On the probabilistic well-posedness of the 3D cubic nonlinear wave equation in negative Sobolev spaces

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Cubic NLW on $\mathbb{T}^3$: deterministic dynamics

We consider the defocusing cubic NLW on $\mathbb{T}^3$:

$$\text{(NLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u + u^3 = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3), \end{cases}$$

where $u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$

- scaling-critical Sobolev regularity: $s_{\text{crit}} = \frac{1}{2}$
  (under the scaling symmetry $u(t,x) \mapsto \lambda u(\lambda t, \lambda x)$ on $\mathbb{R}^3$)

Well/Ill-posedness of NLW in $\mathcal{H}^s(\mathbb{T}^3)$:

- $s \geq 1$: classical global well-posedness based on Sobolev embedding
  $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ and the energy conservation
- $s \geq \frac{1}{2}$: local well-posedness via Strichartz estimates (Lindblad-Sogge '95)
- $s < \frac{1}{2}$: ill-posedness (Christ-Colliander-Tao '03, Burq-Tzvetkov '08, Xia '15, Forlano-Okamoto '20)

  Solution map: $(u_0, u_1) \in \mathcal{H}^s \mapsto (u(t), \partial_t u(t)) \in \mathcal{H}^s$ is **discontinuous everywhere** in $\mathcal{H}^s(\mathbb{T}^3)$

Discontinuity of the solution map = lack of good approximation property

Q1: Can we still construct unique solutions in $\mathcal{H}^s(\mathbb{T}^3)$, $s < \frac{1}{2}$?

A1: $0 < s < \frac{1}{2}$: probabilistic global well-posedness of NLW with general random initial data (Burq-Tzvetkov '08, '14)

Q2: Can we prove almost sure local well-posedness in $\mathcal{H}^s(\mathbb{T}^3)$ for $s < 0$?

- We consider the random initial data:

$$u_0^\omega = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \qquad \text{and} \qquad u_1^\omega = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x},$$

  where $\langle n\rangle = \sqrt{1+|n|^2}$ and $\{g_n, h_n\}_{n\in\mathbb{Z}^3}$ are independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$

- We have $(u_0^\omega, u_1^\omega) \in \mathcal{H}^s(\mathbb{T}^3)$ almost surely for $s < \alpha - \frac{3}{2}$:

$$\|u_0^\omega\|^2_{L^2(\Omega, H^s(\mathbb{T}^3))} = \sum_{n\in\mathbb{Z}^3} \underbrace{\mathbb{E}(|g_n|^2)}_{=1} \langle n\rangle^{2(s-\alpha)} < \infty \quad \text{iff} \quad 2(s-\alpha) < -3$$

- We work with $\alpha \le \frac{3}{2}$ so that $(u_0^\omega, u_1^\omega) \in \mathcal{H}^s(\mathbb{T}^3) \setminus \mathcal{H}^0(\mathbb{T}^3)$ a.s. for $s < 0$

Why do we care about this specific (u_0^ω, u_1^ω)?

- The distribution of (u_0^ω, u_1^ω) is given by the following Gaussian probability measure on $\mathcal{D}(\mathbb{T}^3) \times \mathcal{D}(\mathbb{T}^3)$:

  $$\mu_\alpha(du, dv) = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^\alpha}^2} du \otimes e^{-\frac{1}{2}\|v\|_{H^{\alpha-1}}^2} dv$$

  In particular, when $\alpha = 1$, μ_1 corresponds to the base Gaussian part of the Gibbs measure for NLW ($= \Phi_3^4$-measure $\otimes$ white noise on (u, v))

Let us consider the truncated Fourier series $(u_{0,N}^\omega, u_{1,N}^\omega) = \mathbf{P}_{\leq N}(u_0^\omega, u_1^\omega)$:

$$u_{0,N}^\omega(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \qquad u_{1,N}^\omega(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x}$$

Then, we have the following almost sure local well-posedness of the **renormalized NLW**.

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Probabilistic local well-posedness of renormalized NLW

Theorem 1 (Oh-P.-Tzvetkov '19)

Let $\frac{5}{4} < \alpha \leq \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. Then, there exists a sequence $\{\alpha_N\}_{N\in\mathbb{N}}$, $\alpha_N > 0$, $\alpha_N \to \infty$ as $N \to \infty$, such that the following holds true.

Denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth global solutions to

$$\begin{cases} \partial_t^2 u_N - \Delta u_N + u_N^3 - \alpha_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (u^\omega_{0,N}, u^\omega_{1,N}). \end{cases}$$

Then, for almost every $\omega \in \Omega$, the sequence $\{u^\omega_N\}_{N\in\mathbb{N}}$ converges to some (unique) limiting distribution u in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$ as $N \to \infty$.

We formally write that u is a "solution" to the "limit equation":

$$\begin{cases} \partial_t^2 u - \Delta u + u^3 - \infty \cdot u = 0 \\ (u, \partial_t u)|_{t=0} = (u^\omega_0, u^\omega_1) \end{cases}$$

and we'll explain later in the talk in which sense this holds.

- This is the standard formulation of the solution theory in singular stochastic PDEs (since the limit equation is formal).

- Let ρ be a smooth mollification kernel and $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. Then, there exists a sequence $\{\alpha_\varepsilon\}_{\varepsilon>0}$ of diverging constants, depending on ρ, such that the smooth solution u_ε to

$$\begin{cases} \partial_t^2 u_\varepsilon - \Delta u_\varepsilon + u_\varepsilon^3 - \alpha_\varepsilon u_\varepsilon = 0 \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (\rho_\varepsilon * u_0^\omega, \rho_\varepsilon * u_1^\omega) \end{cases}$$

  converges (in probability) to some limit u as $\varepsilon \to 0$.

  The limit u does *not* depend on the choice of a mollification kernel ρ and agrees with the limit in Theorem 1.

Instability of NLW in negative Sobolev spaces

We consider

$$\widetilde{u}^{\omega}_{0,N}(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle_N \langle n\rangle^{\alpha-1}} e^{in\cdot x} \qquad \text{and} \qquad \widetilde{u}^{\omega}_{1,N}(x) = \sum_{|n|\le N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x},$$

where $\langle n\rangle_N = \sqrt{C_N + |n|^2}$ for some suitable choice of $C_N > 0$, $C_N \to \infty$ as $N \to \infty$. Note that

$$(\widetilde{u}^{\omega}_{0,N}, \widetilde{u}^{\omega}_{1,N}) \to (0, u^{\omega}_1) \text{ in } \mathcal{H}^s(\mathbb{T}^3) \text{ a.s. for } s < \alpha - \frac{3}{2}, \text{ as } N \to \infty$$

Theorem 2 (Oh-P.-Tzvetkov '19)

Let $\frac{5}{4} < \alpha < \frac{3}{2}$ and $(w_0, w_1) \in \mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$. Denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth global solutions to the defocusing (un-renormalized) NLW with the random initial data

$$(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\widetilde{u}^{\omega}_{0,N}, \widetilde{u}^{\omega}_{1,N}).$$

Then, for almost every $\omega \in \Omega$, the sequence $\{u^{\omega}_N\}_{N\in\mathbb{N}}$ converges to 0 as space-time distributions on $[-T_\omega, T_\omega] \times \mathbb{T}^3$ as $N \to \infty$.

When $\alpha = \frac{3}{2}$, the same result holds but only along a subsequence $\{N_k\}_{k\in\mathbb{N}}$.

- For *every* (w_0, w_1) sufficiently smooth, Theorem 2 shows that

$$(u_N, \partial_t u_N)|_{t=0} \longrightarrow (w_0, w_1 + u_1^\omega) \qquad \text{almost surely in } \mathcal{H}^s(\mathbb{T}^3)$$

but $$u_N \longrightarrow 0 \qquad \text{as space-time distributions}$$

- This phenomenon is known as *triviality* in singular stochastic PDEs. Namely, unless we renormalize the equation, the dynamics trivializes to $u \equiv 0$; See Hairer-Ryser-Weber '12, Oh-Okamoto-Robert '20.
See also Guo-Oh '18 (1-d cubic NLS) and Chapouto '21 (complex-valued mKdV) in the deterministic PDE context (with deterministic initial data)

- Let C_N be as above. A slight modification of the proof of Theorem 1 shows that the solution v_N to the renormalized NLW (with $\alpha_N = \alpha_N(C_N) \to \infty$) with $(v_N, \partial_t v_N)|_{t=0} = (w_0, w_1) + (\widetilde{u}_{0,N}^\omega, \widetilde{u}_{1,N}^\omega)$ converges to a non-trivial limit (i.e. a "solution" to the limit equation with initial data $(w_0, w_1 + u_1^\omega)$)

- Theorem 2 holds only in the defocusing case. In the focusing case, we expect a different kind of instability (such as instantaneous blowup) but we do not know how to prove it at this point

Why this renormalization? (back to Picard iteration)

We consider the Picard iteration scheme for NLW

- For short times, we hope to represent a solution of NLW as

$$u^\omega = \sum_{j=1}^{\infty} Q_{2j+1}(u_0^\omega, u_1^\omega),$$

where Q_{2j+1} is homogeneous of order $2j+1$ in (u_0^ω, u_1^ω):

$$Q_1(u_0^\omega, u_1^\omega) = S(t)(u_0^\omega, u_1^\omega),$$
$$Q_3(u_0^\omega, u_1^\omega) = -\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \big(S(\tau)(u_0^\omega, u_1^\omega)\big)^3 d\tau,$$

and $S(t)(u_0^\omega, u_1^\omega)$ denotes the solution of the linear wave equation with initial data (u_0^ω, u_1^ω).

- We have $Q_1 \notin H^s$ a.s. for $s \geq \alpha - \frac{3}{2}$

- For $\frac{3}{2} < \alpha < \frac{5}{2}$, the probabilistic Strichartz estimates shows almost surely,

$$\|Q_3(u_0^\omega, u_1^\omega)\|_{L_T^\infty H_x^1(\mathbb{T}^3)} \leq \|S(t)(u_0^\omega, u_1^\omega)\|^3_{L_T^3 L_x^6(\mathbb{T}^3)} < \infty$$

$\Longrightarrow$ Q_3 is a.s. more regular than the random initial data

Why this renormalization? (continued)

We now consider $\alpha \leq \frac{3}{2}$. Let $z_N = S(t)(u^\omega_{0,N}, u^\omega_{1,N}) = \mathbf{P}_N z$.

- $z_N(t) \longrightarrow z(t)$ in $W^{s,p}(\mathbb{T}^3) \setminus W^{\alpha-\frac{3}{2},p}(\mathbb{T}^3)$ almost surely, $s < \alpha - \frac{3}{2}$
  In particular, the limit $z(t)$ is merely a distribution of negative regularity

- $(z_N(t))^3$ does **not** converges to any limit and hence $Q_3(u^\omega_{0,N}, u^\omega_{1,N})$ does not converge to any limit

  $\Longrightarrow$ Picard iteration scheme fails (unless we renormalize the nonlinearity)

(For a technical reason) let $z_{1,N}$ be the solution to the linear Klein-Gordon equation $\mathcal{L} z_{1,N} = 0$, where $\mathcal{L} = \partial_t^2 - \Delta + 1$ with the truncated initial data

$$(z_{1,N}, \partial_t z_{1,N})|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega)$$

- With $g_n^t(\omega) = \cos(t\langle n\rangle)\, g_n(\omega) + \sin(t\langle n\rangle)\, h_n(\omega)$, we have

$$z_{1,N}(t,x,\omega) = \sum_{|n|\leq N} \frac{g_n^t(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}$$

- Fix $(t,x) \in \mathbb{R} \times \mathbb{T}^3$. Then, $z_{1,N}(t,x)$ is a mean-zero Gaussian r.v. with variance

$$\sigma_N := \mathbb{E}\big[(z_{1,N}(t,x))^2\big] = \sum_{|n|\leq N} \frac{\mathbb{E}\big[|g_n^t(\omega)|^2\big]}{\langle n\rangle^{2\alpha}} = \sum_{|n|\leq N} \frac{1}{\langle n\rangle^{2\alpha}} \longrightarrow \infty$$

as $N \to \infty$, for $\alpha \leq \frac{3}{2}$ (this shows that the limit of $z_{1,N}$ is not a function)

We now introduce the *Wick renormalization* (from Euclidean QFT):

$$:z_{1,N}^k(t,x): \stackrel{\text{def}}{=} H_k(z_{1,N}(t,x);\sigma_N),$$

where $H_k(x;\sigma)$ is the Hermite polynomial of degree k with parameter $\sigma > 0$.
More concretely, we set

$$\boxed{:z_{1,N}^2: = z_{1,N}^2 - \sigma_N \qquad \text{and} \qquad :z_{1,N}^3: = z_{1,N}^3 - 3\sigma_N z_{1,N}}$$

Then, by stochastic analysis, we can show that

$:z_{1,N}^k:$ converges to some limit (denoted by $:z_1^k:$)
in $W^{s,p}(\mathbb{T}^3)$ a.s. for any $s < k(\alpha - \frac{3}{2})$ and $1 \le p \le \infty$

Wick renormalization plays an important role in the construction of the Φ_d^k-measures in Euclidean quantum field theory and also in the study of singular stochastic PDEs

First order expansion

Renormalized NLW: $\partial_t^2 u_N - \Delta u_N + u_N^3 - \alpha_N u_N = 0$

- Set $\alpha_N = 3\sigma_N + 1$. Then, u_N satisfies

$$\mathcal{L} u_N + u_N^3 - 3\sigma_N u_N = 0$$

- First order expansion (Bourgain '96, Da Prato-Debussche '02):

$$u_N = z_{1,N} + v_N,$$

where v_N satisfies

$$\begin{aligned} \mathcal{L} v_N &= -(v_N + z_{1,N})^3 + 3\sigma_N (v_N + z_{1,N}) \\ &= -v_N^3 - 3v_N z_{1,N} - 3v_N \underbrace{(z_{1,N}^2 - \sigma_N)}_{=:z_{1,N}^2:} - \underbrace{(z_{1,N}^3 - 3\sigma_N z_{1,N})}_{=:z_{1,N}^3:} \end{aligned}$$

It turns out that v_N is more regular than $z_{1,N}$ (in the limiting sense).
In fact, we use deterministic theory to solve the equation for v_N in $C_T H_x^{\frac{1}{2}}$

- The worst term on RHS is $:z_{1,N}^3:$ of Sobolev regularity:

$$\boxed{:z_{1,N}^3: \sim 3(\alpha - \tfrac{3}{2})-}$$

- Using the "one degree of smoothing", the solution v_N is uniformly bounded in $C_T H_x^{\frac{1}{2}}$ if

$$3(\alpha - \tfrac{3}{2}) + 1 > \tfrac{1}{2} \implies \alpha > \tfrac{4}{3}$$

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Second order expansion

- Second order expansion: $\boxed{u_N = z_{1,N} + z_{2,N} + w_N}$, where $z_{2,N}$ is the solution to the following equation:

$$\begin{cases} \mathcal{L}z_{2,N} + :z_{1,N}^3: = 0 \\ (z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0). \end{cases}$$

- We have

$$z_{2,N} = -\mathcal{I}(:z_{1,N}^3:) = \textbf{ Picard 2nd iterate},$$

where $\mathcal{I}$ = Duhamel integral operator for the Klein-Gordon equation

- Thanks to the "one degree of smoothing", $z_{2,N}$ has Sobolev regularity

$$z_{2,N} \sim 3(\alpha - \frac{3}{2}) + 1 - \qquad \text{(which is positive for } \alpha > \frac{7}{6})$$

- Renormalized NLW becomes

$$\mathcal{L}w_N + (w_N + z_{2,N})^3 + 3z_{1,N}(w_N + z_{2,N})^2 + 3 :z_{1,N}^2: (w_N + z_{2,N}) = 0$$

So, **we removed the worst term** $:z_{1,N}^3:$ **from the** v_N**-equation!!**

Where does the restriction $\alpha > \frac{5}{4}$ come from?

- Recall that the Sobolev regularity of $:z_{1,N}^2:$ is $\boxed{:z_{1,N}^2: \sim 2(\alpha - \frac{3}{2})-}$
- Consider the toy model: $\mathcal{L}w_N + 3 :z_{1,N}^2: w_N = 0$
- $:z_{1,N}^2: w_N$ inherits the regularity of $:z_{1,N}^2:$ for sufficiently smooth w_N

$$\boxed{:z_{1,N}^2: w_N \sim 2(\alpha - \tfrac{3}{2})-}$$

- The solution w_N is uniformly bounded in $C_T H_x^{\frac{1}{2}}$ if $2(\alpha - \frac{3}{2}) + 1 > \frac{1}{2}$

  $\implies \quad \alpha > \frac{5}{4}$ = the condition on α in Theorem 1

Sketch of the proof of Theorem 1

1. Use stochastic analysis to construct the stochastic terms

$$:z_{1,N}^k:,\ k=2,3, \qquad z_{2,N}=-\mathcal{I}(:z_{1,N}^3:), \qquad :z_{1,N}^2:z_{2,N}$$

2. Perform a construction argument for the w_N-equation via the Strichartz estimates and product estimates

$$\mathcal{L}w_N+(w_N+z_{2,N})^3+3z_{1,N}w_N^2+6z_{1,N}z_{2,N}w_N+3z_{1,N}z_{2,N}^2$$
$$+3:z_{1,N}^2:w_N+3:z_{1,N}^2:z_{2,N}=0$$

Step 1: Construction of the stochastic terms

Basic stochastic analysis gives (with uniform bounds in N)

- $:z_{1,N}^k:\sim k(\alpha-\frac{3}{2})-$ (namely, $:z_{1,N}^k:\in C_TW_x^{k(\alpha-\frac{3}{2})-,\infty}$, a.s.)
- $z_{2,N}=-\mathcal{I}(:z_{1,N}^3:)\sim 3(\alpha-\frac{3}{2})+1-=3\alpha-\frac{7}{2}-$ (>0 for $\alpha>\frac{7}{6}$)
- The products $z_{1,N}z_{2,N}$ and $z_{1,N}z_{2,N}^2$ make sense deterministically provided $(\alpha-\frac{3}{2})+(3\alpha-\frac{7}{2})>0$ (so for $\alpha>\frac{5}{4}$)

Step 1: Construction of the stochastic terms (continued)

The product $:z_{1,N}^2: z_{2,N}$

- makes sense deterministically provided $2(\alpha - \frac{3}{2}) + (3\alpha - \frac{7}{2}) = 5\alpha - \frac{13}{2} > 0$, so for $\alpha > \frac{13}{10}$
- for $\frac{5}{4} < \alpha \leq \frac{13}{10}$ we make sense of it using stochastic analysis
- in either case, when $z_{2,N}$ has positive regularity (for $\alpha > \frac{7}{6}$), the product inherits the regularity of $:z_{1,N}^2:$ $\boxed{:z_{1,N}^2: z_{2,N} \sim 2(\alpha - \frac{3}{2})-}$

Step 2: Contraction argument

- Strichartz estimates (to handle w_N^3)
- Product estimates: fractional Leibniz rule and the following estimate:

Let $0 \leq s \leq 1$ and $1 < p, q, r < \infty$ such that $s \geq 3\big(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\big)$. Then,

$$\|\langle\nabla\rangle^{-s}(fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle\nabla\rangle^{-s} f\|_{L^p(\mathbb{T}^3)} \|\langle\nabla\rangle^{s} g\|_{L^q(\mathbb{T}^3)}$$

Eg:

$$\begin{aligned}\|\langle\nabla\rangle^{-\frac{1}{2}}(:z_{1,N}^2: w_N)\|_{L^2(\mathbb{T}^3)} &\lesssim \|\langle\nabla\rangle^{-\frac{1}{2}} :z_{1,N}^2: \|_{L^6(\mathbb{T}^3)} \|\langle\nabla\rangle^{\frac{1}{2}} w_N\|_{L^2(\mathbb{T}^3)} \\ &\lesssim \| :z_{1,N}^2: \|_{W^{2(\alpha-\frac{3}{2})-,\infty}(\mathbb{T}^3)} \|w_N\|_{H^{\frac{1}{2}}(\mathbb{T}^3)}\end{aligned}$$

Step 1: Construction of the stochastic terms (continued)

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Meaning of the "solution" u

- Recall that the solution map: $(u_0, u_1) \mapsto (u, \partial_t u)$ does not extend continuously to $\mathcal{H}^s(\mathbb{T}^3)$ for $s < \frac{1}{2}$
- With the second order expansion, we decompose the ill-defined solution map into two steps:
  1. Construction of an *enhanced data set* by stochastic analysis
  2. Deterministic continuous map from enhanced data set to $(w, \partial_t w)$

$$(u_0^\omega, u_1^\omega) \xmapsto{①} (z_1, :z_1^2:, z_2 = -\mathcal{I}(:z_1^3:), :z_1^2: z_2)$$
$$\xmapsto{②} (w, \partial_t w) \longmapsto u = z_1 + z_2 + w \text{ (and } \partial_t u)$$

  where w satisfies

$$\mathcal{L}w + (w + z_2)^3 + 3z_1 w^2 + 6z_1 z_2 w + 3z_1 z_2^2 + 3 :z_1^2: w + 3 :z_1^2: z_2 = 0$$

- Uniqueness of $u = z_1 + z_2 + w$:
  1. uniqueness of z_1 and z_2 as limits of $z_{1,N}$ and $z_{2,N}$
  2. uniqueness of w as a solution to the w-equation

On triviality

- Recall

$$\widetilde{u}^{\omega}_{0,N}(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle_N \langle n\rangle^{\alpha-1}} e^{in\cdot x} \qquad \text{and} \qquad \widetilde{u}^{\omega}_{1,N}(x) = \sum_{|n|\le N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x},$$

where $\langle n\rangle_N = \sqrt{C_N + |n|^2}$ for some positive $C_N \to \infty$ as $N \to \infty$.

- Given $N \in \mathbb{N}$, define the linear Klein-Gordon operator $\mathcal{L}_N$ by setting

$$\mathcal{L}_N = \partial_t^2 - \Delta + C_N.$$

Then, u_N satisfies the following equation:

$$\begin{cases} \mathcal{L}_N u_N + u_N^3 - C_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\widetilde{u}^{\omega}_{0,N}, \widetilde{u}^{\omega}_{1,N}). \end{cases}$$

- Let $\widetilde{z}_{1,N}$ be the solution to $\mathcal{L}_N \widetilde{z}_{1,N} = 0$ with initial data $(\widetilde{u}^{\omega}_{0,N}, \widetilde{u}^{\omega}_{1,N})$.

Then, for each fixed $(t,x) \in \mathbb{R} \times \mathbb{T}^3$, $\widetilde{z}_{1,N}(t,x)$ is a mean-zero Gaussian random variable with variance $\widetilde{\sigma}_N$ given by

$$\widetilde{\sigma}_N = \mathbb{E}\big[(\widetilde{z}_{1,N}(t,x))^2\big] = \sum_{|n|\le N} \frac{1}{\langle n\rangle_N^2 \langle n\rangle^{2(\alpha-1)}}$$

Now, we implicitly define $C_N > 0$ by

$$
\begin{aligned}
C_N = 3\widetilde{\sigma}_N &= 3 \sum_{|n| \le N} \frac{1}{\langle n \rangle_N^2 \langle n \rangle^{2(\alpha-1)}} \\
&= 3 \sum_{|n| \le N} \frac{1}{(C_N + |n|^2)\langle n \rangle^{2(\alpha-1)}}
\end{aligned}
$$

- As $N \to \infty$, we have $C_N = 3\sigma_N +$ smaller error $\sim \begin{cases} N^{3-2\alpha} & \text{for } \alpha < \frac{3}{2}, \\ \log N & \text{for } \alpha = \frac{3}{2} \end{cases}$
- Then u_N satisfies

$$
\begin{cases} \mathcal{L}_N u_N + u_N^3 - 3\widetilde{\sigma}_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\widetilde{u}_{0,N}^\omega, \widetilde{u}_{1,N}^\omega). \end{cases}
$$

  Namely, we introduced an **artificial renormalization**

- By repeating the proof of Theorem 1, we have uniform (in N) almost sure local well-posedness for u_N (or rather $w_N = u_N - \widetilde{z}_{1,N} + \mathcal{I}_N(:\widetilde{z}_{1,N}^3:)$)

1. Let $\mathcal{I}_N = \mathcal{L}_N^{-1}$ be the Duhamel integral operator given by

$$\mathcal{I}_N F(t) = \int_0^t \frac{\sin((t-t')\sqrt{C_N - \Delta})}{\sqrt{C_N - \Delta}} F(t')dt'.$$

- Non-homogeneous Strichartz estimate holds for $\mathcal{I}_N$ uniformly in $C_N \geq 1$
- Also, $\|\mathcal{I}_N(F)\|_{C_T L_x^2} \lesssim C_N^{-\delta} \|F\|_{L_t^1 H_x^{-1+2\delta}}$

$\Longrightarrow$ If $\{F_N\}_{N\in\mathbb{N}}$ is bounded in $L_t^1 H_x^{-1+2\delta}$, then $\mathcal{I}_N(F_N) \to 0$ in $C_T L_x^2$ as $N \to \infty$ (since $C_N \to \infty$)

2. Let $S_N(t)(w_0, w_1) = \cos(t\sqrt{C_N - \Delta})w_0 + \frac{\sin(t\sqrt{C_N - \Delta})}{\sqrt{C_N - \Delta}} w_1.$

- $\|S_N(t)(w_0, w_1)\|_{L_T^4 L_x^4} \leq C \|(w_0, w_1)\|_{\mathcal{H}^{\frac{3}{4}}}$ for $0 < T \leq 1$, uniformly in $C_N \geq 1$
- $S_N(t)(w_0, w_1)$ tends to 0 in the space-time distributional sense as $N \to \infty$

Together with the uniform (in N) boundedness of w_N and

$$(\widetilde{z}_{1,N}, :\widetilde{z}_{1,N}^2:, \mathcal{I}_N(:\widetilde{z}_{1,N}^3:), :\widetilde{z}_{1,N}^2 : \mathcal{I}_N(:\widetilde{z}_{1,N}^3:))$$

(in appropriate norms), **1** and **2** yield that $w_N \to 0$ as a space-time distribution as $N \to \infty$.

Comments and open problems

- The 2nd order expansion gave an improvement over the 1st order expansion. However, the worst term in the 2nd order expansion is $3 :z_1^2: w$ involving the unknown w and thus it cannot be removed by a higher order expansion
- Oh-Wang-Zine '21 improved Theorem 1 to $\alpha > 1$ by multilinear smoothing (including random operators). Their argument is based on the analysis by Bringmann '20 on the Hartree NLW $\partial_t^2 u + (1-\Delta)u - (D^{-\beta}u^2)u = 0$, $\beta > 0$, with random initial data with $\alpha = 1$.
- The case $\alpha = 1$ for the cubic NLW on $\mathbb{T}^3$ corresponds to the hyperbolic Φ_3^4-model. Very challenging open question!
- A.s. global well-posedness of renormalized NLW in negative Sobolev spaces (extending Burq-Tzvetkov '14)? On $\mathbb{T}^2$, Gubinelli-Koch-Oh-Tolomeo '20 proved a.s. GWP of SNLW with random initial data slightly outside $L^2(\mathbb{T}^2)$
- An analogue of Theorem 1 on $\mathbb{R}^3$? The random data is not translation invariant and thus $\sigma_N = \sigma_N(x)$, i.e. depend on $x \in \mathbb{R}^3$