Advanced PDE II

(MIGSAA Ph.D. course & SMSTC advanced course)

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Course website: SMSTC website, http://www.macs.hw.ac.uk/∼op4/Teaching.html

Lecture: 9:00-11:00 Wednesday (Jan. 11- Mar. 18)
Room 107 (on the ground floor) in ICMS, 15 South College Street

Assessment: 3 homework assignments to be submitted in weeks 4, 7, and 10. (The problem sheets will be handed out in weeks 3, 6, and 9.)

This course is dedicated to the study of hyperbolic PDEs in Sobolev spaces. We will mainly focus on nonlinear wave equations and symmetric hyperbolic systems. The course will begin with some preliminary ideas including the method of characteristics, finite speed of propagation, and finite time blowup. Then, there will be a brief discussion of classical methods including the explicit solution formula of D'Alembert and Kirchhoff and the CauchyKovalevskaya existence and uniqueness theorem.

The course will then turn to Sobolev space methods. These include energy estimates, Klainerman-Sobolev inequality, the vector field method, and well-posedness of semilinear wave equations using Sobolev estimates.

In the final two weeks we will cover special topics at the instructor's discretion. Possible special topics are small data global well-posedness of quasilinear wave or Klein-Gordon equations (study started independently in works of Klainerman and Shatah from the 1980s), the global well-posedness of the defocusing energy critical semilinear wave equation (work of Shatah-Struwe, 1994), or a survey of geometric hyperbolic PDE such as the Yang-Mills and wave maps equations.

Basic topics:

- Method of characteristics
- Formation of shocks for the inviscid Burger's equation
- Explicit formulas for solutions of the linear wave equation (d'Alembert, Kirchoff, Poisson)
- Cauchy-Kowalevski theorem
- Review of Gronwall's inequality, Sobolev spaces, Picard iteration, definition of an initial value problem, local well-posedness, global well-posedness, Green's theorem
- Review of Fourier analysis
- Vector fields
- Energy estimates
- Finite speed of propagation
- Klainerman-Sobolev inequality (preceded by a review of Sobolev embeddings)

- Local well-posedness for quasi-linear wave equations and global well-posedness in a subcritical situation
- Symmetric hyperbolic systems

Some more advanced topics (only some will be covered):

- Small data global well-posedness for quasi-linear wave equations on $\mathbb{R}^n$, $n \geq 4$
- Small data global well-posedness for quasi-linear Klein-Gordon equation on $\mathbb{R}^3$
- Null forms and small data global well-posedness for quasi-linear wave equations on $\mathbb{R}^3$

Special topics in the last two weeks: to be determined

References

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[4] L. C. Evans, *Partial differential equations*, Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[5] G. Folland, *Introduction to partial differential equations*, Second edition. Princeton University Press, Princeton, NJ, 1995.

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[7] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], 26. Springer-Verlag, Berlin, 1997.

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[9] F. John, *Nonlinear wave equations, formation of singularities*, Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989. University Lecture Series, 2. American Mathematical Society, Providence, RI, 1990.

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[11] S. Klainerman, *Brief History of the Vectorfield method*, special lecture in honour of F. John's 100 anniversary, https://web.math.princeton.edu/ seri/homepage/serilectures.html.

[12] A. Lawrie, *Nonlinear wave equations*, online expository notes, https://math.berkeley.edu/ alawrie/.

[13] J. Luk, *Introduction to nonlinear wave equations*, online lecture notes for a course taught at University of Cambridge, https://www.dpmms.cam.ac.uk/ jl845/.

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[16] J. Rauch, *Hyperbolic partial differential equations and geometric optics*, Graduate Studies in Mathematics, 133. American Mathematical Society, Providence, RI, 2012.

[17] H. Ringström, *The Cauchy problem in general relativity*, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2009.

[18] J. Shatah, *Regularity results for semilinear and geometric wave equations*, Mathematics of gravitation, Part I (Warsaw, 1996), 69–90, Banach Center Publ., 41, Part I, Polish Acad. Sci., Warsaw, 1997.

[19] J. Shatah, M. Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics, 2. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998.

[20] C. Sogge, *Lectures on non-linear wave equations*, Second edition. International Press, Boston, MA, 2008.
[21] W. Strauss, *Partial differential equations. An introduction*, Second edition. John Wiley & Sons, Ltd., Chichester, 2008.
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