The Szego equation seen as the resonant dynamics of a non-linear wave equation

Oana Pocovnicu

Université Paris-Sud 11, Orsay, France

September 8th, 2011

Workshop HANDDY
Berder

The cubic Szegö equation

$$\text{(SE)} \qquad i\partial_t u = \Pi_+(|u|^2 u), \qquad u(t,x) \in \mathbb{C}, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where Π_+ is the Szegö projector onto non-negative frequencies, was recently introduced by Gérard and Grellier who study it on $\mathbb{T}$

- mathematical model of a non-dispersive Hamiltonian equation
- completely integrable
- It exhibits growth of high Sobolev norms $\|u(t)\|_{H^s} \to \infty$ if $t \to \infty$ and $s > 1/2$. More precisely, there are solutions ($u_0 = \frac{1}{x+i} - \frac{2}{x+2i}$) such that

$$\|u(t)\|_{H^s} \sim t^{2s-1}.$$

The Szegö equation as the first approximation of NLW

Theorem (P '11)

Let $W_0 \in H^s_+(\mathbb{R})$, $s > \frac{1}{2}$. *Let* v *be the solution of the NLW on* $\mathbb{R}$

(NLW)
$$\begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \varepsilon W_0. \end{cases}$$

Denote by u *the solution of the Szegö equation*

$$\begin{cases} i\partial_t u = \Pi_+(|u|^2 u) \\ u(0) = \varepsilon W_0. \end{cases}$$

Assume that $\|u(t)\|_{H^s} \leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$ *for* $0 \leq \alpha \leq \frac{1}{2}$ *and* $\delta > 0$ *small.*

Then, if $0 \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$ *we have that*

$$\|v(t) - e^{-i|D|t}u(t)\|_{H^s} \leq C\varepsilon^{2-C_0\delta}.$$

Growth of high Sobolev norms for solutions of NLW

Corollary (P '11)

Let $0 < \varepsilon \ll 1$, $s > \frac{1}{2}$, and $\delta > 0$ sufficiently small. Let $W_0 \in H^s_+(\mathbb{R})$ be the rational function $W_0 = \frac{1}{x+i} - \frac{2}{x+2i}$. Denote by v be the solution of the NLW equation on $\mathbb{R}$

$$\text{(NLW)} \qquad \begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \varepsilon W_0. \end{cases}$$

Then, for $\frac{1}{2\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}} \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}}$, we have that

$$\frac{\|v(t)\|_{H^s(\mathbb{R})}}{\|v(0)\|_{H^s(\mathbb{R})}} \geq C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{4s-2}{4s-1}} \gg 1.$$

Remark: In order to show arbitrarily large growth of the solution, one needs an approximation at least for a time $0 \leq t \leq \frac{1}{\varepsilon^{2+\beta}}$, where $\beta > 0$.

- If $v_0 \in H_+^{1/2}(\mathbb{R})$, $\|v_0\|_{H^{1/2}} = \varepsilon \Longrightarrow 2H(v(t)) - M(v(t)) = 2H(v_0) - M(v_0)$:

$$2(|D|v_-(t), v_-(t)) + \frac{1}{2}\|v(t)\|_{L^4}^4 = \frac{1}{2}\|v_0\|_{L^4}^4 = O(\varepsilon^4).$$

  Thus $\|v_-(t)\|_{\dot{H}^{1/2}(\mathbb{T})} = O(\varepsilon^2)$.

- Moreover,

$$\|v_-(t)\|_{H^{1/2}(\mathbb{T})}^2 = \sum_{k \leq -1} (1 + |k|^2)^{1/2} |\hat{v}(k)|^2$$
$$\leq 2 \sum_{k \leq -1} |k||\hat{v}(k)|^2 \leq 2\|v_-(t)\|_{\dot{H}^{1/2}(\mathbb{R})} = O(\varepsilon^4).$$

  and thus $\|v_-(t)\|_{H^{1/2}(\mathbb{T})} = O(\varepsilon^2)$.

- On $\mathbb{R}$, we ONLY have $\|v_-(t)\|_{\dot{H}^{1/2}(\mathbb{R})} = O(\varepsilon^2)$

The renormalization group (RG) method

- It is most often used to find a long-time approximate solution to a perturbed equation
- It was introduced by Chen, Goldenfeld, and Oono (1994) in theoretical physics
- The RG method was justified mathematically for ODEs by De Ville, Harkin, Holzer, Josic, Kaper; Ziane and for some PDEs (Navier-Stokes equations, Swift-Hohenberg equation, quadratic NLS) by Temam, Moise, Petcu, Wirosoetisno, Abou Salem
- Gérard and Grellier (2011) proved analogous results on the torus $\mathbb{T}$ using the theory of Birkhoff normal forms

- Change of variables $w(t) = \frac{1}{\varepsilon} e^{i|D|t} v(t)$ in NLW:

$$\text{(NLW')} \qquad \begin{cases} \partial_t w = -i\varepsilon^2 e^{i|D|t}(|e^{-i|D|t}w|^2 e^{-i|D|t}w) =: \varepsilon^2 f(w,t) \\ w(0) = W_0. \end{cases}$$

- Naive perturbation expansion:

$$w(t) = w^{(0)}(t) + \varepsilon^2 w^{(1)}(t) + \varepsilon^4 w^{(2)}(t) + \dots$$

- Taylor expansion:

$$\begin{aligned} f(w,t) &= f(w^{(0)},t) + f'(w^{(0)},t)(w(t) - w^{(0)}(t)) + \dots \\ &= f(w^{(0)},t) + \varepsilon^2 f'(w^{(0)},t) w^{(1)}(t) + \dots \end{aligned}$$

- Identifying the powers of ε:

$$\begin{cases} \partial_t w^{(0)} = 0 \\ \partial_t w^{(1)} = f(w^{(0)}(t),t) \\ \dots \end{cases}$$

- Then,

$$w(t) = W_0 + \varepsilon^2 w^{(1)}(t) + O(\varepsilon^4) = W_0 + \varepsilon^2 \int_0^t f(W_0,s)ds + O(\varepsilon^4).$$

$$\mathcal{F}\big(f(w,t)\big)(\xi) = -\,i \iint_{\xi=\xi_1-\xi_2+\xi_3} e^{it\phi(\xi,\xi_1,\xi_2,\xi_3)}\hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

where $\phi(\xi,\xi_1,\xi_2,\xi_3) := |\xi| - |\xi_1| + |\xi_2| - |\xi_3|$.

$$f(w,t) = f_{\text{res}}(w) + f_{\text{osc}}(w,t),$$

$$f_{\text{res}}(w) := -i\mathcal{F}^{-1} \iint_{\{\phi=0,\xi=\xi_1-\xi_2+\xi_3\}} \hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

$$f_{\text{osc}}(w,t) := -i\mathcal{F}^{-1} \iint_{\{\phi\neq0,\xi=\xi_1-\xi_2+\xi_3\}} e^{it\phi(\xi,\xi_1,\xi_2,\xi_3)}\hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3.$$

Then, $w(t) = W_0 + \varepsilon^2 t f_{\text{res}}(W_0) + \varepsilon^2 \int_0^t f_{\text{osc}}(W_0,s)ds + O(\varepsilon^4)$.

The term $W_0 + \varepsilon^2 t f_{\text{res}}(W_0)$ is a secular term. We consider the renormalization group equation:

$$\begin{cases} \partial_t W = \varepsilon^2 f_{\text{res}}(W) \\ W(0) = W_0 \end{cases}$$

An approximation for the solution will be:

$$w_{\text{app}}(t) = W(t) + \varepsilon^2 \underbrace{\int_0^t f_{\text{osc}}(W(t),s)ds}_{=:F_{\text{osc}}(W(t),t)}.$$

Special property of NLW: many resonances

The set $\{\phi(\xi,\xi_1,\xi_2,\xi_3)=0\}\subset\mathbb{R}^2$ has non-zero measure for fixed ξ.
It is the subset of $\mathbb{R}^2$ such that ξ_1, ξ_2, and ξ_3 have the same sign as ξ and $\xi=\xi_1-\xi_2+\xi_3$ (or $\xi_1=\xi$ or $\xi_3=\xi$).

$$\begin{aligned} f_{\text{res}}(w) =& -i\mathcal{F}^{-1}\iint_{\{\phi=0,\xi=\xi_1-\xi_2+\xi_3\}} \hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3 \\ =& -i\mathcal{F}^{-1}\mathbf{1}_{\xi\geq 0}\iint_{\xi=\xi_1-\xi_2+\xi_3} \hat{w}_+(\xi_1)\overline{\hat{w}_+}(\xi_2)\hat{w}_+(\xi_3)d\xi_1 d\xi_2 d\xi_3 \\ & -i\mathcal{F}^{-1}\mathbf{1}_{\xi< 0}\iint_{\xi=\xi_1-\xi_2+\xi_3} \hat{w}_-(\xi_1)\overline{\hat{w}_-}(\xi_2)\hat{w}_-(\xi_3)d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Thus, $f_{\text{res}}(w)=-i\big(\Pi_+(|w_+|^2w_+)+\Pi_-(|w_-|^2w_-)\big)$.
We choose W_0 such that $\Pi_-(W_0)=0$. Projecting onto the negative frequencies:

$$\begin{cases} i\partial_t W_- = \varepsilon^2\Pi_-(|W_-|^2W_-) \\ W_-(0)=0. \end{cases}$$

Then $W_-(t)=0$ for all $t\in\mathbb{R}$ and $W(t)=W_+(t)$ satisfies:

$$\begin{cases} i\partial_t W = \varepsilon^2\Pi_+(|W|^2W) \\ W(0)=W_0. \end{cases}$$

Estimates on $F_{\text{osc}}(W,t)$

Claim 1

$\|F_{\text{osc}}(W(t),t)\|_{L^2(\mathbb{R})} \leq C\sqrt{t}$

$$\widehat{f_{\text{osc}}}(W(t),s,\xi) = -i\mathbf{1}_{\xi<0}\iint_{\xi=\xi_1-\xi_2+\xi_3} e^{is\phi(\xi,\xi_1,\xi_2,\xi_3)}\hat{W}(t,\xi_1)\overline{\hat{W}}(t,\xi_2)\hat{W}(t,\xi_3)\mathbf{1}_{\xi_1,\xi_2,\xi_3\geq 0}d\xi_1 d\xi_2 d\xi_3$$

If $\xi<0$, $\xi_1,\xi_2,\xi_3\geq 0$, then $\phi(\xi,\xi_1,\xi_2,\xi_3)=|\xi|-|\xi_1|+|\xi_2|-|\xi_3|=-2\xi$.

Then,

$$\widehat{F_{\text{osc}}}(W(t),t,\xi)=\int_0^t \widehat{f_{\text{osc}}}(W(t),s,\xi)ds = \frac{e^{-2it\xi}-1}{2\xi}\mathcal{F}(|W|^2W)(\xi)\mathbf{1}_{\xi<0}.$$

$$\begin{aligned}\left\|F_{\text{osc}}(W(t),t)\right\|^2_{L^2(\mathbb{R})} &= \left\|\widehat{F_{\text{osc}}}(W(t),t)\right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^0 \frac{\sin^2(t\xi)}{\xi^2}\left|\mathcal{F}(|W|^2W)(\xi)\right|^2 d\xi \\ &\leq \left\|\mathcal{F}(|W|^2W)\right\|^2_{L^\infty(\mathbb{R})}\int_{-\infty}^0 \frac{\sin^2(t\xi)}{\xi^2}d\xi \\ &\leq \left\||W|^2W\right\|^2_{L^1(\mathbb{R})} t\int_0^\infty \frac{\sin^2\eta}{\eta^2}d\eta \leq Ct\|W\|^6_{L^3(\mathbb{R})} \leq Ct\|W(t)\|^6_{H^{1/2}_+} \\ &\leq Ct.\end{aligned}$$

Claim 2

$\|F_{\text{osc}}(W(t),t)\|_{\dot{H}^s(\mathbb{R})} \leq C\|W\|_{H^s}^3$ for $s \geq 1$

$$\begin{aligned}\|F_{\text{osc}}(W(t),t)\|_{\dot{H}^s(\mathbb{R})}^2 &= \int_{-\infty}^{0} \xi^{2s}\frac{\sin^2(t\xi)}{\xi^2}\left|\mathcal{F}(|W|^2W)(\xi)\right|^2 d\xi \\ &\leq \int_{-\infty}^{0} \xi^{2(s-1)}\left|\mathcal{F}(|W|^2W)(\xi)\right|^2 d\xi \\ &\leq \left\||W|^2W\right\|_{\dot{H}^{s-1}(\mathbb{R})}^2 \leq \left\||W|^2W\right\|_{H^s(\mathbb{R})}^2 \leq \|W\|_{H^s(\mathbb{R})}^6.\end{aligned}$$

Therefore, $\|F_{\text{osc}}(W(t),t)\|_{H^s(\mathbb{R})} \leq C(\sqrt{t} + \|W\|_{H^s(\mathbb{R})}^3)$

Claim 3

If $\|W(t)\|_{H^s(\mathbb{R})} \leq C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$ and $0 \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$, we have that

$$\|w_{\text{app}}(t)\|_{H^s} \leq C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$$

$$\|w_{\text{app}}(t)\|_{H^s} = \|W + \varepsilon^2 F_{\text{osc}}(W,t)\|_{H^s} \leq \|W\|_{H^s} + \varepsilon^2 C\sqrt{t} + \varepsilon^2\|W\|_{H^s}^3 \leq C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$$

Proof of the theorem

Set $z(t) := w(t) - w_{\text{app}}(t)$. Using Duhamel's formula we have:

$$z(t) = \varepsilon^2 \int_0^t \left(f(w(s),s) - f(w_{\text{app}}(s),s)\right) ds - \varepsilon^2 \int_0^t \Big(f(W(s),s) - f(w_{\text{app}}(s),s)\Big) ds$$
$$- \varepsilon^4 \int_0^t D_W F_{\text{osc}}(W(s),s) \cdot f_{\text{res}}(W(s)) ds =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

$$\begin{aligned}
\|\mathrm{I}\|_{H^s} &\leq \varepsilon^2 \int_0^t \|z(\tau)\|_{H^s} \left(\|w(\tau)\|_{H^s}^2 + \|w_{\text{app}}(\tau)\|_{H^s}^2\right) d\tau \\
&\leq C\varepsilon^2 \int_0^t \|z(\tau)\|_{H^s} \left(\|z(\tau)\|_{H^s}^2 + \|w_{\text{app}}(\tau)\|_{H^s}^2\right) d\tau \\
&\leq C\varepsilon^2 \int_0^t \|z(\tau)\|_{H^s} \left(\|z(\tau)\|_{H^s}^2 + \left(\log(\tfrac{1}{\varepsilon^\delta})\right)^{2\alpha}\right) d\tau.
\end{aligned}$$

Using $W(s) - w_{\text{app}}(s) = -\varepsilon^2 F_{\text{osc}}(W(s),s)$,

$$\begin{aligned}
\|\mathrm{II}\|_{H^s} &\leq \varepsilon^4 t \|F_{\text{osc}}(t, W(t))\|_{H^s} \left(\|W\|_{H^s}^2 + \|w_{\text{app}}\|_{H^s}^2\right) \\
&\leq C\varepsilon^4 t(\sqrt{t} + \|W\|_{H^s}^3) \left(\log(\tfrac{1}{\varepsilon^\delta})\right)^{2\alpha}
\end{aligned}$$

Assuming that $\|z(t)\|_{H^s} \leq 1$ and $0 \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$, we have:

$$\|z(t)\|_{H^s} \leq C\varepsilon^2\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{2\alpha}\int_0^t \|z(\tau)\|_{H^s}d\tau + C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{3}{2}(1-2\alpha)}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{2\alpha}.$$

By Gronwall's inequality it follows that

$$\|z(t)\|_{H^s} \leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{3}{2}-\alpha} e^{C\varepsilon^2\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{2\alpha}t} \leq C_*\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{3}{2}-\alpha} e^{C\log(\frac{1}{\varepsilon^\delta})}$$
$$\leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{3}{2}-\alpha}\frac{1}{\varepsilon^{C\delta}} \leq C\varepsilon^{1-C_0\delta}.$$

This yields

$$\|w(t) - W(t) - \varepsilon^2 F_{\text{osc}}(W(t),t)\|_{H^s(\mathbb{R})} \leq C\varepsilon^{1-C_0\delta}.$$

Since

$$\|\varepsilon^2 F_{\text{osc}}(W(t),t)\|_{H^s(\mathbb{R})} \leq C\varepsilon^2(\sqrt{t} + \|W\|_{H^s}^3) \leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{2}(1-2\alpha)} \leq C\varepsilon^{1-C_0\delta}$$

we obtain

$$\|w(t) - W(t)\|_{H^s(\mathbb{R})} \leq C\varepsilon^{1-C_0\delta}.$$

With the change of variables $v = \varepsilon e^{-i|D|t}w$ and $u = \varepsilon W$, we obtain:

$$\|v(t) - e^{-i|D|t}u(t)\|_{H^s(\mathbb{R})} \leq C\varepsilon^{2-C_0\delta}.$$

Theorem (Second order approximation)

Let $W_0 \in H^s_+(\mathbb{T})$, $s > 1/2$, *be such that the solution of the Szegö equation with initial condition* εW_0 *is bounded by* $\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$.

Denote by v *the solution of the NLW equation on* $\mathbb{T}$ *with initial condition* εW_0.

Let $\mathcal{W} \in C(\mathbb{R}, H^s_+(\mathbb{T}))$ *be the solution of the following equation on* $\mathbb{T}$*:*

$$\begin{cases} i\partial_t \mathcal{W} = \Pi_+(|\mathcal{W}|^2\mathcal{W}) - \Pi_+(|\mathcal{W}|^2 \frac{1}{D}\Pi_-(|\mathcal{W}|^2\mathcal{W})) - \frac{1}{2}\Pi_+(\mathcal{W}^2 \frac{1}{D}\overline{\Pi_-(|\mathcal{W}|^2\mathcal{W})}) \\ \mathcal{W}(0) = \mathcal{W}_0 = \varepsilon W_0. \end{cases}$$

Consider

$$v_{\mathrm{app}}(t) = e^{-i|D|t}\big(\mathcal{W}(t) + F_{\mathrm{osc}}(\mathcal{W}(t), t)\big).$$

Then, if $0 \le t \le \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$, *we have*

$$\|v(t) - v_{\mathrm{app}}(t)\|_{H^s} \le \varepsilon^{5 - C_0\delta}.$$