Global well-posedness of the Gross-Pitaevskii equation on $\mathbb{R}^4$

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Gross–Pitaevskii equation on $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u\big|_{t=0} = u_0, \end{cases}$$

with the non-vanishing boundary condition :

$$\lim_{|x|\to\infty} |u(x)| = 1.$$

Hamiltonian with Ginzburg-Landau energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^d} (|u|^2 - 1)^2 \, dx.$$

Global well-posedness (GWP) results

- $\mathbb{R}$: Zhidkov (1987) in Zhidkov spaces :

$$X^k(\mathbb{R}) := \{u \in L^\infty(\mathbb{R}) : \partial^\alpha u \in L^2(\mathbb{R}), 1 \leq |\alpha| \leq k\}.$$

  Gallo (2004) in the Zhidkov space $X^1(\mathbb{R})$

- $\mathbb{R}^2$, $\mathbb{R}^3$: Béthuel, Saut (1999) in $1 + H^1$
  Goubet (2007) in the Zhidkov space $X^2(\mathbb{R}^2)$
  Gallo (2008) in $u_0 + H^1$ if $E(u_0) < \infty$
  Gérard (2006) in the energy space

$$\mathcal{E}_{\mathrm{GP}} := \{u : E(u) < \infty\}$$

- $\mathbb{R}^4$: Gérard (2006) in the energy space such that $\nabla u \in L^2_{t,\mathrm{loc}} L^4_x$
  in the case of small energy data

Remark : New difficulty on $\mathbb{R}^4$ in the case of large energy data :
the cubic nonlinearity is energy-critical ($\dot{H}^1$-critical)

The energy space

- On $\mathbb{R}^3$, $\mathbb{R}^4$ Gérard (2006) also proved :

$$\mathcal{E}_{\mathrm{GP}}(\mathbb{R}^d) = \left\{ u = \alpha + v \,\middle|\, |\alpha| = 1, v \in \dot{H}^1(\mathbb{R}^d), |v|^2 + 2\,\mathrm{Re}(\bar{\alpha} v) \in L^2(\mathbb{R}^d) \right\}.$$

- $u_0 = \alpha + v_0 \in \mathcal{E}_{\mathrm{GP}}$ implies $u(t) = \alpha + v(t) \in \mathcal{E}_{\mathrm{GP}}$ for all $t \in \mathbb{R}$.

- If $\alpha = e^{i\theta}$, by the gauge invariance $u \mapsto e^{-i\theta} u$, we can assume $\theta = 0$, $\alpha = 1$. Then, $u = 1 + v$ and v satisfies

$$\begin{cases} i\partial_t v + \Delta v = |v|^2 v + 2\,\mathrm{Re}(v) v + |v|^2 + 2\,\mathrm{Re}(v), \\ v\big|_{t=0} = v_0 := u_0 - 1. \end{cases}$$

- On $\mathbb{R}^4$: $v \in \dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4) \implies |v|^2 \in L^2 \implies \mathrm{Re}\, v \in L^2(\mathbb{R}^4)$.

$$\mathcal{E}_{\mathrm{GP}}(\mathbb{R}^4) = \left\{ u = 1 + v : v \in H^1_{\mathrm{real}}(\mathbb{R}^4) + i\dot{H}^1_{\mathrm{real}}(\mathbb{R}^4) \right\}$$

$$E(1+v) = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla v|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^4} \left(|v|^2 + 2\,\mathrm{Re}(v) \right)^2 dx.$$

Global well-posedness for arbitrarily large data on $\mathbb{R}^4$

Theorem (Killip, Oh, P., Vişan 2011)

The Gross–Pitaevskii equation is globally well-posed in the energy space $\mathcal{E}_{\mathrm{GP}}(\mathbb{R}^4)$.

Two ingredients in the proof :

- Global well-posedness of energy-critical defocusing nonlinear Schrödinger equation on $\mathbb{R}^4$:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t w + \Delta w = |w|^2 w \\ w(0) = w_0 \in \dot{H}^1(\mathbb{R}^4). \end{cases}$$

- Perturbation theory : treat the equation of v as

$$\begin{cases} i\partial_t v + \Delta v = |v|^2 v + e, \\ v\big|_{t=0} = v_0, \end{cases}$$

where the error $e := 2\,\mathrm{Re}(v)v + |v|^2 + 2\,\mathrm{Re}(v)$ is small on small intervals.

Scaling invariance

- If w is solution of the cubic NLS equation :

$$i\partial_t w + \Delta w = |w|^2 w$$

  then $w^\lambda(t,x) := \lambda w(\lambda^2 t, \lambda x)$ is also solution of cubic NLS.

- Notice that $\|w_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{1+s-\frac{d}{2}} \|w_0\|_{\dot{H}^s(\mathbb{R}^d)}$

- If $w_0 \in \dot{H}^s(\mathbb{R}^d)$, then the equation is
  - critical if $\|w_0^\lambda\|_{\dot{H}^s} = \|w_0\|_{\dot{H}^s}$, that is $s = s_c = \frac{d}{2} - 1$
  - subcritical if $\|w_0^\lambda\|_{\dot{H}^s} \to \infty$ as $\lambda \to \infty$, that is $s > s_c = \frac{d}{2} - 1$
  - supercritical if $\|w_0^\lambda\|_{\dot{H}^s} \to 0$ as $\lambda \to \infty$, that is $s < s_c = \frac{d}{2} - 1$

- Cubic NLS on $\mathbb{R}^4$ is critical for $w_0 \in \dot{H}^{s_c}(\mathbb{R}^4) = \dot{H}^1(\mathbb{R}^4)$
  Quintic NLS on $\mathbb{R}^3$ is also critical for $w_0 \in \dot{H}^{s_c}(\mathbb{R}^3) = \dot{H}^1(\mathbb{R}^3)$

Stricharz estimates

- Dispersive estimate : $\|e^{it\Delta}w_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}}}\|w_0\|_{L^1(\mathbb{R}^d)}$
- For any interval $I \subset \mathbb{R}$, we define the Strichartz norm

$$\|w\|_{S(I)} = \|w\|_{S(I\times\mathbb{R}^d)} := \sup \|w\|_{L_t^q L_x^r(I\times\mathbb{R}^d)},$$

  where the supremum is taken over all admissible pairs (q,r), $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$
- $N(I \times \mathbb{R}^d)$ denotes the dual space of $S(I \times \mathbb{R}^d)$
- Homogeneous Strichartz estimate :

$$\left\|e^{it\Delta}w_0\right\|_{S(I\times\mathbb{R}^d)} \lesssim \|w_0\|_{L_x^2(\mathbb{R}^d)}$$

- Inhomogeneous Strichartz estimate :

$$\left\|\int_{t_0}^{t} e^{i(t-t')\Delta}F(t')\,dt'\right\|_{S(I\times\mathbb{R}^d)} \lesssim \|F\|_{N(I\times\mathbb{R}^d)}.$$

- Admissible pairs on $\mathbb{R}^4$: $(\infty, 2)$, $(2,4)$, $(6, \frac{12}{5})$
- By the Sobolev embedding $\dot{W}^{1,\frac{12}{5}}(\mathbb{R}^4) \subset L^6(\mathbb{R}^4)$, we have

$$\|w\|_{L_{t,x}^6(I\times\mathbb{R}^4)} \leq \|\nabla w\|_{L_t^6 L^{\frac{12}{5}}(I\times\mathbb{R}^4)}.$$

The energy-critical NLS

- Locally well-posed : Cazenave-Weissler (1989)
- Globally well-posed for small data :
  Duhamel formula :
  $$w(t) = e^{it\Delta}w_0 - i\int_0^t e^{i(t-s)\Delta}|w|^2 w(s)ds$$
  Using Strichartz estimates :
  $$\begin{aligned}\|\nabla w\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)} &\lesssim \|\nabla e^{it\Delta}w_0\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)} + \|w^2\nabla w\|_{L_t^2 L_x^{\frac{4}{3}}(I\times\mathbb{R}^4)} \\ &\lesssim \|\nabla e^{it\Delta}w_0\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)} + \|w\|^2_{L^6_{t,x}(I\times\mathbb{R}^4)}\|\nabla w\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)} \\ &\lesssim \|\nabla e^{it\Delta}w_0\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)} + \|\nabla w\|^3_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)}.\end{aligned}$$
  - if $\|\nabla e^{it\Delta}w_0\|_{L_t^6 L_x^{\frac{12}{5}}([0,T]\times\mathbb{R}^4)}$ is small, we can close the argument :
    T depends on the profile of the initial data w_0, not only on $E(w_0)$
  - Since $\|\nabla e^{it\Delta}w_0\|_{L_t^6 L_x^{\frac{12}{5}}(\mathbb{R}\times\mathbb{R}^4)} \leq \|w_0\|_{\dot{H}^1(\mathbb{R}^4)}$ we can also close the argument when $\|w_0\|_{\dot{H}^1}$ is small $\Longrightarrow$ GWP for small data
- $\|w\|_{L^6_{t,x}([T_{\min},T_{\max}]\times\mathbb{R}^4)} \leq C \Longrightarrow$ global well-posedness

Main results on defocusing energy-critical NLS

- Bourgain (1999) : GWP + scattering, quintic NLS on $\mathbb{R}^3$ with radial data
  - induction on energy
  - localized Morawetz estimate

$$\int_I \int_{|x| \lesssim |I|^{1/2}} \frac{|w|^6}{|x|} dx \lesssim |I|^{1/2} E(u)$$

  by localizing $\int_I \int \frac{|w|^6}{|x|} dx \lesssim (\sup_{t \in I} \|w(t)\|_{\dot{H}^{1/2}})^2$ of Lin and Strauss (1978)

- Grillakis (2000) : global regularity for quintic NLS on $\mathbb{R}^3$ with radial data
- Colliander, Keel, Staffilani, Takaoka, and Tao (2008) : removed the radial assumption on $\mathbb{R}^3$
- Ryckman, Vişan (2007) : GWP and scattering for cubic NLS on $\mathbb{R}^4$
- Vişan (2010) : simpler method for GWP + scattering for cubic NLS on $\mathbb{R}^4$
- Kenig, Merle (2006) : focusing energy-critical NLS on $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$: GWP + scattering for radial data with energy and kynetic energy smaller than those of the stationary solution

Cubic NLS on $\mathbb{R}^4$ (Ryckman, Vişan)

- Goal : prove the existence of a global solution $w \in C_t \dot{H}^1_x \cap L^6_{t,x}$ such that
$$\|w\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \leq C(E(w_0))$$
- Define
$$M(E) := \sup \left\{ \|w\|_{L^6_{t,x}(I\times\mathbb{R}^4)} : I \subset \mathbb{R} \text{ compact}, \ w \text{ solution with } E(u) \leq E \right\}$$
- By contradiction : assume there exists a critical energy E_{crit} such that $M(E_{\text{crit}}) = \infty$
- Then there exists a blowup solution w_* with $E(w_*) = E_{\text{crit}}$,
$$\int_0^{T_{max}} \int_{\mathbb{R}^4} |w_*|^6 dxdt = \int_{T_{min}}^0 \int_{\mathbb{R}^4} |w_*|^6 dxdt = \infty$$
which is almost periodic :
$$\int_{|x-x(t)|\geq C(\eta)/N(t)} |\nabla w_*|^2 dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi|^2 |\hat{w}_*(t,\xi)|^2 d\xi \leq \eta.$$

- Frequency-localized Morawetz inequality (ONLY true for w_*) :

$$\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|P_{\geq N} w_*(t,x)|^2 |P_{\geq N} w_*(t,y)|^2}{|x-y|^3} dxdydt \lesssim \eta N^{-3}.$$

  - obtained by localizing in frequency the interaction Morawetz estimate

$$\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|w(t,x)|^2 |w(t,y)|^2}{|x-y|^3} dxdydt \lesssim \|w\|^3_{L^\infty_t L^2(I\times\mathbb{R}^4)} \|w\|_{L^\infty_t \dot{H}^1(I\times\mathbb{R}^4)}$$

  - $\|P_{\geq N} w_*\|_{L^2} \leq \frac{1}{N} \|P_{\geq N} w_*\|_{\dot{H}^1} \leq \frac{\eta}{N}$ by the frequency localization of w_*

- This shows that

$$\|w_{*,\mathrm{hi}}\|_{L^3_{t,x}(I_0\times\mathbb{R}^4)} \lesssim C\eta^{1/3}$$

- As a consequence, we obtain $\|w_*\|_{L^6_{t,x}} \leq C \Longrightarrow$ contradiction !

Cubic NLS on $\mathbb{R}^4$ (Vişan)

- By contradiction and using concentration-compactness $\Longrightarrow$ there exists a blowup solution w_* which is almost periodic :

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} |\nabla w_*|^2 dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi|^2 |\hat{w}_*(t,\xi)|^2 d\xi \leq \eta.$$

- After rescaling, we can assume $N(t) \geq 1$
- Rapid frequency-cascade scenario : $\int_0^{T_{max}} N(t)^{-1} dt < \infty$

  Quasi-soliton scenario : $\int_0^{T_{max}} N(t)^{-1} dt = \infty$

- In both cases we get a contradiction using long time Strichartz estimates in the spirit of Dodson :

$$\|\nabla P_{\leq M} w_*\|_{L_t^2 L_x^4 (I\times\mathbb{R}^4)} \lesssim 1 + M^{3/2} \Big(\int_I N(t)^{-1} dt \Big)^{1/2}$$

- For the quasi-soliton scenario we also use the frequency-localized Morawetz estimate

Perturbation theory

Lemma (Colliander, Keel, Staffilani, Takaoka, and Tao 2008)

Let I be a compact time interval and v be a solution on $I \times \mathbb{R}^4$ of the perturbed equation $i\partial_t v + \Delta v = |v|^2 v + e$. Suppose

$$\|v\|_{L^6_{t,x}(I\times\mathbb{R}^4)} \le L,$$
$$\|v\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^4)} \le E_0.$$

Then there exists $\varepsilon_0 = \varepsilon_0(E_0, L) > 0$ such that, if for some $0 < \varepsilon \le \varepsilon_0$ and for some $t_0 \in I$ we have

$$\|v(t_0) - w(t_0)\|_{\dot{H}^1(\mathbb{R}^4)} \le \varepsilon$$
$$\|\nabla e\|_{N(I\times\mathbb{R}^4)} \le \varepsilon,$$

then, there exists a solution w to cubic NLS on $I \times \mathbb{R}^4$ with data $w(t_0)$ at time t_0 with the properties

$$\|\nabla w - \nabla v\|_{S(I\times\mathbb{R}^4)} \le C(E_0, L)\varepsilon$$
$$\|\nabla w\|_{S(I\times\mathbb{R}^4)} \le C(E_0, L),$$

where $C(E_0, L) > 0$ is a non-decreasing function of E_0 and L.

Strategy : GWP of Gross-Pitaevskii equation on $\mathbb{R}^4$

- We prove that there exists $T = T(E(v_0))$ such that if the solution v exists on $[0, T]$, then

$$\|v\|_{L^6_{t,x}([0,T]\times\mathbb{R}^4)} \leq C(E(v_0))$$

- This shows that there exists only a finite number of subintervals

  $I_k \subset [0, T]$ such that $[0, T] = \bigcup_{k=1}^{N} I_k$ and

$$\|v\|_{L^6_{t,x}(I_k\times\mathbb{R}^4)} \sim \eta$$

- By a contraction mapping argument, we can then prove local well-posednes on each subinterval I_k, and thus on $[0, T]$
- We built a local solution on $[0, T_1]$, where $T_1 = T = T(E(v_0))$. Next, we build a solution on $[T_1, T_2]$. By conservation of energy we have

$$T_2 - T_1 = T(E(v(T_1))) = T(E(v_0)),$$

  which gives a solution on $[T, 2T]$. Recursively, we extend it to $[0, \infty)$.

Proof

- For w, the solution of cubic NLS, we have $\|\nabla w\|_{L_t^6 L_x^{\frac{12}{5}}(\mathbb{R}\times\mathbb{R}^4)} \leq C(E(v_0))$
- We divide $[0,\infty)$ into $J = J(\eta, E(v_0))$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|\nabla w\|_{L_t^6 L_x^{\frac{12}{5}}(I_j\times\mathbb{R}^4)} \sim \eta$$

- The linear evolution will still be small

$$\|\nabla e^{i(t-t_j)\Delta} w(t_j)\|_{L_t^6 L_x^{\frac{12}{5}}(I_j\times\mathbb{R}^4)} \leq 2\eta$$

On $I_0 = [0, t_1]$ we have :

- By the Strichartz estimates and using $v_0 = v(t_0) = w(t_0)$, we have

$$\begin{aligned}\|\nabla v\|_{L_t^6 L_x^{\frac{12}{5}}(I_0)} &\lesssim \|\nabla e^{it\Delta} v_0\|_{L_t^6 L_x^{\frac{12}{5}}(I_0)} + \|v^2 \nabla v\|_{L_t^2 L_x^{\frac{4}{3}}(I_0)} \\ &\quad + \|v\nabla v\|_{L_t^{\frac{6}{5}} L_x^{\frac{12}{7}}(I_0)} + \|\nabla v\|_{L_t^1 L_x^2(I_0)} \\ &\lesssim 2\eta + \|\nabla v\|^3_{L_t^6 L_x^{\frac{12}{5}}(I_0)} + T^{1/2}\|\nabla v\|^2_{L_t^6 L_x^{\frac{12}{5}}(I_0)} + TE(v_0)^{1/2}\end{aligned}$$

- If $\eta \ll 1$ and $T = T(\eta, E(v_0))$ small enough, then

$$\|v\|_{L^6_{t,x}(I_0)} \leq \|\nabla v\|_{L_t^6 L_x^{\frac{12}{5}}(I_0)} \leq 3\eta \Longrightarrow \text{control on the } L^6_{t,x} - \text{norm}$$

- By the conservation of energy :

$$\|v\|^2_{L_t^\infty \dot{H}^1} \leq 2E(v) = 2E(v_0) \Longrightarrow \text{ control on the } L_t^\infty \dot{H}_x^1 - \text{norm}$$

- $v(0) = w(0) \Longrightarrow$ the initial data on $I_0 = [0, t_1]$ are trivially close

- The error is $e = 2v\,\mathrm{Re}(v) + |v|^2 + 2\,\mathrm{Re}(v)$. Then,

$$\begin{aligned}
\|\nabla e\|_{N(I_0)} &\lesssim \|v \nabla v\|_{L_t^{\frac{6}{5}} L_x^{\frac{12}{7}}(I_0 \times \mathbb{R}^4)} + \|\nabla v\|_{L_t^1 L_x^2(I_0 \times \mathbb{R}^4)} \\
&\lesssim T^{1/2} \|\nabla v\|^2_{L_t^6 L_x^{\frac{12}{5}}(I_0 \times \mathbb{R}^4)} + T E(v_0)^{1/2} \\
&\lesssim T^{1/2} \eta^{1/2} + T E(v_0)^{1/2}
\end{aligned}$$

- If ε and $T = T(\eta, \varepsilon, E(v_0))$ are sufficiently small we have

$$\|\nabla e\|_{N(I_0)} \leq \varepsilon \Longrightarrow \text{ small error}$$

- By the Perturbation Lemma on I_0 : $\|\nabla w - \nabla v\|_{S(I_0)} \leq C(E(v_0))\varepsilon$. Thus,

$$\|w(t_1) - v(t_1)\|_{\dot{H}^1} \leq \|\nabla w - \nabla v\|_{S(I_0)} \leq C(E(v_0))\varepsilon$$

$\Longrightarrow$ the initial data on $I_1 = [t_1, t_2]$ are close

- At each step the bound for $\|w(t_j) - v(t_j)\|_{\dot{H}^1}$ may grow, but its ultimate size is $C(J, E(v_0))\varepsilon$ and we still have

$$C(J, E(v_0))\varepsilon \leq \varepsilon_0, \text{ if } \varepsilon \text{ is sufficiently small}$$

- Recursively, we obtain $\|\nabla v\|_{L_t^6 L_x^{12/5}(I_j)} \leq 3\eta$ for all $1 \leq j \leq J$. Thus,

$$\|v\|_{L^6_{t,x}([0,T]\times\mathbb{R}^4)} \leq \|\nabla v\|_{L_t^6 L_x^{12/5}([0,T]\times\mathbb{R}^4)} \leq J(\eta, E(v_0)) \cdot 3\eta \leq C(E(v_0)).$$

- Finally, notice that we chose $T = T(\varepsilon, \eta, E(v_0)) = T(E(v_0))$

Cubic-quintic NLS with non-vanishing BC on $\mathbb{R}^3$

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)(|u|^2 - r_1^2)u, \\ u\big|_{t=0} = u_0, \end{cases}$$

with $r_1^2 < 1$ and boundary condition :

$$\lim_{|x|\to\infty} |u(x)| = 1.$$

If $u = 1 + v$, then v satisfies

$$\begin{cases} i\partial_t v + \Delta v = |v|^4 v + \mathcal{R}(v), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ v\big|_{t=0} = v_0. \end{cases}$$

where

$$\mathcal{R}(v) = |v|^4 + 4|v|^2 v \operatorname{Re}(v) + 4|v|^2 \operatorname{Re}(v) + \gamma |v|^2 v + 4v \operatorname{Re}(v)^2 \\ + \gamma |v|^2 + 4 \operatorname{Re}(v)^2 + 2\gamma v \operatorname{Re}(v) + 2\gamma \operatorname{Re}(v).$$

and $\gamma = 1 - r_1^2 > 0$.
The quintic nonlinearity is energy-critical in $\mathbb{R}^3$.

Global well-posedness of cubic-quintic NLS on $\mathbb{R}^3$

Hamiltonian equation of energy

$$E(1+v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2\,dx + \frac{\gamma}{4}\int_{\mathbb{R}^3}\left(|v|^2 + 2\,\mathrm{Re}(v)\right)^2 dx + \frac{1}{6}\int_{\mathbb{R}^3}\left(|v|^2 + 2\,\mathrm{Re}(v)\right)^3 dx.$$

The energy is NOT sign definite $\Longrightarrow$ it does NOT control $\|v\|_{\dot{H}^1}$.

However, we will see that $E(v)$ and $\|\mathrm{Re}(v)\|_{L^2}$ control $\|v\|_{\dot{H}^1}$.

We then define the energy space as :

$$\mathcal{E}_{\mathrm{CQ}} := \{u = 1 + v : |E(1+v)| < \infty, \|\mathrm{Re}(v)\|_{L^2(\mathbb{R}^3)} < \infty\}.$$

Theorem (Killip, Oh, P., Vişan 2011)

The cubic-quintic NLS is globally well-posed in the energy space $\mathcal{E}_{\mathrm{CQ}}(\mathbb{R}^3)$.

Proof : As before we treat our equation as a perturbation of the energy-critical quintic NLS on $\mathbb{R}^3$.

Control of $\|v\|_{\dot{H}^1(\mathbb{R}^3)}$

Lemma

There exists $C_0 = C_0(\gamma) > 0$ *such that*

$$\int_{\mathbb{R}^3} |\nabla v|^2 \, dx + \int_{\mathbb{R}^3} |v|^6 \, dx + \gamma \int_{\mathbb{R}^3} |v|^4 \, dx \lesssim E(v) + C_0 \int_{\mathbb{R}^3} |\mathrm{Re}(v)|^2 \, dx.$$

Idea of proof :

$$|v|^4 \leq 2\big(|v|^2 + 2\,\mathrm{Re}(v)\big)^2 + 8|\mathrm{Re}(v)|^2 \leq 8\Big[\tfrac{1}{4}\big(|v|^2 + 2\,\mathrm{Re}(v)\big)^2 + |\mathrm{Re}(v)|^2\Big].$$

Corollary

$$\mathcal{E}_{CQ}(\mathbb{R}^3) = 1 + \big(H^1_{\mathrm{real}}(\mathbb{R}^3) + i\dot{H}^1_{\mathrm{real}}(\mathbb{R}^3)\big) \cap L^4(\mathbb{R}^3).$$

Corollary

$$\|v\|^2_{L^\infty_t \dot{H}^1_x(\mathbb{R}\times\mathbb{R}^3)} \leq M(v) := E(v) + C_0 \int_{\mathbb{R}^3} |\mathrm{Re}(v)|^2$$

<u>Problem</u> : $M(v)$ is not a conserved quantity

$M(v)$ can be controled on any finite time interval

Lemma

Let v be a solution to the cubic-quintic NLS on a time interval $[0, \tau]$. Then,

$$M(v(t)) \leq M(v_0)e^{C_1 t}$$

for all $t \in [0, \tau]$ and some $C_1 > 0$.

Proof : We have that

$$\partial_t \int_{\mathbb{R}^3} |\mathrm{Re}(v)|^2 \, dx = 2 \int_{\mathbb{R}^3} \mathrm{Re}(v) \, \partial_t \, \mathrm{Re}(v) \, dx$$

$$= -2 \int_{\mathbb{R}^3} \mathrm{Re}(v) \, \mathrm{Im}(\Delta v) \, dx + 2 \int_{\mathbb{R}^3} \mathrm{Re}(v) \, \mathrm{Im} \left(|v|^4 v + \mathcal{R}(v) \right) dx$$

By interpolation and Young's inequality, we bound the RHS by $C_1 M(v)$.

Strategy in the case of cubic-quintic NLS

- We rely on

$$\|v\|_{L^\infty \dot{H}^1([0,\tau]\times\mathbb{R}^3)} \leq M(v_0)e^{C_1\tau} = C\big(E(v_0), \|\operatorname{Re}(v_0)\|_{L^2}, \tau\big) =: N(v_0, \tau).$$

- Goal : for and $0 < \tau < \infty$, there exists a solution on $[0, \tau]$. We do not attempt to prove directly the existence of a solution on $[0, \infty)$, as we did for the Gross-Pitaevskii equation.

- We prove that if such a solution exists, then there is $T = T(N(v_0, \tau)) > 0$ such that

$$\|\nabla v\|_{S([T_0, T_0+T]\times\mathbb{R}^3)} \leq C(N(v_0, \tau))$$

as long as $[T_0, T_0 + T] \subset [0, \tau]$.

Scattering for the GP equation in the case of large data

- Gross-Pitaevskii equation possesses traveling waves solutions

$$u(x,t) = u_0(x_1 - ct, x_2, x_3, x_4)$$

  that do NOT scatter

- The formation of traveling waves requires a minimal energy in $\mathbb{R}^d$, $d \geq 3$ (Bethuel, Gravejat, Saut 2009, de Laire 2009)

- Solutions with sufficiently small energy scatter (Gustafson, Nakanishi, Tsai 2006)

- Can one prove scattering up to the minimal energy of a traveling wave ?

Work in progress : Scattering for cubic-quintic NLS

- Cubic-Quintic NLS with zero boundary condition :

$$\text{(CQ)} \qquad \begin{cases} i\partial_t v + \Delta v = |v|^4 v - |v|^2 v \\ v(0) = v_0 \in H^1(\mathbb{R}^3) \end{cases}$$

- Conserved quantities – mass and energy :

$$M(v) = \int |v|^2 dx, \qquad E(v) = \int \Big(\frac{|\nabla v|^2}{2} + \frac{|v|^6}{6} - \frac{|v|^4}{4} \Big) dx.$$

- Soliton solutions of the form $v(t,x) = e^{i\omega t} u(x)$, where u satisfies

$$\Delta u - |u|^4 u + |u|^2 u - \omega u = 0.$$

Theorem (Work in progress with Killip, Oh, Vişan)

If $v_0 \in H^1(\mathbb{R}^3)$ has positive energy, smaller than the energy of any soliton of mass $M(v_0)$, then the corresponding solution of the (CQ) equation scatters.