Invariant Gibbs dynamics for fractional wave equations in negative Sobolev spaces

Oana Pocovnicu

Heriot-Watt University
Maxwell Institute for Mathematical Sciences, Edinburgh

June 4th, 2026

London Analysis Seminar

A class of fractional nonlinear wave equations on $\mathbb{T}^2$

We consider the defocusing fractional nonlinear wave equations (fNLW) on $\mathbb{T}^2$:

$$\begin{cases} \partial_t^2 u + (1-\Delta)^{\alpha} u + u^{2m+1} = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R},$$

where $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$, $m \geq 1$ integer, $\alpha < 1$ and

$$(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-\alpha}(\mathbb{T}^2) \text{ for } s < 0.$$

With the notation $v := \partial_t u$, fNLW is a **Hamiltonian evolution**:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H_\alpha}{\partial(u,v)},$$

associated to the **energy**:

$$H_\alpha(u,v) := \frac{1}{2}\left(\|u\|^2_{H^\alpha_x(\mathbb{T}^2)} + \|v\|^2_{L^2_x(\mathbb{T}^2)} \right) + \frac{1}{2m+2} \|u\|^{2m+2}_{L^{2m+2}_x(\mathbb{T}^2)}.$$

Beyond the limit of deterministic analysis

In the case $\alpha = 1$ of the defocusing nonlinear wave equation (NLW)

$$\partial_t^2 u + (1 - \Delta)u + u^{2m+1} = 0,$$

it is known that NLW is ill-posed in $\mathcal{H}^s(\mathbb{T}^2)$ for $s < 0$ (Christ-Colliander-Tao '03, Xia '15, Oh-Okamoto-Tzvetkov '22, Forlano-Okamoto '20).

At least in the case of $\alpha < 1$ sufficiently close to 1, we expect that fNLW is also ill-posed in $\mathcal{H}^s(\mathbb{T}^2)$ for $s < 0$.

We will adopt a probabilistic point of view to understand the well-posedness of fNLW in $\mathcal{H}^s(\mathbb{T}^2)$ for $s < 0$.

The aim is to go beyond the limit of deterministic analysis. This idea goes back to works of Bourgain '96, Burq-Tzvetkov '08, Colliander-Oh '12.

Goals of the talk

- **Construction of the Gibbs measure** associated with the renormalized fNLW
- **Almost sure local well-posedness of the renormalized fNLW** with respect to *Gibbsian* random initial data (of negative Sobolev regularity)
- **To extend globally-in-time the local-in-time solutions** previously obtained, using Bourgain's invariant measure argument
- **A counterexample**

Part I. **Construction of the Gibbs measure associated with the renormalized fNLW**

Gibbs measure for fNLW on $\mathbb{T}^2$

- fNLW is a Hamiltonian PDE:

$$H_\alpha(u,v) := \frac{1}{2}\Big(\|u\|^2_{H^\alpha_x(\mathbb{T}^2)} + \|v\|^2_{L^2_x(\mathbb{T}^2)}\Big) + \frac{1}{2m+2}\|u\|^{2m+2}_{L^{2m+2}_x(\mathbb{T}^2)}.$$

- $H_\alpha(u,v)$ is conserved under the flow of fNLW

- Gibbs measure: "$dP^{2m+2}_{\alpha,2} = Z^{-1}\exp(-H_\alpha(u,v))du \otimes dv$" is "expected" to be ***invariant***

- Gibbs measure as a weighted Gaussian measure:

$$dP^{2m+2}_\alpha = e^{-\frac{1}{2m+2}\int u^{2m+2}dx} d\vec{\mu}_\alpha,$$

where $\vec{\mu}_\alpha$ is the Gaussian measure on $\mathcal{D}'(\mathbb{T}^2) \times \mathcal{D}'(\mathbb{T}^2)$ given by:

$$d\vec{\mu}_\alpha = \underbrace{Z_0^{-1}e^{-\frac{1}{2}\int |(1-\Delta)^{\alpha/2}u|^2 dx}du}_{\text{Ornstein-Uhlenbeck measure on } \mathbb{T}^2} \otimes \underbrace{Z_1^{-1}e^{-\frac{1}{2}\int v^2 dx}dv}_{\text{white noise measure on } \mathbb{T}^2}.$$

- Formally, we write:

$$d\vec{\mu}_\alpha = Z_0^{-1} e^{-\frac{1}{2}\int |(1-\Delta)^{\alpha/2} u|^2 dx} du \otimes Z_1^{-1} e^{-\frac{1}{2}\int v^2 dx} dv$$
$$= Z_0^{-1} Z_1^{-1} \prod_{n\in\mathbb{Z}^2} e^{-\frac{1}{2}\langle n\rangle^{2\alpha}|\hat{u}_n|^2} e^{-\frac{1}{2}|\hat{v}_n|^2} d\hat{u}_n d\hat{v}_n,$$

where $\hat{u}_n$ and $\hat{v}_n$ are the Fourier coefficients of u and v respectively.

- Rigorously, $\vec{\mu}_\alpha$ is the induced probability measure induced by the map

$$\omega \mapsto (u^\omega(x), v^\omega(x)) := \left(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \right),$$

where $\{g_{0,n}, g_{1,n}\}_{n\in\mathbb{Z}^2}$ are independent standard $\mathbb{C}$-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$, $g_{j,-n} = \overline{g_{j,n}}$, $\mathbb{E}(|g_{j,n}|^2) = 2$, $j = 0, 1$.

- (u^ω, v^ω) are rough, of negative Sobolev regularity:

$$(u^\omega, v^\omega) \in \mathcal{H}^s(\mathbb{T}^2) \text{ for } s < -(1-\alpha)(< 0) \text{ almost surely}$$

$$\mathbb{E}(\|u^\omega\|^2_{H^s(\mathbb{T}^2)}) = \mathbb{E}\Big(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s} \frac{|g_{0,n}(\omega)|^2}{\langle n\rangle^{2\alpha}}\Big) = \sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s} \frac{2}{\langle n\rangle^{2\alpha}}$$
$$= 2\sum_{n\in\mathbb{Z}^2} \frac{1}{\langle n\rangle^{2\alpha-2s}} < \infty \text{ iff } 2\alpha - 2s > 2$$

  and also $(u^\omega, v^\omega) \notin \mathcal{H}^{-(1-\alpha)}(\mathbb{T}^2)$ almost surely.
- $\vec{\mu}_\alpha$ is supported on $\mathcal{H}^s(\mathbb{T}^2)$, $s < -(1-\alpha)$
- $\int_{\mathbb{T}^2} u^{2m+2} dx = \infty$ $\vec{\mu}_\alpha$-almost surely
- $e^{-\frac{1}{2m+2}\int u^{2m+2}dx} d\vec{\mu}_\alpha$ cannot define a probability measure
- we need to renormalize $\int u^{2m+2} dx$ to construct the Gibbs measure

Wick renormalization

- For a typical element u under μ_α, $u \notin L^2(\mathbb{T}^2)$ almost surely and, with $\mathbf{P}_N f := \sum_{n\in\mathbb{Z}^2, |n|\le N} \widehat{f}(n)e^{in\cdot x}$,

$$\sigma_{\alpha,N} := \mathbb{E}[(\mathbf{P}_N u)^2(x)] = \sum_{\substack{n\in\mathbb{Z}^2\\ |n|\le N}} \frac{2}{\langle n\rangle^{2\alpha}} \sim N^{2-2\alpha} \to \infty \text{ as } N \to \infty.$$

- We introduce the Wick ordered monomial $:(\mathbf{P}_N u)^k:$ by

$$:(\mathbf{P}_N u)^k(x): := H_k(\mathbf{P}_N u(x); \sigma_{\alpha,N}),$$

where $H_k(x;\sigma)$ is the **Hermite polynomial** of degree k defined via:

$$F(t,x;\sigma) := e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma),$$

$$H_0(x;\sigma) = 1, \qquad H_1(x;\sigma) = x, \qquad H_2(x;\sigma) = x^2 - \sigma,$$
$$H_3(x;\sigma) = x^3 - 3\sigma x, \qquad H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2.$$

Construction of the Gibbs measure

Theorem (Construction of the Gibbs measure, Forcella-P. '22)

Let $\alpha \in \left(1 - \frac{1}{2m+2}, 1\right)$ and $1 \leq p \leq \infty$.

Then, $\{\int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}(x): dx\}_N$ converges in $L^p(\mu_\alpha)$ as $N \to \infty$ and we set

$$\int_{\mathbb{T}^2} :u^{2m+2}: dx := \lim_{N\to\infty} \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx.$$

We can then rigorously define the Gibbs measure as

$$dP_\alpha^{2m+2} = Z^{-1} e^{-\frac{1}{2m+2}\int :u^{2m+2}: dx} d\vec{\mu}_\alpha.$$

- The condition on α is optimal if one wishes to obtain a Gibbs measure that is absolutely continuous with respect to μ_α.
- To prove this theorem, we use the variational approach introduced by Barashkov-Gubinelli '20.

Boué-Dupuis variational formula

- Let $W(t)$ be a cylindrical Brownian motion: $W(t) = \sum_{n\in\mathbb{Z}^2} B_n(t)e^{in\cdot x}$, where $\{B_n\}_{n\in\mathbb{Z}^2}$ are independent complex-valued Brownian motions
- $Y(t) := \langle\nabla\rangle^{-\alpha}W(t)$, $Y_N(t) := \mathbf{P}_N Y$, and set $Y_N := Y_N(1)$. Note that $\mathrm{Law}(Y_N) = (\mathbf{P}_N)_*\mu_\alpha$
- Drifts: $\mathbb{H}_a := \{\theta \in L^2([0,1]; L^2(\mathbb{T}^2)); \text{progressively measurable P-a.s.}\}$

Lemma (Boué-Dupuis formula (B-D '98, Üstünel '14))

Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}^2) \to \mathbb{R}$ is measurable such that $\mathbb{E}\big[|F(Y_N(1))|^p\big] < \infty$ and $\mathbb{E}\big[|e^{-F(Y_N(1))}|^q\big] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$-\log \mathbb{E}\Big[e^{-F(Y_N)}\Big] = \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[F(Y_N + \Theta_N) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2\,dt\bigg],$$

where $\Theta_N := \mathbf{P}_N \int_0^1 \langle\nabla\rangle^{-\alpha}\theta(t')dt'$ and $\mathbb{E} = \mathbb{E}_P$.

On the optimality of measure construction result

- We set $G_N(u) := \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}(x): dx$.
- Goal of the proof: a uniform in N bound on $\int e^{-\frac{1}{2m+2}G_N(u)} d\mu_\alpha(u)$.
- Thanks to the Boué-Dupuis variational formula (with $F = \frac{1}{2m+2}G_N$), this reduces to proving a uniform in N lower bund on

$$\inf_{\theta \in \mathbb{H}_a} \mathbb{E}\Big[G_N(Y_N + \Theta_N)\Big]$$

- We have

$$G_N(Y_N + \Theta_N) = \sum_{\ell=0}^{2m+2} \binom{2m+2}{\ell} \int_{\mathbb{T}^2} :Y_N^{2m+2-\ell}: \Theta_N^\ell \, dx$$

- Since $\mathbb{E}\Big[\int_{\mathbb{T}^2} :Y_N^{2m+2}: \, dx\Big] = 0$, the worst term is

$$\mathbb{E}\Big[\int_{\mathbb{T}^2} :Y_N^{2m+1}: \Theta_N \, dx\Big]$$

- Since $:Y_N^{2m+1}:$ has regularity $(2m+1)(\alpha-1)-\varepsilon$, $\varepsilon > 0$, $\Theta_N \in H^\alpha$, we can control it iff

$$(2m+1)(\alpha-1) + \alpha > 0 \Longleftrightarrow \alpha > 1 - \frac{1}{2m+2}.$$

Further heuristics

- We expect that $\rho_\alpha := Z^{-1} e^{-\frac{1}{2m+2}\int :u^{2m+2}: dx}\, d\mu_\alpha$ is absolutely continuous with respect to the shifted measure

$$\mathrm{Law}\left(\sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} + (1-\Delta)^{-\alpha} : \left(\sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} \right)^{2m+1} : \right)$$

- The first term corresponds to the typical element in the support of μ_α and has regularity $s_{\text{Gaussian}} := \alpha - \frac{d}{2} - \varepsilon$, $\varepsilon > 0$.

- The shift term has regularity $s_{\text{shift}} := (2m+1)(\alpha - \frac{d}{2}) + 2\alpha - \varepsilon$, $\varepsilon > 0$.

Case 1: If $s_{\text{shift}} > \alpha$, i.e. $\alpha > \frac{2m+1}{2m+2} \cdot \frac{d}{2}$, we expect that ρ_α **is absolutely continuous w.r.t** μ_α.

Case 2: If $s_{\text{shift}} \leq s_{\text{Gaussian}}$, i.e. $\alpha \leq \frac{m}{m+1} \cdot \frac{d}{2} =: \alpha_{\text{crit}}$, we expect **triviality results**. For $\alpha = 1$, $m = 1$, $d = 4$, such a triviality result is known for the critical Φ^4_4 model; see Aizenman-Duminil-Copin '21.

Case 3: If $\alpha_{\text{crit}} < \alpha \leq \frac{2m+1}{2m+2} \cdot \frac{d}{2}$, we expect that **the Gibbs measure exists, but is singular w.r.t.** $\vec{\mu}_\alpha$. See Barashkov-Gubinelli '20, Oh-Okamoto-Tolomeo '24.

Further heuristics

- We expect that $\rho_\alpha := Z^{-1} e^{-\frac{1}{2m+2}\int :u^{2m+2}: dx}\, d\mu_\alpha$ is absolutely continuous with respect to the shifted measure

$$\mathrm{Law}\left(\sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} + (1-\Delta)^{-\alpha} : \left(\sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} \right)^{2m+1} : \right)$$

- The first term corresponds to the typical element in the support of μ_α and has regularity $s_{\mathrm{Gaussian}} := \alpha - \frac{d}{2} - \varepsilon$, $\varepsilon > 0$.

- The shift term has regularity $s_{\mathrm{shift}} := (2m+1)(\alpha - \frac{d}{2}) + 2\alpha - \varepsilon$, $\varepsilon > 0$.

Case 1: If $s_{\mathrm{shift}} > \alpha$, i.e. $\alpha > \frac{2m+1}{2m+2} \cdot \frac{d}{2}$, we expect that ρ_α **is absolutely continuous w.r.t** μ_α.

Case 2: If $s_{\mathrm{shift}} \leq s_{\mathrm{Gaussian}}$, i.e. $\alpha \leq \frac{m}{m+1} \cdot \frac{d}{2} =: \alpha_{\mathrm{crit}}$, we expect **triviality results**. For $\alpha = 1$, $m = 1$, $d = 4$, such a triviality result is known for the critical Φ^4_4 model; see Aizenman-Duminil-Copin '21.

Case 3: If $\alpha_{\mathrm{crit}} < \alpha \leq \frac{2m+1}{2m+2} \cdot \frac{d}{2}$, we expect that **the Gibbs measure exists, but is singular w.r.t.** $\vec{\mu}_\alpha$. See Barashkov-Gubinelli '20, Oh-Okamoto-Tolomeo '24.

Part II. **Almost sure local well-posedness of the renormalized fNLW with Gibbsian initial data**

Probabilistic approach to local well-posedness

- We consider the Gibbsian random initial data:

$$(u_0^\omega, u_1^\omega) := \Bigg(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \Bigg),$$

- (u_0^ω, u_1^ω) are rough, of negative Sobolev regularity:

$$(u_0^\omega, u_1^\omega) \in \mathcal{H}^s(\mathbb{T}^2) \text{ for } s < -(1-\alpha) \text{ almost surely},$$
$$(u_0^\omega, u_1^\omega) \notin \mathcal{H}^{-(1-\alpha)}(\mathbb{T}^2) \text{ almost surely.}$$

- Since $(u_0^\omega, u_1^\omega) \notin \mathcal{H}^0(\mathbb{T}^2) = L^2(\mathbb{T}^2) \times H^{-\alpha}(\mathbb{T}^2)$ a.s., the random linear solution

$$z_1^\omega := \cos(t\langle\nabla\rangle^\alpha) u_0^\omega + \frac{\sin(t\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} u_1^\omega$$

is not in $L^2(\mathbb{T}^2)$ a.s. $\Longrightarrow$ there is an issue with making sense of z_1^{2m+1}
$\Longrightarrow$ there is an issue with making sense of u^{2m+1}!

- This leads us to considering the formal renormalized fNLW

$$\begin{cases} \partial_t^2 u + (1-\Delta)^\alpha u \; + :u^{2m+1}: \; = 0, \\ (u, \partial_t y)|_{t=0} = (u_0^\omega, u_1^\omega). \end{cases}$$

- First order expansion: $u = z_1 + w$, where we expect that w has positive regularity.
- Since Hermite polynomials satisfy $H_k(x+y;\sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(y;\sigma) x^{k-\ell}$:

$$:(z_1+w)^{2m+1}: = \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} :z_1^\ell: w^{2m+1-\ell}.$$

So, w satisfies

$$\begin{cases} \partial_t^2 w + (1-\Delta)^\alpha w \; + \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} :z_1^\ell: w^{2m+1-\ell} = 0 \\ (w, \partial_t w)|_{t=0} = (0,0). \end{cases}$$

- A solution to the above equation is defined using **Duhamel's formula**:

$$w(t) = \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} :z_1^\ell: w^{2m+1-\ell} dt'$$

Theorem (Almost sure local well-posedness, Forcella-P. '23)

For $m = 1$, let $\alpha \in (\frac{5}{6}, 1)$. For $m \geq 2$, let $\alpha \in \left(1 - \frac{1}{2m^2+2m+1}, 1\right)$.

Then the renormalized fNLW admits a.s. a local-in-time solution in $C_t H_x^{-(1-\alpha)-\varepsilon}$, for any $\varepsilon > 0$.

- Strichartz estimates are a crucial tool in nonlinear dispersive PDEs; the first ones were proved by R. Strichartz in 1978
- For $p, q \geq 2$ and $I \subset \mathbb{R}$ interval, the space $L_t^p(I; L_x^q(\mathbb{T}^2))$ is endowed with the norm:

$$\|f\|_{L_t^p(I;L_x^q)} := \big\| \|f(t,x)\|_{L_x^q(\mathbb{T}^2)} \big\|_{L_t^p(I)} = \left(\int_I \left(\int_{\mathbb{T}^2} |f(t,x)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}$$

- Let $p, q \geq 2$ satisfying some admissibility conditions, $\gamma_{p,q} > 0$, and

$$\begin{cases} \partial_t^2 v + (1-\Delta)^\alpha v = F \\ (v, \partial_t v)|_{t=0} = (v_0, v_1), \end{cases} \qquad (t,x) \in I \times \mathbb{T}^2.$$

Proposition (Stricharz estimates, Dinh '17)

Let I a bounded interval, $\alpha \in (0,1)$, (p,q) a fractional admissible pair, $(v_0, v_1) \in H^{\gamma_{p,q}} \times H^{\gamma_{p,q}-\alpha}$. Then

$$\|v\|_{L_t^p(I;L_x^q)} \lesssim \|(v_0, v_1)\|_{\mathcal{H}_x^{\gamma_{p,q}}} + \|F\|_{L_t^1(I;H_x^{\gamma_{p,q}-\alpha})}.$$

- We consider the 'Strichartz space':

$$X^s := L_t^\infty([0,T]; H_x^s) \cap L_t^p([0,T]; W_x^{s-\gamma_{p,q},q})$$

where $s < 1$, (p,q) is an admissible pair, and $\|f\|_{W^{s-\gamma_{p,q},q}} := \|\langle\nabla\rangle^{s-\gamma_{p,q}} f\|_{L^q}$.

- Proof: we perform a fixed point argument for w in X^s

- To estimate the random terms, we use:

$$:z_1^{\ell}: \in C(\mathbb{R}_+; W^{-\ell(1-\alpha)-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\ell \in \mathbb{N}$ and any $\varepsilon > 0$, almost surely.

- The **roughest term** in w is:

$$\int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} :z_1^{2m+1}: dt' =: z_2.$$

- By the Strichartz estimates, we have

$$\begin{aligned}\|z_2\|_{X^s} &\lesssim \| :z^{2m+1}: \|_{L^1_T H^{s-\alpha}_x} \lesssim T \|\langle\nabla\rangle^{s-\alpha} :z^{2m+1}: \|_{L^\infty_T L^\infty_x} \\ &\lesssim T \|\langle\nabla\rangle^{-(2m+1)(1-\alpha)-\varepsilon} :z^{2m+1}: \|_{L^\infty_T L^\infty_x},\end{aligned}$$

provided $s - \alpha < -(2m+1)(1-\alpha) \iff s < \alpha(2m+2) - (2m+1)$.

- From the analysis in other cases, we also have: $s > \max\{1 - \frac{\alpha}{m}, 2m(1-\alpha)\}$

- So, for $m = 1$, we get $2(1-\alpha) < 4\alpha - 3 \iff \alpha > \frac{5}{6}$

- While for $m \geq 2$, we get $1 - \frac{\alpha}{m} < \alpha(2m+2) - (2m+1) \iff \alpha > 1 - \frac{1}{2m^2+2m+1}$

Second order expansion

Theorem (Improved a.s. local well-posedness, Forcella-P. '23)

Let $m \geq 2$ and let $\alpha \in \left(1 - \frac{1}{2m^2+m+1}, 1\right)$.

Then the renormalized fNLW admits a.s. a local-in-time solution in $C_t H_x^{-(1-\alpha)-\varepsilon}$.

- We further expand u as $u = z_1 + z_2 + w_2$, where $z_2 =: \mathcal{I}(:z_1^{2m+1}:)$. In particular, z_2 satisfies:

$$\begin{cases} \partial_t^2 z_2 + (1-\Delta)^\alpha z_2 + :z_1^{2m+1}: = 0 \\ (z_2, \partial_t z_2)|_{t=0} = (0,0), \end{cases}$$

and so w_2 satisfies:

$$\begin{cases} \partial_t^2 w_2 + (1-\Delta)^\alpha w_2 + \sum_{\ell=0}^{2m} \binom{2m+1}{\ell} :z_1^\ell:(z_2 + w_2)^{2m+1-\ell} = 0 \\ (w_2, \partial_t w_2)|_{t=0} = (0,0). \end{cases}$$

- We proceed as for the previous theorem, but this time we need to estimate products of stochastic objects $:z_1^\ell: z_2^{2m+1-\ell}$

Proposition (Oh-Zine '22)

Let $k \in \mathbb{N}$, $0 \le k_1 \le k-1$ and $0 \le k_2 \le k$ integers. Assume that $\alpha > 1 - \frac{1}{2k+1}$. Then,

$$:z_1^{k_1}: \big(\mathcal{I}(:z_1^k:)\big)^{k_2} \in C(\mathbb{R}_+; W^{-k_1(1-\alpha)-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

- The **roughest term** is $:z_1^{2m}: z_2 =: z_1^{2m} : \mathcal{I}(:z_1^{2m+1}:)$, of regularity $-2m(1-\alpha)-\varepsilon$
- as expected, this is more regular than $:z_1^{2m+1}:$ (which has regularity $-(2m+1)(1-\alpha)-\varepsilon$) $\Longrightarrow$ wider range of α
- In Oh-P.-Tzvetkov '22, we considered the cubic NLW on $\mathbb{T}^3$ and so we *only* needed to analyze $:z_1^2: \mathcal{I}(:z_1^3:)$
- Here we have a general nonlinearity, so a more intricate analysis is needed

- The map $(u_0^\omega, u_1^\omega) \longmapsto u$ is *ill-posed* since the initial data are very rough
- We decompose the (ill-posed) solution map into

$$(u_0^\omega, u_1^\omega) \overset{①}{\longmapsto} \underbrace{(z_1, :z_1^2:, \dots, :z_1^{2m}:, z_2)}_{\text{enhanced data set}} \overset{②}{\longmapsto} w_2 \longmapsto \boxed{u = z_1 + z_2 + w_2}$$

1. 1st step involves stochastic analysis, generating an *enhanced data set*
2. 2nd step involves only deterministic analysis. Moreover, the map in the 2nd step is continuous (with appropriate topology on the data set)

Part III. **Extending the local-in-time solutions previously obtained globally-in-time**

- We typically use a **conservation law** to extend globally-in-time local-in-time solutions. However, there is no known conservation law at such low regularities!

- Idea: use the formal invariance of the Gibbs measure for the renormalized equation to construct a.s. global dynamics

- Circular argument?

$$\begin{tikzcd} \text{Invariance} \arrow[r, bend left] & \text{a.s. GWP} \arrow[l, bend left] \end{tikzcd}$$

- Bourgain '94: Extended local-in-time solutions to global ones by

  invariance of ***finite-dimensional*** Gibbs measures

  (in place of conservation laws) and approximation argument

  $\Longrightarrow$ a.s. GWP on the statistical ensemble & invariance of Gibbs measure

- Bourgain's invariant measure argument can also be used here to obtain a.s. global well-posedness of fNLW

Comments

Oh-Thomann '20 proved similar results in the case $\alpha = 1$ of the NLW:

- fNLW is only α-smoothing, while NLW is 1-smoothing
- for fNLW, we have regularity $:z_1^\ell: \sim -\ell(1-\alpha) - \varepsilon$, which leads to a restriction on α or, in other words, if we fix α, on a restriction on the degree $2m+1$ of the nonlinearity we can handle
- in the case of NLW $:z_1^\ell: \sim -\varepsilon$, and Oh-Thomann can handle any nonlinearity

Sun-Tzvetkov-Xu '22: weak universality for a class of nonlinear fractional wave equations (includes the a.s. global well-posedness of the renormalised cubic fNLW)

Oh-Zine '22: Almost sure local well-posedness for fNLW using Sobolev embedding

Further questions

- Can we extend the range of α in our local well-posedness result by using multilinear smoothing and, after that, the paracontrolled approach (Gubinelli-Imkeller-Parkowski '15, Gubinelli-Koch-Oh '18, Oh-Okamaoto-Tolomeo '20, '21, Bringman '20, Bringman-Deng-Nahmod-Yue '22)?
- Can we work with rougher initial data: $\sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle^\beta} e^{in\cdot x}$ with $\beta < \alpha$?
- What can be done in the focusing case? (For the measure construction in the case of focusing NLW: Bourgain '95, Brydges-Slade '96, Oh-Seong-Tolomeo '20)

Part IV. **A counterexample**

So far we considered

$$(u_0^\omega, u_1^\omega) := \bigg(\sum_{n \in \mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \sum_{n \in \mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \bigg),$$

and, for any $j \geq 2$, we defined $:z_1^j:$ as a suitable limit of $\{:z_N^j:\}_N$, where $z_N := \mathbf{P}_N z_1$ and z_1 is the linear solution of the fractional wave equation.

In particular, this also holds for the case $\alpha = 1$ of the wave equation.

Question: Let $s < 0$. Can we still construct $:z^j:$ in the case $\alpha = 1$ for

$$(u_0^\omega, u_1^\omega) := \bigg(\sum_{n \in \mathbb{Z}^d} g_{0,n}(\omega) a_n e^{in\cdot x}, \sum_{n \in \mathbb{Z}^d} g_{1,n}(\omega) b_n e^{in\cdot x} \bigg)$$

with $(u_0, u_1) = \big(\sum_{n \in \mathbb{Z}^d} a_n e^{in\cdot x}, \sum_{n \in \mathbb{Z}^d} b_n e^{in\cdot x} \big) \in \mathcal{H}^s(\mathbb{T}^d) \setminus H^0(\mathbb{T}^d)$?

Answer: **No**, unless we impose additional Fourier-Lebesgue regularity?

So far we considered

$$(u_0^\omega, u_1^\omega) := \left(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \right),$$

and, for any $j \geq 2$, we defined $:z_1^j:$ as a suitable limit of $\{:z_N^j:\}_N$, where $z_N := \mathbf{P}_N z_1$ and z_1 is the linear solution of the fractional wave equation.

In particular, this also holds for the case $\alpha = 1$ of the wave equation.

Question: Let $s < 0$. Can we still construct $:z^j:$ in the case $\alpha = 1$ for

$$(u_0^\omega, u_1^\omega) := \left(\sum_{n\in\mathbb{Z}^d} g_{0,n}(\omega) a_n e^{in\cdot x}, \sum_{n\in\mathbb{Z}^d} g_{1,n}(\omega) b_n e^{in\cdot x} \right)$$

with $(u_0, u_1) = \left(\sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot x}, \sum_{n\in\mathbb{Z}^d} b_n e^{in\cdot x}\right) \in \mathcal{H}^s(\mathbb{T}^d) \setminus H^0(\mathbb{T}^d)$?

Answer: No, unless we impose additional Fourier-Lebesgue regularity?

For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T}^d)$:

$$\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^p(\mathbb{Z}^d)}$$

and we set $\vec{\mathcal{F}L}^{s,p}(\mathbb{T}^d) = \mathcal{F}L^{s,p}(\mathbb{T}^d) \times \mathcal{F}L^{s-1,p}(\mathbb{T}^d)$.

Theorem (Oh-Okamoto-P.-Tzvetkov '23)

Let $s < 0$.

(i) Let $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d) \setminus \mathcal{H}^0(\mathbb{T}^d)$, $j \geq 2$ integer, $\sigma < js$ and $p > 2$ satisfying suitable conditions.

If $(u_0, u_1) \in \vec{\mathcal{F}L}^{s,p}(\mathbb{T}^d)$, then given any $r \geq 1$ and $T > 0$, $\{:z_N^j:\}_N$ converges to a limit $:z^j:$ in $C([0,T]; W^{\sigma,r}(\mathbb{T}^d))$.

(ii) Given an integer $j \geq 2$, there exists

$(u_0, u_1) \in \left(\bigcap_{s<0} \mathcal{H}^s(\mathbb{T}^d)\right) \setminus \vec{\mathcal{F}L}^{0,\frac{2j}{j-1}}(\mathbb{T}^d)$ such that the following holds for any $\sigma \in \mathbb{R}$, almost surely.

Given any $t \in \mathbb{R}$ and any $T > 0$, the truncated Wick power $:z_N^j(t):$ does not converge to any limit in $C([0,T]; H^\sigma(\mathbb{T}^d))$ or as distributions on $\mathbb{T}^d$.