{H}^1$. Let $M > 0$ such that $\int_{|x|>M} |\nabla \tilde{u}|^2 < \eta$. Then we have

$$\eta > \int_{|x|>M} |\nabla \tilde{u}|^2 = \lim_{n\to\infty} \int_{|x|>M} |\nabla \tilde{u}(t_n)|^2 \overset{\text{big enoguh } n}{\geq} \eta,$$

contradiction.

The other inequality is analogous: if it fails, then on a sequence

$$\int_{|\xi|\geq n} \left|\xi \widehat{\tilde{u}(t_n)}\right|^2 \geq \eta,$$

and

$$\eta > \int_{|\xi|\geq M} \left|\xi \widehat{\tilde{u}}\right|^2 = \lim_{n\to\infty} \int_{|\xi|\geq M} \left|\xi \widehat{\tilde{u}(t_n)}\right|^2 \overset{\text{big enoguh } n}{\geq} \eta.$$

(2) As before, by contradiction on a suitable sequence we have

$$\int_{|x|\leq \frac{1}{n}} |\nabla u(t_n)(x)|^2 \,\mathrm{d}x \geq \eta$$

and

$$\eta > \int_{|x|\leq \frac{1}{M}} |\nabla u(x)|^2 \,\mathrm{d}x = \lim_{n\to\infty} \int_{|x|\leq \frac{1}{M}} |\nabla u(t_n)(x)|^2 \,\mathrm{d}x \overset{\text{big enoguh } n}{\geq} \eta,$$

and, for the last inequality

$$\int_{|\xi|\leq \frac{1}{n}} \left|\xi \widehat{\tilde{u}(t_n)}\right|^2 \geq \eta$$

on a sequence and

$$\eta > \int_{|\xi|\leq \frac{1}{M}} \left|\xi \widehat{\tilde{u}}\right|^2 = \lim_{n\to\infty} \int_{|\xi|\leq \frac{1}{M}} \left|\xi \widehat{\tilde{u}(t_n)}\right|^2 \overset{\text{big enoguh } n}{\geq} \eta.$$

□

Lemma 3.5 (Local constancy of $N(t)$)**.** *Let u be an almost periodic solution to NLS. Then there exists $\delta = \delta(u)$ such that for every t_0 in I we have:*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I \tag{31}$$

$$N(t) \sim_u N(t_0) \text{ whenever } |t - t_0| \leq \delta N(t_0)^2. \tag{32}$$

Proof. Let $\{\tilde{u}(t_n)\}$ a precompact family of rescaling and translations of $u(t_n)$, which existence is given by almost periodicity modulo symmetries.
Suppose by contradiction that (31) fails. Then there exists t_n such that $t_n \in I$ and $t_n \pm \frac{1}{n} N(t_n)^{-2} \notin I$. Up to subsequences, the sign is constant, we will consider just $+$ (the $-$ is analogous).
Let $v_n : I_n \times \mathbb{R}^3 \to \mathbb{C}$ be the maximal solution with initial data $\tilde{u}(t_n)$. Since v_n is given by rescaling and translations of $u(t - t_n)$, we have that $0 \in I_n$, $\frac{1}{n} \notin I_n$. Up to subsequences, $\tilde{u}_n(t_n) \to v_0$ in $\dot{H}^1$. Let $v : I \times \mathbb{R}^3 \to \mathbb{C}$ be the solution to NLS starting from v_0. Therefore, by Theorem 2.10, if $K = [0, b]$ is a compact subinterval of I, then $K \subset I_n$ for n big enough. But $b > \frac{1}{n} \notin I_n$, contradiction.
Now take δ smaller then the one determined for (31) to hold, and suppose (32) fails. Then there exists t_n, t'_n such that $s_n := (t_n - t'_n) N(t_n)^2 \to 0$ but $\frac{N(t_n)}{N(t'_n)} \to 0$ or $\frac{N(t_n)}{N(t'_n)} \to \infty$. As before, let v_n the solution to NLS with initial data $\tilde{u}(t_n)$, and let v be the solution to NLS with initial data $v_0 = \lim_{n \to \infty} \tilde{u}(t_n)$. Because of $\frac{N(t_n)}{N(t'_n)} \to 0$ or $\frac{N(t_n)}{N(t'_n)} \to \infty$, we have that $v_n(s_n) \rightharpoonup 0$ in $\dot{H}^1$. By Duhamel, Strichartz, Hölder, Lemma 2.6 and Theorem 2.10, we have that

$$\|v_n(s_n) - v_0\| \leq \|v_n(s_n) - v_n(0)\|_{\dot{H}^1} + \|v_n(0) - v_0\|_{\dot{H}^1}$$

$$\lesssim \|v_n\|^4_{L^{10}_{t,x}([0,s_n] \times \mathbb{R}^3)} \|v_n\|_{L^{10}_x \dot{W}^{1,\frac{30}{13}}_x([0,s_n] \times \mathbb{R}^3)} + \|v_n(0) - v_0\|_{\dot{H}^1}$$

$$\lesssim \|v\|^4_{L^{10}_{t,x}([0,s_n] \times \mathbb{R}^3)} C\left(E, 2\|v\|_{L^{10}_{t,x}([0,s_n] \times \mathbb{R}^3)}\right) + \|v_n(0) - v_0\|_{\dot{H}^1} \to 0$$

as $n \to \infty$. Therefore, by coherence of the limits, we have that $v_0 = 0$, so $u(t_n) = 0$ and $E(u(t_n)) = E(u_0) = 0$, which implies $u_0 = 0$, contradiction. $\square$

Remark 3.6. Covering I with intervals $I_0 \ni 0 = t_0, I_{t_1} \ni t_1, \ldots, I_{t_n} \ni t_n, \ldots$ as in (31) with $|t_{n+1} - t_n| \gtrsim N(t_n)^{-2}$ and changing $N(t)$ in such a way that $N(t) \equiv N(t_n)$ on $[t_n, t_{n+1}]$, the set $\{u(t)\}$ is still precompact modulo this new family of scaling (and the old family of symmetries). Therefore, we can assume $N(t)$ constant on intervals $J_k = [t_k, t_{k+1}]$ of size $\sim N_k^{-2}$.

Corollary 3.7. *Let $u : (0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ be a maximal forward in time solution to NLS. Then*

$$\text{If } T_{\max} < \infty, \text{ then } N(t) \gtrsim_u |T_{\max} - t|^{-\frac{1}{2}} \tag{33}$$

$$\text{If } T_{\max} = \infty, \text{ then } \forall t, t_0, N(t) \gtrsim_u \min\{N(t_0), |t - t_0|^{-\frac{1}{2}}\} \tag{34}$$

Proof. If $T_{\max} < +\infty$, then because of (31) we have $T_{\max} > t_0 + \delta N(t_0)^{-2} \Rightarrow N(t_0) \gtrsim_u |T_{\max} - t_0|^{-\frac{1}{2}}$.
If $T_{\max} = \infty$, we either have $t > t_0 + \delta N(t_0)^{-2}$ or $N(t) \sim N(t_0)$, from which respectively $N(t) \gtrsim_u |t - t_0|^{-\frac{1}{2}}$ or $N(t) \gtrsim_u N(t_0)$. □

Lemma 3.8. *Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution to NLS, such that I contains at least one full interval J_k. Let (q, r) be Strichartz admissible ($\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$). Then*

$$\|\nabla u\|^q_{L^q_t L^r_x} \lesssim_u 1 + \int_I N(t)^2 \mathrm{d}t \lesssim_u \int_I N(t)^2 \mathrm{d}t.$$

Proof. Let $J_k = [t_k, t_{k+1}]$ an interval such that $N \equiv N_k$ on J_k. Because of Remark 3.6, $\int_{J_k} N_k^2 \gtrsim_u 1$. Using Strichartz, Hölder, Sobolev and Bernstein inequality, we have that

$$\begin{aligned}
\|u\|_{L^{10}_t \dot{W}^{\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} &\lesssim \left\|S(t)(u_{t_0})_{>C(\eta)N_k}\right\|_{L^\infty_t \dot{H}^1_x([t_k,t]\times\mathbb{R}^3)} \\
&\quad + \left\|S(t)(u_{t_0})_{\le C(\eta)N_k}\right\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} + \|u\|^5_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} \\
&\lesssim \left\|(u_{t_0})_{>C(\eta)N_k}\right\|_{\dot{H}^1} + |J_k|^{\frac{1}{10}} C(\eta)^{\frac{1}{5}} N_k^{\frac{1}{5}} \|u_{t_0}\|_{\dot{H}^1} + \|u\|^5_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} \\
&\le \eta + |J_k|^{\frac{1}{10}} C(\eta)^{\frac{1}{5}} N_k^{\frac{1}{5}} \|u_{t_0}\|_{\dot{H}^1} + \|u\|^5_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} \\
&\lesssim_u |J_k|^{\frac{1}{10}} C(\eta)^{\frac{1}{5}} N_k^{\frac{1}{5}} + \|u\|^5_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)}
\end{aligned}$$

if $\eta \lesssim \int_{J_k} N_k^2$.
Since $\|u\|_{L^{10}_t \dot{W}^{\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)}$ is continuos in t, this implies that $\|u\|_{L^{10}_t \dot{W}^{\frac{30}{13}}_x(J_k\times\mathbb{R}^3)} \lesssim_u |J_k|^{\frac{1}{10}} C(\eta)^{\frac{1}{5}} N_k^{\frac{1}{5}}$, and taking the 10-th power, this implies

$$\|u\|^{10}_{L^{10}_t \dot{W}^{\frac{30}{13}}_x([t_k,t]\times\mathbb{R}^3)} \lesssim_u \int_I N(t)^2.$$

For the general case, it is enough to prove the statement for $q = 2, r = 6$, and then using Hölder combined with $\|\nabla u\|_{L^\infty_t L^2_x} = \|u\|_{\dot{H}^1} \gtrsim_u 1$ (which is true because the energy $E(u)$ is conserved), and recalling that $\int_I N(t)^2 \mathrm{d}t \gtrsim_u 1$ as well. We have that, by the usual Duhamel, Strichartz, Hölder and Sobolev,

$$\|u\|^2_{L^2_t \dot{W}^{1,6}_x(I\times\mathbb{R}^3)} \lesssim \|u_0\|^2_{\dot{H}^1} + \|u\|^{10}_{L^{10}_t \dot{W}^{1,\frac{30}{13}}_x} \lesssim_u 1 + \int_I N(t)^2 \lesssim_u \int_I N(t)^2.$$

□

Proposition 3.9 (No waste Duhamel)**.** *Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a maximal solution to NLS periodic modulo symmetries. Then $S(-t)u(t) \rightharpoonup 0$ weakly in $\dot{H}^1$. Therefore, we have*

$$u(t) = i \lim_{T\to T_{\max}} \int_t^T S(t-s)|u(s)|^4 u(s)\mathrm{d}s, \tag{35}$$

where the limit is intended weakly in $\dot{H}^1$.

Proof. If $T_{\max} < +\infty$, then by Corollary 3.7 we have $N(T) \to \infty$ as $T \to T_{\max}$, so $S(-T)u(T) \rightharpoonup 0$ as $T \to T_{\max}$.
If $T_{\max} = +\infty$, let $\phi \in C^\infty_c$. We want to prove that

$$\langle \nabla u(T), \nabla S(T)\phi\rangle \to 0$$

as $T \to T_{\max}$. Up to subsequences, because of precompactness,

$$N(T)^{-\frac{1}{2}} u\left(T, N(T)^{-1}(x - x(t))\right) \to \overline{u} \text{ in } \dot{H}^1 \text{ as } T \to \infty,$$

so

$$\langle \nabla u(T), \nabla S(T)\phi\rangle - \left\langle \nabla\overline{u}, N(t)^{-\frac{3}{2}}(\nabla S(T)\phi)\left(N(t)^{-1}(x - x(t))\right)\right\rangle \to 0$$

as $T \to \infty$. If $|x(t)| \to \infty$, then $N(t)^{-\frac{3}{2}}(\nabla S(T)\phi)\left(N(t)^{-1}(x - x(t))\right) \rightharpoonup 0$, so we have the thesis. Otherwise, because of Lemma 2.2 and Corollary 3.7 we have that for every ball $B = \{|x| \le r\}$ and T big enough,

$$\left\| N(T)^{-\frac{3}{2}}(\nabla S(T)\phi)\left(N(T)^{-1}(x - x(t))\right)\right\|_{L^2(B)}$$

$$\lesssim \left\| N(T)^{-\frac{3}{2}}(2iT)^{-\frac{3}{2}} e^{i|N(T)^{-1}(x-x(T))|^2/4T} \frac{N(T)^{-1}(x - x(T))}{2T}\hat{\phi}\left(\frac{N(t)^{-1}(x - x(T))}{2T}\right)\right\|_{L^2(B)}$$

$$\overset{|x|,|x(t)| \text{ bounded and } N(T)\lesssim T^{-\frac{1}{2}},}{\lesssim} \left\| T^{-\frac{5}{4}}\|\phi\|_{L^1}\right\|_{L^2(B)} \to 0$$

as $T \to \infty$. Therefore, $N(t)^{-\frac{3}{2}}(\nabla S(T)\phi)\left(N(t)^{-1}(x - x(t))\right) \rightharpoonup 0$, and we have the thesis also in this case. □

Now, up to changing our blowup solution, we can add a further property to u, namely, that $N(t) \ge 1$ $\forall t$.

Theorem 3.10. *Suppose Theorem 1.1 fails to be true. Thene there exists an almost periodic modulo symmetries solution* $u : J \times \mathbb{R}^d \to \mathbb{C}$ *such that* $\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} = \infty$ *and* $N(t) \geq 1$ *for all* $t \in J$.

Proof. Let $v : I \times \mathbb{R}^3 \to \mathbb{C}$ as in Theorem 3.3. We will construct u starting from this v. Let $J_1 \subset J_2 \subset \cdots \subset J$ an exhaustion in compact intervals of J. Since $v \in C_t(\dot{H}^1)$, $\{u(t)|t \in J_n\}$ is compact in $\dot{H}^1$, and therefore $N(t)$ is bounded from above and below on J_n. Let $t_k \in J_n$ such that $2N(t_n) \geq N(t)$ for all $t \in J_n$. Let $v_n : I_n \times \mathbb{R}^3 \to \mathbb{C}$ be the normalization of v centered in t_n:

$$v_n(t,x) = N(t)^{\frac{1}{2}} v(N(t_n)^2(t-t_n), N(t_n)(x+x(t))),$$

With $I_n = \{t|t_n + N(t_n)^{-2}t \in J_n\}$. Up to subsequences, we have that $v_n(0) \to u_0$. Let $u : I = (-T_-, T_+) \times \mathbb{R}^3 \to \mathbb{C}$ be the maximal solution with initial data u_0. Let $K \subset I$ be an interval such that $\|u\|_{L^{10}_{t,x}(K\times\mathbb{R}^3)} < \infty$. Then from Theorem 2.10, we have $\|v_n\|_{L^{10}_{t,x}(K\cap I_n\times\mathbb{R}^3)} \leq C(K,u)$ for n big enough. Since moreover $\|v_n\|_{L^{10}_{t,x}(I_n\times\mathbb{R}^3)} = \|v\|_{L^{10}_{t,x}(J_n\times\mathbb{R}^3)} \to \infty$, we have that $I_n \not\subset K$ for n large enough. Since they are both intervals containing 0, this leads to two cases:

(1) $K \cap \mathbb{R}^+ \subset I_n$ or
(2) $K \cap \mathbb{R}^- \subset I_n$.

Therefore, up to further subsequences and time reversal, we can assume $(0,t) \subset I_n$ for every $t < T_+$ and n big enough (notice that $\|u\|_{L^{10}_{t,x}([0,t]\times\mathbb{R}^3)} < \infty$). Let us rename $I = (0, T_+)$ and restrict u to $I \times \mathbb{R}^3$. We have that

(1) u is almost periodic: $u(t) = \lim_{n\to\infty} v_n(t) = N(t)^{\frac{1}{2}} v(N(t_n)^2(t-t_n), N(t_n)(x+x(t)))$, so the family $\{u(t)\}$ is contained in the closure of $\{v(t)\}$ modulo symmetries.
(2) $\|u\|_{L^{10}_{t,x}} = \infty$: if not, because of Lemma 2.8, $T_+ = +\infty$, $K = \mathbb{R}^+$ is a valid choice and so $I_n \supset \mathbb{R}^+$. Therefore, by Theorem 2.10, there exists a constant C such that

$$C > \|v_n\|_{L^{10}_{t,x}(\mathbb{R}^+\times\mathbb{R}^3)} = \|v\|_{L^{10}_{t,x}(\{t\geq t_n\}\times\mathbb{R}^3)} = \infty,$$

contradiction,
(3) $N_u(t)$ is bounded from below: obviously, $N_u(t)$ is bounded from below on $[0,t]$ for all $t < T_+$. Let now $t \to T_+$. Then we have

$$\eta \geq \int_{|\xi|\leq c(\eta)N(t)} |\xi\hat{v}(t)|^2 = \int_{|\xi|\leq c(\eta)\frac{N(t_n+N(t_n)^{-2}t)}{N(t_n)}} |\xi\hat{v}_n(t)|^2$$
$$\overset{t\in J_n}{\geq} \int_{|\xi|\leq \frac{1}{2}c(\eta)} |\xi\hat{v}_n(t)|^2 \overset{n \text{ big enough}}{\sim} \int_{|\xi|\leq \frac{1}{2}c(\eta)} |\xi\hat{u}(t)|^2.$$

But since $\|u\|_{\dot{H}^1} \gtrsim 1$ and $\int_{|\xi|\le C(\delta)N_u(t)} |\xi\hat{u}(t)|^2 \le \delta$, we cannot have $N(t) \to 0$ on a sequence (otherwise, $C(\delta)N(t) \le \frac{1}{2}c(\eta)$ and we get a contradiction).

Now, since $N_u(t) \ge N_0$, up to rescaling the solution u, we can make $N_u(t) \ge 1$. □

Therefore, in order to prove Theorem 1.1, we can restrict our attention to proving the non existence of this very specific solution to NLS:

Theorem 3.11. *Suppose Theorem 1.1 fails to be true. Then there exists an almost periodic solution* $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ *such that*

$$\|u\|_{L^{10}_{t,x}} = +\infty,$$

Moreover, we may write $[0, T_{\max}) = \bigcup_k J_k$ *with* J_k *being intervals where* $N(t) \equiv N_k$ *is constant,* $|J_k| \gtrsim N_k^{-2}$*, and* $N_k \ge 1$ *for every* k*.*

In order to prove that such a solution cannot exist, we define

$$K = \int_0^{T_{\max}} N(t)^{-1}, \tag{36}$$

and we will split the argument on the two possiibilities:

- $K < +\infty$, the rapid frequency cascade case,
- $K = \infty$, the quasisoliton case.

4. Impossibility of rapid frequency cascade

The main technical tool for this part is the following

Theorem 4.1 (Long-time Strichartz estimate)**.** *Let* $u : (T_{\min}, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ *be a maximal almost periodic solution to NLS as in Theorem 3.11 and let* $I \subset (T_{\min}, T_{\max})$ *be a finite interval which is a union of finitely many intervals* J_k*. Then for any fixed* $6 < q < \infty$ *and any frequency* $N > 0$*,*

$$A(N) := \left(\sum_{M \le N} \|\nabla u_M\|^2_{L^2_t L^6_x(I\times\mathbb{R}^3)} \right)^{\frac{1}{2}} = \|\nabla u_{\le N}\|_{l^2_N L^2_t L^6_x} \tag{37}$$

and

$$\tilde{A}_q(N) := N^{\frac{3}{2}} \left\| \sup_{M \ge N} M^{\frac{3}{q}-1} \|u_M(t)\|_{L^q_x(\mathbb{R}^3)} \right\|_{L^2_t(I)} \tag{38}$$

obey

$$A(N) + \tilde{A}_q(N) \lesssim_u 1 + N^{\frac{3}{2}} K^{\frac{1}{2}}, \tag{39}$$

where $K := \int_I N(t)^{-1} \mathrm{d}t$*. The implicit constant is independent on the interval* I*.*

Proof. First of all, notice that

$$\|\nabla u_{\leq N}\|_{L^2_t L^6_x} \sim \|\nabla u_{\leq N}\|_{L^2_t L^6_x l^2_M} \lesssim \|\nabla u_{\leq N}\|_{l^2_N L^2_t L^6_x} = A(N) \tag{40}$$

$$\|u_{\leq N}\|_{L^4_t L^\infty_x} \overset{\text{Lemma 2.12}}{\lesssim} A(N)^{\frac{1}{2}} \|u\|_{L^\infty_x \dot{H}^1} \lesssim_u A(N)^{\frac{1}{2}} \tag{41}$$

$$A(N)^2 \overset{\text{Lemma 2.11 + Sobolev}}{\lesssim_u} 1 + \|\nabla u\|^{10}_{L^{10}_t L^{\frac{30}{13}}_x} \overset{\text{Lemma 3.8}}{\lesssim_u} \int_I N(t)^2 \tag{42}$$

$$\begin{aligned} \tilde{A}_q(N) &\overset{\text{Proposition 2.13}}{\lesssim} N^{\frac{3}{2}} \left(\left\| |\nabla|^{-\frac{1}{2}} u_{\geq N} \right\|_{L^\infty_t L^2_x} + \left\| |\nabla|^{-\frac{1}{2}} P_{\geq N} |u|^4 u \right\|_{L^2_t L^{\frac{6}{5}}_x} \right) \\ &\overset{\text{Bernstein}}{\lesssim} 1 + \|\nabla u\|_{L^2_t L^6_x} \|u\|^4_{L^\infty_t L^6_x} \overset{\text{Lemma 3.8}}{\lesssim_u} \left(\int N(t)^2 \right)^{\frac{1}{2}} \end{aligned} \tag{43}$$

therefore, we have the inequality

$$A(N) + \tilde{A}_q(N) \lesssim_u N^{\frac{3}{2}} K^{\frac{1}{2}}$$

whenever

$$N \geq \sup_{t \in I} N(t) \geq \left(\frac{\int_I N(t)^2}{\int_I N(t)^{-1}} \right)^{\frac{1}{3}}.$$

In order to prove the thesis for N small we need and induction on scales argument, based on the following recurrence relation:

Lemma 4.2. *For η small enough,*

$$A(N) \lesssim_u 1 + c(\eta)^{-\frac{3}{2}} N^{\frac{3}{2}} K^{\frac{1}{2}} + \eta^2 \tilde{A}_q(2N) \tag{44}$$

$$\tilde{A}_q(N) \lesssim 1 + c(\eta)^{-\frac{3}{2}} N^{\frac{3}{2}} K^{\frac{1}{2}} + \eta A(N) + \eta^2 \tilde{A}_q(2N) \tag{45}$$

where $c(\eta)$ is the same as in Proposition 3.4.

Proof. Rename $c = c(t)$. Decomposing u as $u = u_{\leq cN(t} + u_{>cN(t)}$ and then $u = u_{\leq N} + u_{>N}$, we may write

$$F(u) = \emptyset(u^2_{>cN(t)} u^3) + \emptyset(u^2_{\leq cN(t)} u^2_{>N} u) + \emptyset(u^2_{\leq cN(t)} u^2_{\leq N} u).$$

Therefore we have:

$$\begin{aligned} A(N) &\overset{\text{Lemma 2.11 + Sobolev}}{\lesssim} \|\nabla u_{\leq N}\|_{L^\infty_t L^2_x} + \left\| \nabla P_{\leq N} \emptyset(u^2_{>cN(t)} u^3) \right\|_{L^2_t L^{\frac{6}{5}}_x} \\ &\quad + \left\| \nabla P_{\leq N} \emptyset(u^2_{\leq cN(t)} u^2_{>N} u) \right\|_{L^2_t L^{\frac{6}{5}}_x} + \left\| \nabla P_{\leq N} \emptyset(u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L^2_t L^{\frac{6}{5}}_x} \end{aligned} \tag{46}$$

$$\overset{\text{Bernstein}}{\lesssim_u} 1 + N^{\frac{3}{2}} \left\| u^2_{>cN(t)} u^3 \right\|_{L^2_t L^1_x} + N^{\frac{3}{2}} \left\| u^2_{\leq cN(t)} u^2_{>N} u \right\|_{L^2_t L^1_x} + \left\| \nabla P_{\leq N} \emptyset(u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L^2_t L^{\frac{6}{5}}_x}.$$

Similarly,

(47)
$$\tilde{A}_q(N) \overset{\text{Proposition 2.13}}{\lesssim} N^{\frac{3}{2}} \Big(\left\| |\nabla|^{-\frac{1}{2}} u_{\geq N} \right\|_{L_t^\infty L_x^2} + \left\| u^2_{>cN(t)} u^3 \right\|_{L_t^2 L_x^1} + N^{\frac{3}{2}} \left\| u^2_{\leq cN(t)} u^2_{>N} u \right\|_{L_t^2 L_x^1}$$
$$+ \left\| |\nabla|^{-\frac{1}{2}} P_{\geq N} (u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} \Big)$$

$$\overset{\text{Bernstein}}{\lesssim_u} 1 + N^{\frac{3}{2}} \left\| u^2_{>cN(t)} u^3 \right\|_{L_t^2 L_x^1} + N^{\frac{3}{2}} \left\| u^2_{\leq cN(t)} u^2_{>N} u \right\|_{L_t^2 L_x^1} + \left\| \nabla P_{\leq N} (u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L_t^2 L_x^{\frac{6}{5}}}.$$

Therefore, the lemma is proven as soon as we prove

$$N^{\frac{3}{2}} \left\| u^2_{>cN(t)} u^3 \right\|_{L_t^2 L_x^1} \lesssim c^{-\frac{3}{2}} N^{\frac{3}{2}} K^{\frac{1}{2}} \tag{48}$$

$$N^{\frac{3}{2}} \left\| u^2_{\leq cN(t)} u^2_{>N} u \right\|_{L_t^2 L_x^1} \lesssim \eta^2 \tilde{A}_q(2N) \tag{49}$$

$$\left\| \nabla P_{\leq N} (u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} \lesssim \eta A(N). \tag{50}$$

For the first one, consider J_k where $N(t) \equiv N_k$. Then by Hölder, Sobolev, Bernstein and Lemma 3.8,

$$N^{\frac{3}{2}} \left\| u^2_{>cN(t)} u^3 \right\|_{L_t^2 L_x^1(J_k \times \mathbb{R}^3)} \lesssim N^{\frac{3}{2}} N_k^{\frac{3}{2}} \left\| u_{>cN(t)} \right\|^2_{L^4_{t,x}(J_k \times \mathbb{R}^3)} \left\| u \right\|^3_{L_t^2 L_x^6(J_k \times \mathbb{R}^3)}$$
$$\lesssim_u N^{\frac{3}{2}} c^{-\frac{3}{2}} N_k^{-\frac{3}{2}} \left\| \nabla u_{>cN_k} \right\|^2_{L_t^4 L_x^3(J_k \times \mathbb{R}^3)} \lesssim_u N^{\frac{3}{2}} c^{-\frac{3}{2}} N_k^{-\frac{1}{2}} |J_k|^{\frac{1}{2}}.$$

Therefore, squaring and summing over k, we get

$$N^{\frac{3}{2}} \left\| u^2_{>cN(t)} u^3 \right\|_{L_t^2 L_x^1} \lesssim c^{-\frac{3}{2}} N^{\frac{3}{2}} K^{\frac{1}{2}}.$$

For (49), we have by Hölder, Bernstein and Schur's test:

$$\left\| u^2_{>N} u \right\|_{L_t^2 L_x^{\frac{3}{2}}} \lesssim \left\| \sum_{M_1 \geq M_2 \geq M_3 M_2 > N} \left\| u_{M_1} \right\|_{L^2} \left\| u_{M_2} \right\|_{L^q} \left\| u_{M_3} \right\|_{L^{\frac{6q}{q-6}}} \right\|_{L_t^2} \tag{51}$$

$$\lesssim \left\| \sup_{M>N} \left\| M^{\frac{3}{q}-1} u_M(t) \right\|_{L_x^q} \sum_{M_1 \geq M_3} \left(\frac{M_3}{M_1} \right)^{\frac{3}{q}} \left\| \nabla u_{M_1} \right\|_{L_x^2} \left\| \nabla u_{M_2} \right\|_{L_x^2} \right\|_{L_t^2} \lesssim_u N^{-\frac{3}{2}} \tilde{A}_q(2N).$$

Therefore, by Hölder,

$$N^{\frac{3}{2}} \left\| u^2_{\leq cN(t)} u^2_{>N} u \right\|_{L_t^2 L_x^1} \lesssim N^{\frac{3}{2}} \left\| u_{\leq c(\eta) N(t)} \right\|^2_{L_t^\infty L_x^6} \left\| u^2_{>N} u \right\|_{L_t^2 L_x^{\frac{3}{2}}} \lesssim_u \eta^2 \tilde{A}_q(2N).$$

For (50), by Hölder we have

$$\left\| \nabla P_{\leq N} \emptyset (u^2_{\leq cN(t)} u^2_{\leq N} u) \right\|_{L_t^2 L_x^{\frac{6}{5}}}$$

$$\lesssim \|\nabla u_{\leq N}\|_{L^2L^6} \left\|u_{\leq cN(t)}\right\|_{L^\infty L^6} \|u\|_{L^\infty L^6}^3 + \|\nabla u\|_{L_t^\infty L_x^2} \left\|u_{\leq cN(t)}\right\|_{L_t^\infty L_x^6} \|u_{\leq N}\|_{L_t^4 L_x^\infty}^2 \|u\|_{L_t^\infty L_x^6}$$

$$\lesssim_u \eta A(N).$$

□

Given these recursive inequalities, the proof of the theorem follows by a straightforward induction. □

This tool is crucial for obtaining a bound on the mass of the solution u. The precise statement is the following:

Lemma 4.3. *finite mass Let $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ an almost periodic solution to NLS with $\|u\|_{L^{10}_{t,x}} = +\infty$ and*

$$K := \int_0^{T_{\max}} N(t)^{-1} \mathrm{d}t < \infty. \tag{52}$$

Then for all $0 < N < 1$,

$$\|u_{N\leq \cdot \leq 1}\|_{L_t^\infty L_x^2} + \frac{1}{N} \left(\sum_{M<N} \|\nabla u_M\|_{L_t^2 L_x^6}^2 \right)^{\frac{1}{2}} \lesssim_u 1. \tag{53}$$

In particular,

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u_{\leq 1}\|_{L_t^\infty L_x^2} + \|u_{>1}\|_{L_t^\infty L_x^2} \lesssim \|u_{\leq 1}\|_{L_t^\infty L_x^2} + \|\nabla u_{>1}\|_{L_t^\infty L_x^2} \lesssim_u 1. \tag{54}$$

Proof. First of all, we have that

$$LHS \lesssim N^{-1} \|u\|_{L^\infty \dot{H}^1} + N^{-1} A(N/2) \lesssim_u N^{-1}(1 + N^{\frac{3}{2}} K^{\frac{1}{2}}) < +\infty,$$

so LHS is finite for every N. Using Proposition 3.9 and Lemma 2.11, we get

$$LHS \lesssim N^{-1} \|\nabla P_{<N} F(u)\|_{L_t^2 L_x^{\frac{6}{5}}} + \|P_{N\leq \cdot \leq 1} F(u)\|_{L_t^2 L_x^{\frac{6}{5}}}.$$

To estimate the nonlinearity, we write down

$$F(u) = \emptyset(u_{>cN(t)}^2 u^3) + \emptyset(u_{\leq cN(t)} u_{<N}^2 u^2) + \emptyset(u_{\leq cN(t)} u_{\leq 1}^2 u_{N\leq \cdot \leq 1} u) + \emptyset(u_{\leq cN(t)} u_{>1}^2 u^2).$$

For the first term, using Bernstein and (48),

$$\frac{1}{N} \left\|\nabla P_{<N} u_{>cN(t)}^2 u^3\right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\|P_{N\leq \cdot \leq 1} u_{>cN(t)}^2 u^3\right\|_{L_t^2 L_x^{\frac{6}{5}}} \lesssim (N^{\frac{1}{2}} + 1) \left\|u_{>cN(t)}^2 u^3\right\| \lesssim_u c^{-\frac{3}{2}} K^{\frac{1}{2}}.$$

For the second term, using Bernstein on the second term and distributing the gradient, (41), (40) Proposition 3.4 and Theorem 4.1,

$$\begin{aligned}
&\frac{1}{N}\left\|\nabla P_{<N}u_{\leq cN(t)}u_{<N}^2u^2\right\|_{L_t^2L_x^{\frac{6}{5}}}+\left\|P_{N\leq\cdot\leq 1}u_{\leq cN(t)}u_{<N}^2u^2\right\|_{L_t^2L_x^{\frac{6}{5}}}\\
&\lesssim \frac{1}{N}\left\|\nabla u_{\leq cN(t)}\right\|_{L_t^\infty L_x^2}\left\|u_{<N}\right\|_{L_t^4L_x^\infty}^2+\frac{1}{N}\left\|u_{\leq cN(t)}\right\|_{L_t^\infty L_x^6}\left\|\nabla u_{<N}\right\|_{L_t^2L_x^6}\left\|u\right\|_{L_t^\infty L_x^6}^3\\
&\qquad+\frac{1}{N}\left\|u_{\leq cN(t)}\right\|_{L_t^\infty L_x^6}\left\|u_{<N}\right\|_{L_t^4L_x^\infty}^2\left\|\nabla u\right\|_{L_t^\infty L_x^2}\left\|u\right\|_{L_t^\infty L_x^6}\\
&\lesssim_u \eta(1+K^{\frac{1}{2}})LHS.
\end{aligned}$$

For the third term, using Bernstein, Theorem 4.1, (41), and Proposition 3.4,

$$\begin{aligned}
&N^{-1}\left\|\nabla P_{<N}(u_{\leq cN(t)}u_{\leq 1}^2u_{N\leq\cdot\leq 1}u)\right\|_{L_t^2L_x^{\frac{6}{5}}}+\left\|P_{N\leq\cdot\leq 1}(u_{\leq cN(t)}u_{\leq 1}^2u_{N\leq\cdot\leq 1}u)\right\|_{L_t^2L_x^{\frac{6}{5}}}\\
&\lesssim \left\|u_{\leq cN(t)}\right\|_{L_t^\infty L_x^6}\left\|u_{\leq 1}\right\|_{L_t^4L_x^\infty}^2\left\|u_{1\leq\cdot\leq 1}\right\|_{L_t^\infty L_x^2}\left\|u\right\|_{L_t^\infty L_x^6}\\
&\lesssim_u \eta(1+K^{\frac{1}{2}})LHS.
\end{aligned}$$

For the last term, using Bernstein, Theorem 4.1, (51), and Proposition 3.4,

$$\begin{aligned}
&N^{-1}\left\|\nabla P_{<N}u_{\leq cN(t)}u_{>1}^2u^2\right\|_{L_t^2L_x^{\frac{6}{5}}}+\left\|P_{N\leq\cdot\leq 1}u_{\leq cN(t)}u_{>1}^2u^2\right\|_{L_t^2L_x^{\frac{6}{5}}}\\
&\lesssim (N^{\frac{1}{2}}+1)\left\|u_{\leq cN(t)}u_{>1}^2u^2\right\|_{L_t^2L_x^1}\\
&\lesssim \left\|u_{\leq cN(t)}\right\|_{L_t^\infty L_x^6}\left\|u_{>1}^2u\right\|_{L_t^2L_x^{\frac{3}{2}}}\left\|u\right\|_{L^\infty L^6}\\
&\lesssim_u \eta(1+K^{\frac{1}{2}}).
\end{aligned}$$

Collecting all the estimates, we have that

$$LHS \lesssim_u \eta(1+K^{\frac{1}{2}})LHS+1+c^{-\frac{3}{2}}K^{\frac{1}{2}}.$$

Therefore, by taking η small enough, we have the thesis. ☐

Now we are ready to prove the main result of this section:

Theorem 4.4. *There are no almost periodic solutions $u:[0,T_{\max})\times\mathbb{R}^3\to\mathbb{C}$ to NLS with $\|u\|_{L_{t,x}^{10}}=\infty$ and*

$$\int_0^{T_{\max}} N(t)^{-1}\mathrm{d}t<+\infty.$$

Proof. By contradiction, let u be such a solution. If $T_{\max}<\infty$, then $N(t)\to\infty$ as $t\to T_{\max}$ because of Corollary 3.7. The same holds when $T_{\max}=\infty$: from Remark 3.6, we have that

$$\sum_{k=1}^{\infty}N_k^{-3}\lesssim\int_0^{T_{\max}}N(t)^{-1}\mathrm{d}t<+\infty,$$

so $N_k \to \infty$ when $k \to \infty$, which implies $N(t) \to \infty$ as $t \to \infty$.
We want to prove that the existence of such a solution is inconsistent with the conservation of mass. We proved in Lemma 4.3 that the mass of u is finite, we will prove that by conservation of mass this has to be 0 (which in turn implies $u \equiv 0$, which contradicts the hypothesis on the L^{10} norm). Using Proposition 3.9, we can estimate

$$\|u_{\leq N}\|_{L_t^\infty L_x^2} \lesssim \|P_{\leq N} F(u)\|_{L_t^2 L_x^{\frac{6}{5}}} \lesssim N^{\frac{1}{2}} \|P_{\leq N} F(u)\|_{L_t^2 L_x^1}.$$

Decompose the nonlinearity as $F(u) = \emptyset(u_{\leq 1}^3 u^2) + \emptyset(u_{>1}^3 u^2)$. By Theorem 4.1, (41), Bernstein, and the finiteness of mass, we get

$$\left\|u_{\leq 1}^3 u^2\right\|_{L_t^2 L_x^1} \lesssim \|u_{\leq 1}\|_{L_t^4 L_x^\infty}^2 \|u_{\leq 1}\|_{L_t^\infty L_x^\infty} \|u\|_{L_t^\infty L_x^2}^2 \lesssim_u 1.$$

Instead, by Theorem 4.1 and (51)

$$\left\|u_{>1}^3 u^2\right\|_{L_t^2 L_x^1} \lesssim \|u\|_{L_t^\infty L_x^6}^2 \left\|u_{>1}^3\right\|_{L_t^2 L_x^{\frac{3}{2}}} \lesssim_u 1.$$

Thus, $\|u_{\leq N}\|_{L_t^\infty L_x^2} \lesssim_u N^{\frac{1}{2}}$. Let $c = c(\eta)$ as in Theorem 3.4. Then we have, for $t \to T_{\max}$, recalling that $N(t) \to \infty$,

$$\begin{aligned}
\|u\|_{L_x^2} &\lesssim \|u_{\leq N}\|_{L^2} + \left\|P_{>N} u_{\leq cN(t)}\right\|_{L^2} + \left\|u_{>cN(t)}\right\|_{L^2} \\
&\lesssim_u N^{\frac{1}{2}} + N^{-1} \left\|u_{\leq cN(t)}\right\|_{\dot{H}^1} + c^{-1} N(t)^{-1} \|u\|_{\dot{H}^1} \\
&\lesssim_u N^{\frac{1}{2}} + N^{-1}\eta + c^{-1} N(t)^{-1}.
\end{aligned}$$

This can be made as small as we want by choosing first N small, then η small depending on N, and lastly t close to $T_{\max}$, depending on $c = c(\eta)$. Therefore for conservation of mass we get $\|u\|_{L^2} = 0$, contradiction. □

5. Impossibility of the quasisoliton case

The main technical result which helps ruling out this case is the following:

Theorem 5.1 (Frequency - Localized interaction Morawetz estimate)**.** *Suppose $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ is an almost periodic solution to NLS such that $N(t) \geq 1$ and let $I \subset [0, T_{\max})$ be a finite union of contiguous intervals J_k as in Remark 3.6. Fix $0 \leq \eta_0 < 1$. For $N > 0$ sufficiently small (depending only on u, η_0),*

$$\int_I \int_{\mathbb{R}^3} |u_{>N}(t,x)|^4 \mathrm{d}x\mathrm{d}t \lesssim_u \eta_0 (N^{-3} + K),$$

where $K := \int_I N(t)^{-1} \mathrm{d}t$. The implicit constant does not depend neither on I nor on η_0.

We will not present a full proof of this Theorem here, but we will go through the main tools and ideas. For the proofs of the various estimates and identities, we refer to Section 5 of [6] and their relative references.

Unlike Theorem 4.1, the argument for Theorem 5.1 does not rely just on linear estimates on the Scrhödinger propagator $S(t)$, but it is strongly dependent on the sign of the nonlinearity. Namely, the whole argument relies on the identity

Proposition 5.2 (Interaction Morawetz identity). *Suppose*

$$i\partial_t\phi = -\Delta\phi + |\phi|^4\phi + \mathcal{F}$$

and let

$$M(t) := 2\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |\phi(y)|^2 a_k(x-y)\Im\left(\phi_k(x)\overline{\phi}(x)\right)\mathrm{d}x\mathrm{d}y. \tag{55}$$

for some weigh $a:\mathbb{R}^3\to\mathbb{R}$. Then

$$\partial_t M(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Bigg(\frac{4}{3}a_{kk}(x-y)|\phi(x)|^6|\phi(y)|^2 \tag{56}$$
$$+ 2a_k(x-y)|\phi(y)|^2\Re\left(\phi_k(x)\overline{\mathcal{F}}(x) - \mathcal{F}_k(x)\overline{\phi}(x)\right) \tag{57}$$
$$+ 4a_k(x-y)\Im\left(\mathcal{F}(y)\overline{\phi}(y)\right)\Im\left(\phi_k(x)\overline{\phi}(x)\right) \tag{58}$$
$$+ 4a_{jk}(x-y)\left(|\phi(y)|^2\overline{\phi}_j(x)\phi_k(x) - \Im\left(\overline{\phi}(y)\phi_j(y)\right)\Im\left(\phi_k(x)\overline{\phi}(x)\right)\right) \tag{59}$$
$$- a_{jjkk}(x-y)|\phi(y)|^2|\phi(x)|^2\Bigg). \tag{60}$$

Here subscripts denote spacial derivatives and repeated index are summed over.

The evidence of this identity is clear by making the choice $a(x) = |x|$, and ϕ a solution to NLS. In this case, $\mathcal{F} = 0$, and by the fundamental theorem of calculus we have

$$8\pi\int_I\int_{\mathbb{R}^3}|\phi(t,x)|^4\mathrm{d}x\mathrm{d}t \le 2\left\|M(t)\right\|_{L^\infty_t} \le 4\left\|\phi\right\|^3_{L^\infty_t L^2_x}\left\|\phi\right\|_{L^\infty_t \dot{H}^1_x}.$$

Here the L^4 norm arises from the term (60), and the terms (56) and (59) are positive (the other two terms are 0). Unfortunately, we cannot obtain an L^4 bound by this inequality, since a minimal blowup solution to NLS does not need to be in L^2. In order to circumvent this problem, we restrict our attention to $\phi = u_{>N}$. This produces many error terms, and in particular there is an error term arising from (58) which is not easy to bound. A possible solution to this is to truncate a, taking as a a radial

function such that:
(61)
$$a(0)=0, \quad a_r \geq 0, \quad a_{rr} \leq 0, \quad a_r = \begin{cases} 1 & \text{if } r \leq R, \\ 1 - J^{-1}\log(\frac{r}{R}) & \text{if } eR \leq r \leq e^{J-J_0}R, \\ 0 & \text{if } e^J R \leq r, \end{cases}$$

where $J_0 \geq 1$, $J \geq 2J_0$, and R are parameters to be determined, bound by the relation $e^J RN = 1$. Moreover, we extend a_r where is not defined in such a way that $|\partial_r^k a_r| \lesssim_k J^{-1} r^{-k}$ for each $k \geq 1$.
Let

$$B_I := \int_I \int \int_{|x-y|\leq e^{J-J_0}R} \frac{4}{3} a_{kk}(x-y)|u_{>N}(x)|^6 |u_{>N}(y)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{62}$$

It turns out that $B_I \geq 0$, and for N small enough,

- $\|(55)\|_{L_t^\infty} \lesssim_u \eta^3 N^{-3}$,
- $|\int_I (56) - B_I| \lesssim \frac{J_0^2}{J}(K + N^{-3})$,
- For any $\varepsilon > 0$, $\int_I |(57)| \lesssim_{u,\varepsilon} \varepsilon B_I + \eta \|u_{>N}\|^4_{L^4_{t,x}} + (\varepsilon^{-1}\eta + \varepsilon \frac{J_0^2}{J})(N^{-3} + K)$,
- $\int_I |(58)| \lesssim \eta^{\frac{1}{4}} \left(\|u_{>N}\|^4_{L^4_{t,x}} + N^{-3} + K \right)$,
- $\int_I |(59)| \lesssim_u \left(\eta^2 + \frac{J_0}{J}\right)(K + N^{-3}) + \frac{1}{J_0} B_I$,
- $8\pi \|u_{>N}\| - \int_I (60) \lesssim_u \frac{\eta^2 e^{2J}}{J}(K + N^{-3})$.

Therefore, putting everything together,

$$8\pi \|u_{>N}\|^4_{L^4_{t,x}} + B_I \lesssim \left(\varepsilon + \frac{1}{J_0}\right) B_I + \eta^{\frac{1}{4}} \|u_{>N}\|^4_{L^4_{t,x}} + \left(\eta^{\frac{1}{4}} + \frac{\eta}{\varepsilon} + \frac{J_0^2}{J} + \eta^2 \frac{e^{2J}}{J}\right)(N^{-3} + K),$$

from which one can deduce the thesis of Theorem 5.1.
In order to conclude the argument (and therefore proving Theorem 1.1), one need an estimate that states that $\|u_{>N}\|^4_{L^4_x}$ cannot be too small (compared to $N(t)^{-1}$). Indeed we have

Lemma 5.3. *Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ an almost periodic solution to NLS. Then there exists $C(u)$ such that*

$$N(t) \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u|^4 \mathrm{d}x \gtrsim_u 1.$$

Proof. Let $M_0 = \inf_{t\in I} \|u\|^2_{\dot{H}^1}$. We have that $M_0 > 0$, since energy is conserved and $E(u) \lesssim \|u\|^2_{\dot{H}^1} + \|u\|^6_{\dot{H}^1}$. Let $C(u) := C(\frac{M_0}{2})$ as in Proposition 3.4. Suppose by contradiction that the inequality does not hold. Then there exists a sequence of times t_n such that

$$N(t_n)\int_{|x-x(t_n)|\leq\frac{C(u)}{N(t_n)}}|u|^4\mathrm{d}x\leq n^{-1}. \tag{63}$$

Let v_n be the precompact family of rescalings and translates of $u(t_n)$, and up to subsequences, let $v_n\to v$ in $\dot{H}^1$. The condition (63) becomes

$$\int_{|x|<C(u)}|v_n|^4\mathrm{d}x\leq n^{-1}.$$

This implies that $v_n\to 0$ in $L^4(\{|x|<C(u)\})$. Because of compatibility of convergences, this implies that $v(x)=0$ if $|x|<C(u)$. Therefore,

$$M_0\leq\lim_n\|v_n\|_{\dot{H}^1}^2=\|v\|_{\dot{H}^1}^2=\int_{|x|>C(u)}|\nabla v|^2\mathrm{d}x=\lim_{n\to\infty}\int_{|x|>C(u)}|\nabla v_n|^2\mathrm{d}x\leq\frac{M_0}{2},$$

contradiction. □

Now we are ready to prove the impossibility of the quasisoliton case, which is the last possible case remaining for the minimal counterexample to Theorem 1.1.

Theorem 5.4. *There are no almost periodic solutions $u:[0,T_{\max})\times\mathbb{R}^3\to\mathbb{C}$ to NLS with $N(t)\equiv N_k\geq 1$ on the intervals J_k defined in Remark 3.6 which satisfies $\|u\|_{L^{10}_{t,x}}=+\infty$ and*

$$K=\int_0^{T_{\max}}N(t)^{-1}\mathrm{d}t=+\infty.$$

Proof. Suppose by contradiction that such a solution exists. From Theorem 5.1, we have that, for N small enough,

$$\int_I\int_{\mathbb{R}^3}|u_{>N}(t,x)|^4\mathrm{d}x\mathrm{d}t\lesssim_u\eta_0\left(N^{-3}+\int_I N(t)^{-1}\mathrm{d}t\right).$$

Moreover, if N is small enough,

$$N(t)\int_{|x-x(t)|\leq\frac{C(u)}{N(t)}}|u_{<N}|^4\mathrm{d}x\lesssim|C(u)|\,\|u_{\leq N}\|_{L^6_x}^4\leq|C(u)|\left\|u_{\leq NN(t)}\right\|_{L^6_x}^4\lesssim_u\varepsilon.$$

Therefore,

$$\left\|N(t)^{\frac14}u_{>N}\right\|_{L^4\left(\left\{|x|<\frac{C(u)}{N(t)}\right\}\right)}\geq\left\|N(t)^{\frac14}u\right\|_{L^4\left(\left\{|x|<\frac{C(u)}{N(t)}\right\}\right)}-\left\|N(t)^{\frac14}u_{\leq N}\right\|_{L^4\left(\left\{|x|<\frac{C(u)}{N(t)}\right\}\right)}\gtrsim_u 1.$$

Hence we have

$$\int_I N(t)^{-1}\mathrm{d}t\lesssim\int_I\int_{\mathbb{R}^3}|u|^4(t,x)\mathrm{d}x\mathrm{d}t\lesssim_u\eta_0\left(N^{-3}+\int_I N(t)^{-1}\mathrm{d}t\right),$$

from which, choosing η_0 small enough and N accordingly, $\int_I N(t)^{-1}\mathrm{d}t \lesssim_u 1$. Taking the limit as $I \to [0, T_{\max})$, this implies that

$$K = \int_0^{T_{\max}} N(t)^{-1} \lesssim_u 1 < +\infty,$$

contradiction. $\square$

References

[1] J. Bourgain. Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. *J. Amer. Math. Soc.*, 12(1):145–171, 1999.

[2] Thierry Cazenave and Fred B. Weissler. Some remarks on the nonlinear Schrödinger equation in the critical case. In *Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987)*, volume 1394 of *Lecture Notes in Math.*, pages 18–29. Springer, Berlin, 1989.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865, 2008.

[4] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.*, 166(3):645–675, 2006.

[5] Sahbi Keraani. On the blow up phenomenon of the critical nonlinear Schrödinger equation. *J. Funct. Anal.*, 235(1):171–192, 2006.

[6] Rowan Killip and Monica Vişan. Global well-posedness and scattering for the defocusing quintic NLS in three dimensions. *Anal. PDE*, 5(4):855–885, 2012.

[7] Terence Tao. *Nonlinear dispersive equations: local and global analysis*, volume 106. American Mathematical Soc., 2006.

[8] Monica Vişan. Oberwolfach lecture notes. https://www.math.ucla.edu/˜visan/Oberwolfach2012.pdf, 2012. In “Dispersive Equations and Nonlinear Waves”, Oberwolfach Seminars, 45, Birkhauser/Springer Basel AG, Basel, 2014.