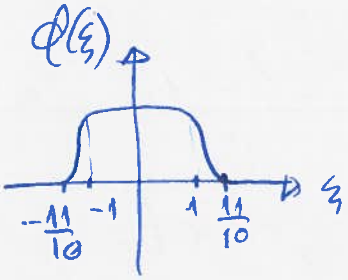
Strichartz estimates for the wave equation

1. Littlewood-Payley theory

Let $\phi(\xi)$ be a radial bump function s.t.

$$\phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1 \\ 0, & \text{if } |\xi| > \frac{11}{10} \end{cases}$$



$\Leftarrow$ for 1D case

Let $\psi(\xi) := \phi(\xi) - \phi(2\xi)$. Then ψ is a bump function supported on $\{\xi;\ \frac{1}{2} \leq |\xi| \leq \frac{11}{10}\}$ and

$$\sum_{k \in \mathbb{Z}} \psi\left(\frac{\xi}{2^k}\right) = 1, \quad \forall \xi \neq 0.$$

We define the Littlewood-Payley projections by

$$\begin{cases} \widehat{P_{\leq N} f}(\xi) := \phi\left(\frac{\xi}{N}\right) \hat{f}(\xi) \\ \widehat{P_{> N} f}(\xi) := \left[1 - \phi\left(\frac{\xi}{N}\right)\right] \hat{f}(\xi) \\ \widehat{P_N f}(\xi) := \psi\left(\frac{\xi}{N}\right) \hat{f}(\xi) \end{cases}$$

On the physical side:

$$P_N f(x) = \mathcal{F}^{-1}\left[\psi\left(\frac{\xi}{N}\right)\hat{f}(\xi)\right](x)$$

$$= \mathcal{F}^{-1}\left[\psi\left(\frac{\xi}{N}\right)\right] * f$$

$$= N^d \left(\mathcal{F}^{-1}\psi\right)(N\cdot) * f$$

$$=: N^d\, m(N\cdot) * f$$

So, P_N is a convolution operator on the physical side.

$\boxed{\text{Theorem}}$ ① For any $f \in L^2$, we have

$$f = \sum_{k\in\mathbb{Z}} P_{2^k} f$$

② (Almost orthogonality) $P_{2^{k_1}} P_{2^{k_2}} = 0$ whenever $|k_1 - k_2| >$

In particular,

$$\|f\|_{L^2}^2 \sim \sum_{k\in\mathbb{Z}} \|P_{2^k} f\|_{L^2}^2, \quad \forall f \in L^2$$

③ (Bernstein's inequalities) For $1 \le p \le q \le \infty$, the following hold:

① $\| |\nabla|^s P_N f\|_{L^p(\mathbb{R}^d)} \sim N^s \|P_N f\|_{L^p(\mathbb{R}^d)}, \forall s \in \mathbb{R}$

② $\| |\nabla|^s P_{\le N} f\|_{L^p(\mathbb{R}^d)} \lesssim N^s \|P_{\le N} f\|_{L^p(\mathbb{R}^d)}, \forall s \ge$

(3) $\|P_N f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p(\mathbb{R}^d)}$

(4) $\|P_{\leq N} f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p(\mathbb{R}^d)}$.

Here, $|D|^s f := \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))$

(4) (Square function estimate) Given $f \in \mathcal{S}$, we define

$$S(f)(x) := \Big(\sum_{k \in \mathbb{Z}} |P_{2^k} f(x)|^2\Big)^{1/2}$$

the correspondind Littlewood-Paley function.

Then,

$$\|S(f)\|_{L^p} \sim \|f\|_{L^p}, \quad \forall\, 1 < p < \infty.$$

(Partial) Proof: (2) $\psi\left(\frac{\xi}{2^{k_1}}\right)$ is supported on $\frac{2^{-k_1}}{2} \leq |\xi| \leq 2^{-k_1}\cdot\frac{11}{10}$

$\psi\left(\frac{\xi}{2^{k_2}}\right)$ is supported on $\frac{2^{-k_2}}{2} \leq |\xi| \leq 2^{-k_2}\cdot\frac{11}{10}$.

Assume, without loss of generality, that $k_1 < k_2$.

If $k_2 - k_1 \geq 2 \Rightarrow 2^{k_2-k_1} \geq 4 > \frac{22}{10}$, which yields $\frac{2^{-k_1}}{2} > \frac{11}{10}\cdot 2^{-k_2}$. Hence, the supports of $\psi\left(\frac{\xi}{2^{k_1}}\right)$ and $\psi\left(\frac{\xi}{2^{k_2}}\right)$ are disjoint and so

$$\psi\left(\frac{\xi}{2^{k_1}}\right)\psi\left(\frac{\xi}{2^{k_2}}\right) = 0, \quad \forall\, \xi \in \mathbb{R}^d.$$

Next, we show that $\exists\, c, C > 0$ s.t.

$$c \cdot \sum_{k \in \mathbb{Z}} \| P_{2^k} f \|_{L^2}^2 \leq \| f \|_{L^2}^2 \leq C \cdot \sum_{k \in \mathbb{Z}} \| P_{2^k} f \|_{L^2}^2 .$$

- $$\| f \|_{L^2}^2 = \Big\| \sum_{k \in \mathbb{Z}} P_{2^k} f \Big\|_{L^2}^2 = \Big\langle \sum_{k \in \mathbb{Z}} P_{2^k} f , \sum_{k' \in \mathbb{Z}} P_{2^{k'}} f \Big\rangle$$

$$= \sum_{|k-k'| \leq 1} \langle P_{2^k} f , P_{2^{k'}} f \rangle$$

$$\overset{C-S}{\leq} \sum_{|k-k'| \leq 1} \| P_{2^k} f \|_{L^2} \| P_{2^{k'}} f \|_{L^2}$$

$$\leq \sum_{|k-k'| \leq 1} \frac{\| P_{2^k} f \|_{L^2}^2 + \| P_{2^{k'}} f \|_{L^2}^2}{2}$$

$$\leq 3 \sum_{k \in \mathbb{Z}} \| P_{2^k} f \|_{L^2}^2 .$$

- $$\sum_{k \in \mathbb{Z}} \| P_{2^k} f \|_{L^2}^2 = \sum_{k \in \mathbb{Z}} \| \widehat{P_{2^k} f} \|_{L^2}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} \Big| \psi\Big(\frac{\xi}{2^k}\Big) \Big|^2 |\hat{f}(\xi)|^2 d\xi$$

$$\leq \sum_{k \in \mathbb{Z}} \int_{2^{k-1} \leq |\xi| \leq 2^k} |\hat{f}(\xi)|^2 d\xi$$

$$\leq 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = 2 \, \| \hat{f} \|_{L^2}^2 = 2 \| f \|_{L^2}^2 .$$

2. Dispersive estimates for the half-wave propagator

For $f, g \in L^1(\mathbb{R}^d)$ we define their convolution by

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy.$$

The following inequality will prove very useful.

Young's inequality: Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.
For any $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ we have

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

We will also be using the asymptotics of certain Bessel functions:

$$J_m(r) := \frac{\left(\frac{r}{2}\right)^m}{\Gamma(m+\frac{1}{2})\, \pi^{1/2}} \int_{-1}^{1} e^{irt} (1-t^2)^{m-\frac{1}{2}}\, dt$$

$$\overset{t=\cos\theta}{=} \frac{\left(\frac{r}{2}\right)^m}{\Gamma(m+\frac{1}{2})\, \pi^{1/2}} \int_0^{\pi} e^{ir\cos\theta} (\sin\theta)^{2m}\, d\theta, \quad m > -\frac{1}{2}$$

Using oscillatory integrals, in particular Van der Corput lemma, one can show that

$$J_m(r) = O(r^{-1/2}) \quad \text{as } r \to \infty.$$

In particular, one has

$$J_{\frac{d-2}{2}}(|x|) = \frac{\left(\frac{|x|}{2}\right)^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-1}{2}\right)\pi^{1/2}} \int_0^{\pi} e^{i|x|\cos\theta} (\sin\theta)^{d-2}\, d\theta$$

$$= \mathcal{O}\left(|x|^{-\frac{1}{2}}\right) \quad \text{as } |x| \to \infty .$$

We will also need the following lemma, whose proof is based on the above asymptotics for $J_{\frac{d-2}{2}}(|x|)$ as $|x| \to \infty$.

$\boxed{\text{Lemma 1}}$ We denote $\check{\sigma}(x) := \int_{S^{d-1}} e^{ix\cdot\omega}\, d\sigma(\omega)$, $d \geq 2$.

Then $\quad |\check{\sigma}(x)| \leq \langle x \rangle^{-\frac{d-1}{2}}$.

Proof: If $|x| \lesssim 1$, then $|\check{\sigma}(x)| \lesssim \int_{S^{d-1}} d\sigma(\omega) \lesssim 1$.

From now on we consider $|x| >> 1$.

We first show that $\check{\sigma}$ is invariant under rotations R (R = orthogonal matrices $R^{\perp}R = RR^{\perp} = I$). Using $Ru \cdot Rv = u \cdot v$, we have

$$\check{\sigma}(Rx) = \int_{S^{d-1}} e^{iRx\cdot\omega}\, d\sigma(\omega) = \int_{S^{d-1}} e^{ix\cdot R^{-1}\omega}\, d\sigma(\omega)$$

$$\underset{\text{of variables}}{\overset{\text{change}}{=}} \int_{S^{d-1}} e^{ix\cdot\tilde{\omega}}\, d\sigma(\tilde{\omega}) = \check{\sigma}(x).$$

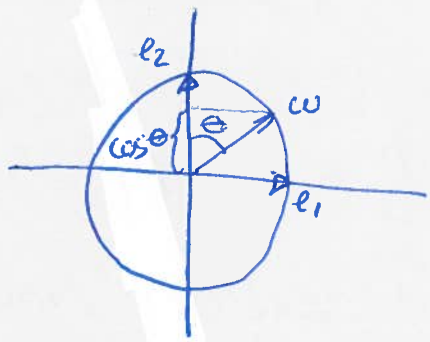
Using the fact that $\check{\sigma}$ is invariant under rotations, we have

$$\check{\sigma}(x) = \check{\sigma}(|x| e_d) = \int_{S^{d-1}} e^{i|x| e_d \cdot \omega} \, d\sigma(\omega)$$

$$\underset{\text{coord.}}{\overset{\text{spherical}}{=}} \int_{S^{d-2}} \int_0^{\pi} e^{i|x|\cos\theta} (\sin\theta)^{d-2} \, d\theta \, d\sigma_{d-2}(\omega)$$

$$\sim J_{\frac{d-2}{2}}(|x|) \cdot |x|^{-\frac{d-2}{2}}$$

$$\lesssim |x|^{-\frac{1}{2}} \cdot |x|^{-\frac{d-2}{2}} \sim |x|^{-\frac{d-1}{2}}, \text{ as } |x| \to \infty$$



2D

$\theta = \measuredangle(\omega, \text{"North Pole"})$

$\boxed{\text{Proposition 1}}$ Let $d \geq 1$ and $N \in 2^{\mathbb{Z}}$ a dyadic number. Then

$$(*) \quad \| e^{\pm it|\nabla|} P_N f \|_{L^\infty} \lesssim (1+|t|N)^{-\frac{d-1}{2}} N^d \| P_N f \|_{L^1}.$$

Proof: It suffices to prove the above inequality for $e^{it|\nabla|}$.

If $d=1$ or $|t| \leq N^{-1}$, we have, by Bernstein's inequalities:

$$\|e^{it|D|} P_N f\|_{L^\infty} \lesssim N^{\frac{d}{2}} \|e^{it|D|} P_N f\|_{L^2}$$
$$= N^{\frac{d}{2}} \|\widehat{P_N f}\|_{L^2}$$
$$= N^{\frac{d}{2}} \|P_N f\|_{L^2}$$
$$\lesssim N^{d} \|P_N f\|_{L^1},$$

which proves (*).

From now on, we assume $d \geq 2$ and $|t| >> N^{-1}$.

Let $\tilde{P}_N = P_{\frac{N}{2}} + P_N + P_{2N}$ be the fattened Littlewood-Paley projection. Note that $\tilde{P}_N P_N = P_N$. Denote by $\tilde{\psi}$ the symbol of the Fourier multiplier $\tilde{P}_1$. Then,

$$e^{it|D|} P_N f = e^{it|D|} \tilde{P}_N P_N f =$$
$$= \mathcal{F}^{-1}\left(e^{it|\xi|} \tilde{\psi}\left(\tfrac{\xi}{N}\right) \widehat{P_N f}(\xi)\right)$$
$$= \mathcal{F}^{-1}\left(e^{it|\xi|} \tilde{\psi}\left(\tfrac{\xi}{N}\right)\right) * P_N f.$$

Then, by Young's inequality, we have

$$\|e^{it|\nabla|}P_N f\|_{L^\infty} = \|\mathcal{F}^{-1}\left(e^{it|\xi|}\tilde{\varphi}\left(\tfrac{\xi}{N}\right)\right) * P_N f\|_{L^\infty}$$

$$\lesssim \|\mathcal{F}^{-1}\left(e^{it|\xi|}\tilde{\varphi}\left(\tfrac{\xi}{N}\right)\right)\|_{L^\infty} \cdot \|P_N f\|_{L^1}.$$

From this, it follows that (*) follows once we prove that

$$\|\mathcal{F}^{-1}\left(e^{it|\xi|}\tilde{\varphi}\left(\tfrac{\xi}{N}\right)\right)\|_{L^\infty} \lesssim N^{\frac{d+1}{2}}|t|^{-\frac{d-1}{2}}.$$

We set

$$I := \mathcal{F}^{-1}\left(e^{it|\xi|}\tilde{\varphi}\left(\tfrac{\xi}{N}\right)\right) = \int_{\mathbb{R}^d} e^{ix\cdot\xi + it|\xi|}\tilde{\varphi}\left(\tfrac{\xi}{N}\right)d\xi$$

$$\overset{\eta=\frac{\xi}{N}}{=} N^d \int_{\mathbb{R}^d} e^{ix\cdot N\eta + itN|\eta|}\tilde{\varphi}(\eta)\,d\eta$$

$$= N^d \int_0^\infty \int_{S^{d-1}} e^{i(Nr x\cdot\omega + tNr)}\tilde{\varphi}(r)\,r^{d-1}\,d\sigma(\omega)\,dr$$

$$= N^d \int_0^\infty e^{itNr}\tilde{\varphi}(r)\,r^{d-1}\,\check{\sigma}(Nrx)\,dr$$

<u>Case $|x| \ll |t|$</u>: We set $\phi(r) := Nr\,x\cdot\omega + Nrt$, and note that $|\phi'(r)| = N|x\cdot\omega + t| \gtrsim N|t|$. In particular, $\phi'(r) \neq 0$, $\forall r$.

Then, integrating by parts κ times, we have

$$|I| = \left| N^d \int_0^\infty \int_{S^{d-1}} e^{i\phi(r)} \tilde{\varphi}(r) r^{d-1} \, d\sigma(\omega) \, dr \right|$$

$$= N^d \left| \int_0^\infty \int_{S^{d-1}} \frac{\partial_r (e^{i\phi(r)})}{i\phi''(r)} \tilde{\varphi}(r) r^{d-1} \, d\sigma(\omega) dr \right|$$

$$\lesssim N^d \cdot (N|t|)^{-\kappa}$$

- If $d =$ odd, take $\kappa = \frac{d-1}{2}$ to obtain the upper bound

$$N^d \cdot (N|t|)^{-\frac{d-1}{2}} = N^{\frac{d+1}{2}} |t|^{-\frac{d-1}{2}}.$$

- If $d =$ even, take $\kappa = \frac{d}{2}$, and then

$$N^d \cdot (N|t|)^{-\frac{d}{2}} = N^{\frac{d}{2}} |t|^{-\frac{d}{2}}$$
$$= N^{\frac{d+1}{2}} |t|^{-\frac{d-1}{2}} (N|t|)^{-\frac{1}{2}}$$
$$\ll N^{\frac{d+1}{2}} |t|^{-\frac{d}{2}} \quad \text{since } N|t| \gg 1$$

Case $|x| \gtrsim |t|$

By the previous lemma, we have

$$|I| = N^d \left| \int_0^\infty e^{itNr} \tilde{\varphi}(r) r^{d-1} \check{\sigma}(Nrx) dr \right|$$

$$\lesssim N^d \int_0^\infty \tilde{\varphi}(r) r^{d-1} (Nr|x|)^{-\frac{d-1}{2}} dr$$

$$\lesssim N^{\frac{d+1}{2}} |x|^{-\frac{d-1}{2}} \lesssim N^{\frac{d+1}{2}} |t|^{-\frac{d-1}{2}}.$$

$\boxtimes$

Next, we will apply the Riesz-Thorin interpolation theorem below to obtain a larger family of dispersive estimates.

Riesz-Thorin interpolation theorem

Let $T: L^{q_j} \to L^{p_j}$ be a linear operator, $j=1,2$, satisfying

$$\|Tf\|_{L^{p_j}} \le A_j \|f\|_{L^{q_j}}, \quad j=1,2.$$

Let p, q be such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for some $0 \le \theta \le 1$. Then, $\|Tf\|_{L^p} \le A_1^\theta A_2^{1-\theta} \|f\|_{L^q}$.

$\boxed{\text{Proposition 2}}$ Let $d \geq 1$ and $N \in 2^{\mathbb{Z}}$ a dyadic number. For all $2 \leq p \leq \infty$, the following holds

$$\| e^{\pm it|D|} P_N f \|_{L^p} \lesssim (1+|t|N)^{-\frac{(d-1)(p-2)}{2p}} N^{\frac{d(p-2)}{2}} \| P_N f \|_{L^{p'}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: Note that

$$\| e^{\pm it|D|} P_N f \|_{L^2} = \| e^{\pm it|\xi|} \widehat{P_N f}(\xi) \|_{L^2}$$
$$= \| \widehat{P_N f} \|_{L^2} = \| P_N f \|_{L^2}.$$

We interpolate this with Proposition 1, using Riesz-Thorin theorem, and the estimate in Proposition 2 follows.

3. Strichartz estimates for the wave equation

Def: We say that (q,r) is a wave-admissible pair if $q, r, d \geq 2$, $(q,r,d) \neq (2,\infty,3)$ and

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}.$$

$\boxed{\text{Theorem 1}}$ (Strichartz estimates for the wave eqn.)

Let (q,r) and $(\tilde{q},\tilde{r})$ be wave-admissible pairs such that $r,\tilde{r}<\infty$ and

$$\text{(H1)} \qquad \frac{1}{q}+\frac{d}{r}=\frac{d}{2}-\gamma$$

$$\text{(H2)} \qquad \frac{1}{\tilde{q}'}+\frac{d}{\tilde{r}'}-2=\frac{d}{2}-\gamma$$

for some $\gamma>0$, and where $\frac{1}{\tilde{q}}+\frac{1}{\tilde{q}'}=1=\frac{1}{\tilde{r}}+\frac{1}{\tilde{r}'}$.

Assume u solves

$$\begin{cases} \partial_t^2 u-\Delta u=F \\ u(0)=u_0 \\ \partial_t u(0)=u_1 \end{cases} \qquad \text{on } I\times\mathbb{R}^d.$$

Then

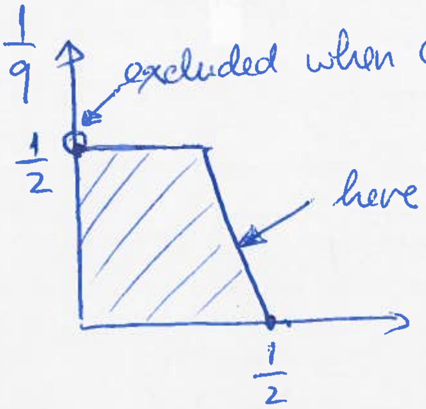
$$\|u\|_{L^q_t L^r_x}\lesssim \|u_0\|_{\dot{H}^\gamma}+\|u_1\|_{\dot{H}^{\gamma-1}}+\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Here, we set

$$\|u\|_{L^q_t L^r_x}:=\Big\|\|u(t,\cdot)\|_{L^r_x}\Big\|_{L^q_t}$$

$$=\left(\int_I\left(\int_{\mathbb{R}^d}|u(t,x)|^r\,dx\right)^{\frac{q}{r}}dt\right)^{\frac{1}{q}}.$$

If (q,r) is a wave-admissible pair, then $\left(\frac{1}{q},\frac{1}{r}\right)$ lie in the following shaded region:



here $\frac{1}{q}+\frac{d-1}{2r}=\frac{d-1}{4}$, pairs satisfying this are called sharp-admissible.

The hypotheses (H1) and (H2) are imposed upon us by dimension counting. We justify (H1) below.

Assume $F\equiv 0$, that is u satisfies

$$\begin{cases}\partial_t^2 u-\Delta u=0\\ u(0)=u_0\\ \partial_t u(0)=u_1.\end{cases}$$

Set $u_\lambda(t,x):=u\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)$. Then,

$$\begin{cases}\partial_t^2(u_\lambda)-\Delta u_\lambda=\frac{1}{\lambda^2}\left(\partial_t^2 u-\Delta u\right)\left(\frac{t}{\lambda},\frac{x}{\lambda}\right)=0\\ u_\lambda(0,x)=u\left(0,\frac{x}{\lambda}\right)=u_0\left(\frac{x}{\lambda}\right)\\ \partial_t u_\lambda(0,x)=\frac{1}{\lambda}(\partial_t u)\left(0,\frac{x}{\lambda}\right)=\frac{1}{\lambda}u_1\left(\frac{x}{\lambda}\right)\end{cases}$$

From the Strichartz estimates applied to u_λ, it follows that:

$$\|u_\lambda\|_{L^q_t L^r_x} \lesssim \|u_\lambda(0)\|_{\dot H^\sigma} + \|\partial_t u_\lambda(0)\|_{\dot H^{\sigma-1}}$$

$$\Longleftrightarrow \|u(\tfrac{t}{\lambda},\tfrac{x}{\lambda})\|_{L^q_t L^r_x} \lesssim \|u_0(\tfrac{x}{\lambda})\|_{\dot H^\sigma} + \|\tfrac{1}{\lambda} u_1(\tfrac{x}{\lambda})\|_{\dot H^{\sigma-1}}$$

$$\Longleftrightarrow \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d} |u(\tfrac{t}{\lambda},\tfrac{x}{\lambda})|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_{\mathbb{R}^d} \left||D|^\sigma \left[u_0(\tfrac{x}{\lambda})\right]\right|^2 dx\right)^{1/2}$$

$$+ \left(\int_{\mathbb{R}^d} \left||D|^{\sigma-1} \left[\tfrac{1}{\lambda} u_1(\tfrac{x}{\lambda})\right]\right|^2 dx\right)^{1/2}$$

With the changes of variables $t' = \frac{t}{\lambda}$, $x' = \frac{x}{\lambda}$, we obtain:

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d} |u(t',x')|^r \lambda^d dx\right)^{\frac{q}{r}} \lambda dt\right)^{\frac{1}{q}}$$

$$\lesssim \frac{1}{\lambda^\sigma}\left(\int_{\mathbb{R}^d} \left|(|D|^\sigma u_0)(x')\right|^2 \lambda^d dx'\right)^{1/2}$$

$$+ \frac{1}{\lambda^{\sigma-1}}\frac{1}{\lambda}\left(\int_{\mathbb{R}^d} \left|(|D|^{\sigma-1} u_1)(x')\right|^2 \lambda^d dx'\right)^{1/2}$$

$$\iff \lambda^{\frac{d}{r}+\frac{1}{q}} \|u\|_{L_t^q L_x^r} \lesssim \lambda^{\frac{d}{2}-\sigma} \left(\|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{\dot{H}^{\sigma-1}} \right)$$

$$\iff \lambda^{\frac{1}{q}+\frac{d}{r}-\frac{d}{2}+\sigma} \|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{\dot{H}^{\sigma-1}}.$$

If $\frac{1}{q}+\frac{d}{r}-\frac{d}{2}+\sigma>0$, we get a contradiction by taking $\lambda\to\infty$.

If $\frac{1}{q}+\frac{d}{r}-\frac{d}{2}+\sigma<0$, we get a contradiction by taking $\lambda\to 0_+$.

Therefore, necessarily $\frac{1}{q}+\frac{d}{r}-\frac{d}{2}+\sigma=0$. $\square$

We will need several building blocks in order to get to the proof of Theorem 1.

A first useful tool is the so-called TT^* argument.

$\boxed{\text{Lemma 2}}$ (TT^* argument) Let H be a Hilbert space, B a Banach space, and $T: H\to B$ a linear operator.

Then, the following statements are equivalent:

(i) $T: H \to B$ is continuous

(ii) $T^*: B' \to H$ is continuous

(iii) $TT^*: B' \to B$ is continuous.

Moreover, $\|TT^*\| = \|T\|^2 = \|T^*\|^2$.

<u>Proof</u>: (i) $\Longrightarrow$ (ii)

$$\|T^* g\|_H = \sup_{\|f\|_H \leq 1} |\langle T^* g, f\rangle| = \sup_{\|f\|_H \leq 1} |{}_{B'}\langle g, Tf\rangle_B|$$

$$\overset{C-S}{\leq} \sup_{\|f\|_H \leq 1} \|g\|_{B'} \|Tf\|_B \overset{(i)}{\leq} \sup_{\|f\|_H \leq 1} \|g\|_{B'} \|T\| \, \|f\|_H$$

$$\leq \|T\| \cdot \|g\|_H$$

and since $\|T\| < \infty$, this proves T^* is continuou[s]

(ii) $\Rightarrow$ (i) is similar to the above argument

So, (i) and (ii) are equivalent and they clearly yield (iii).

We are left with the implication (iii) $\Longrightarrow$ (ii)

$$\|T^* f\|_H^2 = \langle T^* f, T^* f\rangle = {}_{B'}\langle f, TT^* f\rangle_B$$

$$\overset{C-S}{\leq} \|f\|_{B'} \|TT^* f\|_B \overset{(iii)}{\leq} \|TT^*\| \, \|f\|_{B'}^2 .$$

A second useful tool is the following:

Hardy–Littlewood–Sobolev inequality

Let $1 < p, q, r < \infty$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.
Then, for any $f \in L^q(\mathbb{R}^d)$, we have

$$\left\| \frac{1}{|x|^{\frac{d}{p}}} * f \right\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^q(\mathbb{R}^d)}.$$

(The point here is that $\frac{1}{|x|^{\frac{d}{p}}} \notin L^p(\mathbb{R}^d)$, so we cannot use Young's inequality.)

$\boxed{\text{Proposition 3}}$ (Frequency-localized Strichartz estimates for the half-wave propagator)

Let $d \geq 2$ and (q, r) be a wave-admissible pair such that $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma$ for some $\gamma > 0$.
Then:

① $\|e^{\pm it|\nabla|} P_N f\|_{L^q_t L^r_x} \lesssim N^\gamma \|P_N f\|_{L^2}$

② $\left\| \int_{\mathbb{R}} e^{\pm it|\nabla|} P_N F(t,x)\,dt \right\|_{L^2_x} \lesssim N^\gamma \|P_N F\|_{L^{q'}_t L^{r'}_x}$,

where $\frac{1}{q} + \frac{1}{q'} = 1 = \frac{1}{r} + \frac{1}{r'}$.

③ Moreover, if $(\tilde{q},\tilde{r})$ is another wave-admissible pair, then

$$\Big\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \Big\|_{L^q_t L^r_x} \lesssim N^{d-\frac{1}{2}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|P_N F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Proof: We consider the linear operator

$$T: L^2 \longrightarrow L^q_t L^r_x$$

$$T(g) = e^{\pm it|D|} P_N g$$

Next, we determine its adjoint $T^*: L^{q'}_t L^{r'}_x \longrightarrow L^2_x$

$$(g, T^* f)_{L^2} = (Tg, f)_{L^q_t L^r_x} \quad (Tg, f)_{L^{q'}_t L^{r'}_x} = \iint_{\mathbb{R}\,\mathbb{R}^d} (Tg)(s,x)\, \overline{f(s,x)}\, ds\,dx$$

$$= \iint_{\mathbb{R}\,\mathbb{R}^d} e^{\pm is|D|} P_N g\, \overline{f(s,x)}\, ds\,dx$$

$$= \int_{\mathbb{R}} \left(e^{\pm is|D|} P_N g, f(s) \right)_{L^2_x} ds$$

$$= \int_{\mathbb{R}} \left(e^{\pm is|\xi|} \psi\!\left(\tfrac{\xi}{N}\right) \hat{g}(\xi), \hat{f}(s,\xi) \right)_{L^2_\xi} ds$$

$$= \int_{\mathbb{R}} \left(\hat{g}(\xi), e^{\mp is|\xi|} \psi\!\left(\tfrac{\xi}{N}\right) \hat{f}(s,\xi) \right)_{L^2_\xi} ds$$

$$= \int_{\mathbb{R}} (g, e^{\mp is|D|} P_N f(s,\cdot))_{L^2_x} ds$$

$$= \left(g, \int_{\mathbb{R}} e^{\mp is|D|} P_N f(s,\cdot) ds\right)$$

Thus, $T^* f = \int_{\mathbb{R}} e^{\mp is|D|} P_N f(s,\cdot) ds$.

Note that ① is equivalent to showing that T is continuous with $\|T\| = N^\sigma$, while ② is equivalent to T^* continuous with $\|T^*\| =$ By the TT^* argument, it follows that ① and ② are equivalent and to prove them it's sufficient to show that TT^* is continuous and $\|TT^*\| = N^{2\sigma}$.

More precisely, we will show in the following that

$$\|TT^* F\|_{L^q_t L^r_x} = \left\| \int_{\mathbb{R}} e^{\pm i(t-s)|D|} P_N F(s,x) ds \right\|_{L^q_t L^r_x}$$

$$\lesssim N^{2\sigma} \|P_N F\|_{L^{q'}_t L^{r'}_x}.$$

By the dispersive estimates in Proposition 2, we have:

$$\Big\| \int_{\mathbb{R}} e^{\pm i(t-s)|D|} P_N F(s,x)\, ds \Big\|_{L^q_t L^r_x} =$$

$$\leq \Big\| \int_{\mathbb{R}} \| e^{\pm i(t-s)|D|} P_N F(s,x) \|_{L^r_x}\, ds \Big\|_{L^q_t}$$

$$\lesssim \Big\| \int_{\mathbb{R}} (1+|t-s|N)^{-\frac{(d-1)(r-2)}{2r}} N^{\frac{d(r-2)}{r}} \| P_N F(s,\cdot)\|_{L^{r'}_x}\, ds \Big\|_{L^q_t}$$

$$\lesssim N^{\frac{d(r-2)}{r}} \Big\| (1+|t|N)^{-\frac{(d-1)(r-2)}{2r}} * \|P_N F\|_{L^{r'}_x} \Big\|_{L^q_t}$$

$\boxed{\text{Case 1}}$: $\frac{1}{q} + \frac{d-1}{2r} < \frac{d-1}{4}$

In this case, we note that $\frac{(d-1)(r-2)}{2r} \cdot \frac{q}{2} > 1$.

Then, by Young's inequality, we have (with $\frac{1}{q}+1=\frac{2}{q}+$

$$\lesssim N^{\frac{d(r-2)}{r}} \Big\| (1+|t|N)^{-\frac{(d-1)(r-2)}{2r}} \Big\|_{L^{q/2}_t} \Big\| \|P_N F\|_{L^{r'}_x} \Big\|_{L^{q'}_t}$$

$$\overset{t'=tN}{\lesssim} N^{\frac{d(r-2)}{r} - \frac{2}{q}} \Big(\int_{\mathbb{R}} (1+|t'|)^{-\underbrace{\frac{(d-1)(r-2)}{2r}\cdot\frac{q}{2}}_{>1}} dt' \Big)^{\frac{2}{q}} \cdot \|P_N F\|_{L^{q'}_t L^{r'}_x}$$

$$\lesssim N^{\frac{d(r-2)}{r} - \frac{2}{q}} \|P_N F\|_{L^{q'}_t L^{r'}_x} = N^{2\delta} \|P_N F\|_{L^{q'}_t L^{r'}_x}$$

$\boxed{\text{Case 2}}$: $\frac{1}{q}+\frac{d-1}{2r}=\frac{d-1}{4}$

In this case, note that $\frac{(d-1)(r-2)}{2r}=\frac{2}{q}$. Then, using Hardy-Littlewood-Sobolev (with $\frac{1}{q}+1=\frac{2}{q}+\frac{1}{q'}$)

$$\le N^{\frac{d(r-2)}{r}}\left\| (1+|t|N)^{-\frac{2}{q}} * \|P_N F\|_{L_x^{r'}}\right\|_{L_t^q}$$

$$\le N^{\frac{d(r-2)}{r}-\frac{2}{q}}\|P_N F\|_{L_t^{q'}L_x^{r'}} = N^{2\delta}\|P_N F\|_{L_t^{q'}L_x^{r'}}$$

By Cases 1 and 2, we obtain

$$\|TT^*F\|_{L_t^q L_x^{r'}} \lesssim N^{2\delta}\|P_N F\|_{L_t^{q'}L_x^{r'}},$$

which proves ① and ②.

Next, we prove ③.

(a) We first prove ③ with $(\tilde{q},\tilde{r})=(q,r)$.

$$\text{LHS ③} = \left\| \int_{\mathbb{R}} e^{\pm i(t-s)|\nabla|} \mathbb{1}_{[0,t]}(s) P_N F(s,x)\, ds \right\|_{L_t^q L_x^r}$$

$$\le \left\| \int_{\mathbb{R}} \| e^{\pm i(t-s)|\nabla|} \mathbb{1}_{[0,t]}(s) P_N F(s,x)\|_{L_x^r}\, ds \right\|_{L_t^q}$$

$$\lesssim \left\| \int_{\mathbb{R}} (1+|t-s|\cdot N)^{-\frac{(d-1)(r-2)}{2r}} N^{\frac{d(r-2)}{2}} \|\mathbb{1}_{[0,t]}(s) P_N F(s, \cdot)\|_{L_x^{r'}} \right.$$

$$\lesssim N^{\frac{d(r-2)}{2}} \left\| (1+|t|N)^{-\frac{(d-1)(r-2)}{2r}} * \|P_N F(s,\cdot)\|_{L^{r'}_x} \right\|_{L^{q}_t}$$

$$\lesssim N^{d-\frac{2}{q}-\frac{2d}{r}} \|P_N F\|_{L^{q'}_t L^{r'}_x},$$

where we used the dispersive estimate in Proposition 2 and argued as in the proof of ① and ② (Cases 1 and 2).

(b) Next, we show that

$$\left\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \right\|_{L^\infty_t L^2_x} \lesssim N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{r}} \|P_N F\|_{L^{q'}_t L^{r'}_x}$$

$$\left\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \right\|^2_{L^\infty_t L^2_x} =$$

$$= \sup_t \left\langle \int_0^t e^{\pm i(t-s_1)|D|} P_N F(s_1)\,ds_1 , \int_0^t e^{\pm i(t-s_2)|D|} P_N F(s_2)\,ds_2 \right\rangle_{L^2_x}$$

$$\leq \sup_t \int_0^t \left| \left\langle P_N F(s_1), \int_0^t e^{\pm i(s_1-s_2)|D|} P_N F(s_2)\,ds_2 \right\rangle_{L^2_x} \right| ds_1$$

$$\underset{\text{in } x}{\overset{\text{Hölder}}{\leq}} \sup_t \int_{\mathbb{R}} \|P_N F(s_1)\|_{L^{r'}_x} \left\| \int_0^t e^{\pm i(s_1-s_2)|D|} P_N F(s_2)\,ds_2 \right\|_{L^r_x} ds_1$$

$$\underset{\text{in } s_1}{\overset{\text{Hölder}}{\leq}} \|P_N F\|_{L_t^{q'} L_x^{r'}} \sup_t \Big\| \int_0^t e^{\pm i(s_1-s_2)|D|} P_N F(s_2)\,ds_2 \Big\|_{L_{s_1}^q L_x^r}$$

$$= \|P_N F\|_{L_t^{q'} L_x^{r'}} \sup_t \Big\| e^{\pm i(s_1-t)|D|} \int_0^t e^{\pm i(t-s_2)|D|} P_N F(s_2)\,ds_2 \Big\|_{L_{s_1}^q L_x^r}$$

$$\lesssim N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{r}} \|P_N F\|_{L_t^{q'} L_x^{r'}} \sup_t \Big\| e^{\mp it|D|} \int_0^t e^{\pm i(t-s_2)|D|} P_N F(s_2)\,ds_2 \Big\|_{L_x^2}$$

$$= N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{r}} \|P_N F\|_{L_t^{q'} L_x^{r'}} \cdot \Big\| \int_0^t e^{\pm i(t-s_2)|D|} P_N F(s_2)\,ds_2 \Big\|_{L_t^\infty L_x^2}$$

This proves the estimate in (b).

Let us now fix $(\tilde{q}, \tilde{r})$ a wave-admissible pair. Then, by (a) and (b) we've shown:

$$\Big\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \Big\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim N^{d-\frac{2}{\tilde{q}}-\frac{2d}{\tilde{r}}} \|P_N \tilde{F}\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\Big\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \Big\|_{L_t^\infty L_x^2} \lesssim N^{\frac{d}{2}-\frac{1}{\tilde{q}}-\frac{d}{\tilde{r}}} \|P_N \tilde{F}\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

By Riesz-Thorin theorem, we obtain

$$(***) \quad \Big\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,x)\,ds \Big\|_{L_t^q L_x^r} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|P_N F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

for any $q \geq \tilde{q}$ and (q,r) wave-admissible

For (q,r) wave-admissible with $q<\tilde{q}$, we have by an argument analogous to the one we used to prove $(***)$:

$$\left\| \int_{-\infty}^{t} e^{\pm i(t-s)|\nabla|} P_N F(s,x)\,ds \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|F\|_{L_t^{q'} L_x^{r'}}$$

This is equivalent, by duality, to

$$\int_{\mathbb{R}} \left\langle \int_{-\infty}^{t} e^{\pm i(t-s)|\nabla|} P_N F(s,x)\,ds,\ G(t,x) \right\rangle_{L_x^2} dt$$

$$\lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

The RHS in the above estimate can be rewritten as:

$$\int_{\mathbb{R}} \left\langle F(s,x),\ \int_{s}^{\infty} e^{\mp i(t-s)|\nabla|} P_N G(t,x)\,dt \right\rangle_{L_x^2} ds$$

$$\lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

This shows that

$$\left\| \int_{s}^{\infty} e^{\mp i(t-s)|\nabla|} P_N G(t,x)\,dt \right\|_{L_s^{q} L_x^{r}} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

On the other hand, by ① and ②:

$$\Big\| \int_{\mathbb{R}} e^{\mp i(t-s)|D|} P_N G(t,x)\,dt \Big\|_{L^q_s L^r_x} \lesssim N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{r}} \Big\| \int_{\mathbb{R}} e^{\mp it|D|} P_N G(t,x)\,dt \Big\|$$

$$\overset{②}{\lesssim} N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \| G \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

From, the last two estimates we deduce

$$\Big\| \int_{-\infty}^{s} e^{\pm i(s-t)|D|} P_N G(t,x)\,dt \Big\|_{L^q_s L^r_x} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \| G \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

Applying this to $G \mapsto \mathbb{1}_{[0,\infty)}(t) \cdot G(t,x)$, we get

$$\Big\| \int_{0}^{s} e^{\pm i(s-t)|D|} P_N G(t,x)\,dt \Big\|_{L^q_s L^r_x} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}} \| G \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

which proves ③ for $q < \tilde{q}$.

This concludes the proof of Proposition 3.

$\boxtimes$

$\boxed{\text{Corollary}}$ (Strichartz estimates for the half-wave propagator). Under the same hypotheses as in Proposition 3, we have:

1′ $$\| e^{\pm it|D|} f \|_{L_t^q L_x^r} \lesssim \| |D|^\sigma f \|_{L_x^2}$$

2′ $$\Big\| \int_{\mathbb{R}} e^{\mp it|D|} F(t,\cdot)\, dt \Big\|_{L_x^2} \lesssim \| |D|^\sigma F \|_{L_t^{q'} L_x^{r'}}$$

3′ $$\Big\| \int_0^t e^{\pm i(t-s)|D|} F(s,\cdot)\, ds \Big\|_{L_t^q L_x^r} \lesssim \| |D|^{d-\frac{1}{q}-\frac{1}{\tilde q}-\frac{d}{r}-\frac{d}{\tilde r}} F \|_{L_t^{\tilde q'} L}$$

Proof: As in the proof of Prop. 3, we have 1′ $\Longleftrightarrow$ 2′.

To prove 1′ it suffices to prove

α $$\| F \|_{L_t^q L_x^r} \lesssim \Big(\sum_{N \in 2^{\mathbb{Z}}} \| P_N F \|^2_{L_t^q L_x^r} \Big)^{1/2},$$

$$\forall\, 2 \le q \le \infty,\ 2 \le r < \infty.$$

Indeed, by 1 in Prop. 3 and almost orthogonality

$$\| e^{\pm it|D|} f \|_{L_t^q L_x^r} \overset{\alpha}{\lesssim} \Big(\sum_N \| e^{\pm it|D|} P_N f \|^2_{L_t^q L_x^r} \Big)^{1/2}$$

$$\lesssim \Big(\sum_N \| |D|^\sigma P_N f \|^2_{L^2} \Big)^{1/2}$$

$$\sim \| |D|^\sigma f \|_{L^2}.$$

In the following, we prove α.

By the square function estimate and Minkowski's inequality, for $r \geqslant 2$ we have

$$\|f\|_{L^r} \sim \Big\| \Big(\sum_N |P_N f|^2 \Big)^{1/2} \Big\|_{L^r}$$

$$\lesssim \Big(\sum_N \|P_N f\|_{L^r}^2 \Big)^{1/2}$$

Using this, we get for $q \geqslant 2$

$$\|F\|_{L^q_t L^r_x} \lesssim \Big\| \Big(\sum_N \|P_N f\|_{L^r_x}^2 \Big)^{1/2} \Big\|_{L^q_t}$$

$$\overset{\text{Minkowski}}{\lesssim} \Big(\sum_N \|P_N F\|_{L^q_t L^r_x}^2 \Big)^{1/2},$$

which proves α. Hence $1' \Longleftrightarrow 2'$ are proved and we are left with $3'$. To prove $3'$, it's sufficient to show that.

$$\beta \quad \Big(\sum_N \|P_N F\|_{L^{q'}_t L^{r'}_x}^2 \Big)^{1/2} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}.$$

Indeed by ③ in Prop. 3, α, and β:

$$\Big\| \int_0^t e^{\pm i(t-s)|D|} F(s,\cdot)\, ds \Big\|_{L^q_t L^r_x} \overset{\alpha}{\lesssim} \Big(\sum_N \Big\| \int_0^t e^{\pm i(t-s)|D|} P_N F(s,\cdot)\, ds \Big\|_{L^q_t L^r_x}^2 \Big)^{1/2}$$

$$\overset{③\,\text{Prop. 3}}{\lesssim} \Big(\sum_N \Big\| |D|^{d - \frac{1}{q} - \frac{1}{\tilde{q}'} - \frac{d}{r} - \frac{d}{\tilde{r}}} P_N F \Big\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}^2 \Big)^{1/2}$$

$$\overset{(\beta)}{\lesssim} \, \| |\nabla|^{d-\frac{1}{2}-\frac{1}{\tilde{2}}-\frac{d}{r}-\frac{d}{\tilde{r}}} F \|_{L_t^{\tilde{2}'} L_x^{\tilde{r}'}}.$$

It remains to prove (β). We will show, using duality, that $(\beta) \Leftrightarrow (\alpha)$ and, since we already proved (α), this will prove (β).

We define

$$T: L_t^{q'} L_x^{r'} \longrightarrow \ell^2 (L_t^{q'} L_x^{r'})$$

$$T(F) = \{ P_N F \}_{N \in 2^{\mathbb{Z}}}$$

$(\beta) \Leftrightarrow T$ bounded

The adjoint of T is

$$T^*: \ell^2 (L_t^2 L_x^r) \longrightarrow L_t^2 L_x^r$$

$$T^* (\{ G_N \}_{N \in 2^{\mathbb{Z}}}) = \sum_N{}' P_N G_N$$

T^* bounded $\Leftrightarrow$ (γ) $\| \sum_N{}' P_N G_N \|_{L_t^2 L_x^r} \le \left(\sum_N{}' \| G_N \|^2_{L_t^2 L_x^r} \right)^{1/2}$

So far, we have:

$(\beta) \Leftrightarrow T$ bounded $\Leftrightarrow T^*$ bounded $\Leftrightarrow (\gamma)$

Next we show that $(\alpha) \Rightarrow (\gamma)$, which yield $(\alpha) \Rightarrow (\beta)$

$$\| \sum_N{}' P_N G_N \|_{L_t^2 L_x^r} \overset{(\alpha)}{\lesssim} \left(\sum_N{}' \| P_N \sum_M P_M G_M \|^2_{L_t^2 L_x^r} \right)^{1/2}$$

$$= \Big(\sum_N \Big\| \sum_{M \sim N} P_N P_M G_M \Big\|^2_{L^2_t L^r_x} \Big)^{1/2}$$

$$\lesssim \Big(\sum_N \| G_N \|^2_{L^2_t L^r_x} \Big)^{1/2},$$

where we use the boundedness of P_N and the fact that the number of M such that $M \sim N$ is an absolute constant (independent of N).

Finally, to prove $\beta \Rightarrow \alpha$, it's enough to show that $\gamma \Rightarrow \alpha$. Writing $F = \sum_N P_N F = \sum_N P_N \tilde{P}_N F$ and applying γ with $G_N = \tilde{P}_N F$

$$\|F\|_{L^2_t L^r_x} = \Big\| \sum_N P_N (\tilde{P}_N F) \Big\|_{L^2_t L^r_x}$$

$$\lesssim \Big(\sum_N \| \tilde{P}_N F \|^2_{L^2_t L^r_x} \Big)^{1/2}$$

$$\sim \Big(\sum_N \| P_N F \|^2_{L^2_t L^r_x} \Big)^{1/2}.$$

This completes the proof of the corollary. $\boxtimes$

Proof of Theorem 1:

By Duhamel's formula, we have

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1 + \int_0^t \frac{\sin((t-s)|D|)}{|D|}F(s)\,ds$$

$$= \frac{e^{it|D|}+e^{-it|D|}}{2}u_0 + \frac{e^{it|D|}-e^{-it|D|}}{2i}u_1$$

$$+ \int_0^t \frac{e^{i(t-s)|D|}-e^{-i(t-s)|D|}}{2i|D|}F(s)\,ds$$

Then, by the Corollary:

$$\|u\|_{L^q_t L^r_x} \lesssim \|u_0\|_{\dot H^\gamma} + \|u_1\|_{\dot H^{\gamma-1}} + \left\| |D|^{d-\frac{1}{2}-\frac{1}{\tilde q}-\frac{d}{r}-\frac{d}{\tilde r}-1} F \right\|_{L^{\tilde q'}_t L}$$

By (H1): $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma$

By (H2): $\frac{1}{\tilde q'} + \frac{d}{\tilde r'} - 2 = \frac{d}{2} - \gamma \iff$

$$1 - \frac{1}{\tilde q} + d - \frac{d}{\tilde r} - 2 = \frac{d}{2} - \gamma \iff$$

$$\frac{1}{\tilde q} + \frac{d}{\tilde r} = -1 + \frac{d}{2} + \gamma$$

Thus (H1) + (H2) $\implies d - \frac{1}{2} - \frac{1}{\tilde q} - \frac{d}{r} - \frac{d}{\tilde r} - 1 = 0$, and this proves Thm 1. $\square$