A two-soliton with transient turbulent regime for a focusing cubic nonlinear half-wave equation on $\mathbb{R}$

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Dispersion relation

- general linear evolution equation on $\mathbb{R}$:

$$(*) \qquad \partial_t u + P\left(\frac{\partial}{\partial x}\right) u = 0$$

where $P : \mathbb{C} \mapsto \mathbb{C}$ is such that $P(i\mathbb{R}) \subset i\mathbb{R}$.

- plane wave solution $u(x,t) = e^{i(kx-\omega t)} = e^{ik(x-\frac{\omega}{k}t)}$ with $k \in \mathbb{R}$ the wave number, $\omega \in \mathbb{R}$ the angular frequency, and phase velocity $\frac{\omega}{k}$:

$$-i\omega u + P(ik)u = 0$$

- dispersion relation:

$$\text{Phase velocity at wave number } k = \frac{\omega}{k} = \frac{P(ik)}{ik}$$

- using the inverse Fourier transform $u_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\hat{u}_0(k)dk$ and by superposition

$$u(x,t) = \int_{\mathbb{R}} e^{ik\left(x-\frac{P(ik)}{ik}t\right)}\hat{u}_0(k)dk$$

is a solution of $(*)$ with $u(x,0) = u_0(x)$.

- $P(k) = ck, c \in \mathbb{R} \Longrightarrow$ linear advection equation: $\partial_t u + c\partial_x u = 0$

$$\frac{\omega}{k} = \frac{P(ik)}{ik} = c \Longrightarrow \text{all plane waves move with the same velocity } c$$

- $P(k) = -ik^2 \Longrightarrow$ linear Schrödinger equation: $i\partial_t u + \partial_x^2 u = 0$
  - phase velocity:

$$\frac{\omega}{k} = \frac{P(ik)}{ik} = k \Longrightarrow e^{ik\left(x - \frac{P(ik)}{ik}t\right)}\hat{u}_0(k) \text{ at frequency } k \text{ moves faster than those corresponding to smaller frequencies}$$

  - $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ and decay estimate:

$$\|u(t)\|_{L^\infty} \leq \frac{C}{\sqrt{t}}\|u(0)\|_{L^1},$$

  - dispersive equation

- $P(k) = i|k| \Longrightarrow$ linear half-wave equation: $i\partial_t u - |D|u = 0$

$$\frac{\omega}{k} = \frac{P(ik)}{ik} = \frac{|k|}{k} \Longrightarrow \text{phase velocity 1 for positive frequencies and } -1 \text{ for negative frequencies} \Longrightarrow \text{weak dispersion}$$

Example of a nonlinear dispersive PDE

- (defocusing) nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u - \Delta u + |u|^{p-1}u = 0 \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^d) \end{cases}, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}.$$

- Definition of a solution (Duhamel's formula):

$$u(x,t) = e^{-it\Delta}u_0 - i\int_0^t e^{-i(t-t')\Delta}|u|^{p-1}u(x,t')dt',$$

  Here, $e^{-it\Delta}f$ denotes the solution of the linear Schrödinger equation $i\partial_t v - \Delta v = 0$ with $v(0) = f$.

- NLS is a Hamiltonian PDE

$$\partial_t u = i\frac{\partial E}{\partial \bar{u}}$$

  with Hamiltonian

$$E(u(t)) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}dx.$$

- Conservation of Hamiltonian $E(u(t)) = E(u(0))$ and $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ $\Longrightarrow \|u(t)\|_{H^1} \leq C(u_0)$ for all t

Typical problems in the study of dispersive PDEs:

- Local well-posedness (existence, uniqueness of the solution in a space X for a short time $T > 0$, and continuous dependence on the initial data)
- Existence of solutions that blow up in finite time ($T < \infty$)
- Global well-posedness ($T = \infty$)
- Behaviour of global-in-time solutions
    - Scattering: a solution of the nonlinear equation asymptotically behaves like a *linear* solution
    - Solitons: special global solutions of the form $P_{c,\omega}(t,x) = u_0(x - ct)e^{it\omega}$
      Note: $\|P_{c,\omega}(t)\|_{L^\infty} = \|u_0\|_{L^\infty}$ and $\|P_{c,\omega}(t)\|_{H^s} = \|u_0\|_{H^s}$ for all t and s
    - Soliton resolution: solutions decompose into a finite sum of solitons and radiation as $T \to \infty$, in particular $\|u(t)\|_{H^s} \leq C_s$ for all t and s
    - "Weak turbulence" expressed as $\limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty$ for large s

Cubic half-wave equation on $\mathbb{R}$

- Focusing half-wave equation:

(HW) $$i\partial_t u - |D|u = -|u|^2 u, \quad x \in \mathbb{R}, \quad u(t,x) \in \mathbb{C},$$

where $\widehat{|D|f}(\xi) = |\xi|\hat{f}(\xi)$.

- PDEs with nonlocal dispersion appear in physics:
  - models of wave turbulence (Majda-McLaughlin-Tabak 1997),
  - continuum limit of lattice points, gravitational collapse.

- Applying $i\partial_t + |D|$ to both sides of HW $\Longrightarrow$ a nonlinear *wave* equation:

$$\partial_t^2 u - \Delta u = -|u|^4 u + 2|u|^2 |D|u + [|D|, u^2]\bar{u}$$

- HW is a Hamiltonian PDE, $\partial_t u = i\frac{\partial E}{\partial \bar{u}}$, with

$$E(u(t)) := \frac{1}{2}\||D|^{\frac{1}{2}} u(t)\|_{L^2}^2 - \frac{1}{4}\|u(t)\|_{L^4}^4$$

- Conserved Hamiltonian/energy $E(u(t)) = E(u(0))$ and mass:

$$M(u(t)) := \|u(t)\|_{L^2}^2 = M(u(0))$$

Well-posedness theory

- Gérard-Grellier 2010, Krieger-Lenzmann-Raphaël 2012:
  local well-posedness in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$ with blowup alternative
  $$T < \infty \text{ implies } \lim_{t \nearrow T} \|u(t)\|_{H^{\frac{1}{2}}} = \infty$$
- L^2-critical equation, i.e. invariant under the scaling symmetry
  $$u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda t, \lambda x),$$
  which leaves the L^2-norm invariant $\|u_\lambda(t,\cdot)\|_{L^2} = \|u(\lambda^2 t, \cdot)\|_{L^2}$
- Best constant in the Gagliardo-Nirenberg inequality:
  $$\|u\|_{L^4}^4 \leq C_* \||D|^{\frac{1}{2}} u\|_{L^2}^2 \|u\|_{L^2}^2,$$
  is attained by the ground state W and $C_* = \frac{2}{\|W\|_{L^2}^2}$
- **Ground state** $W \in H^{\frac{1}{2}}(\mathbb{R})$: the unique (Frank-Lenzmann 2013) positive, radially symmetric solution of $|D|W + W - |W|^2 W = 0$
- By energy and mass conservation:
  $$E(u_0) = E(u(t)) \geq \frac{1}{2}\left(1 - \frac{\|u_0\|_{L^2}^2}{\|W\|_{L^2}^2}\right) \||D|^{\frac{1}{2}} u(t)\|_{L^2}^2$$
  $\Longrightarrow$ for $u_0 \in H^{\frac{1}{2}}$ with $\|u_0\|_{L^2} < \|W\|_{L^2} \Longrightarrow \|u(t)\|_{H^{\frac{1}{2}}} \leq C(E(u_0), M(u_0))$, $\forall t$

Well-posedness theory (continued)

Krieger-Lenzmann-Raphaël 2012:

- Global well-posedness in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$, for $\|u_0\|_{L^2} < \|W\|_{L^2}$
- HW admits minimal mass blowup solutions (solutions that stop existing in finite time of minimal mass $\|u\|_{L^2} = \|W\|_{L^2}$)

Approximation by the cubic Szegő equation on $\mathbb{R}$

Cubic Szegő equation on $\mathbb{R}$:

$$i\partial_t v = \Pi_+(|v|^2 v), \qquad \text{where} \qquad \widehat{\Pi_+ f}(\xi) = \mathbf{1}_{\xi \geq 0}\hat{f}(\xi).$$

- Introduced by Gérard-Grellier 2008: mathematical toy model of a non-dispersive nonlinear Hamiltonian PDE
- Hamiltonian PDE with Hamiltonian $E(u) = \|u\|_{L^4}^4$.
  Also conserves mass $M(u) := \|u\|_{L^2}^2$ and momentum $P(u) := (Du, u)$
- Globally well-posed in H_+^s, $s \geq \frac{1}{2}$:
  $$H_+^s(\mathbb{R}) := \{f \in H^s(\mathbb{R}) : \text{supp } \hat{f} \subset [0, \infty)\}$$
- Completely integrable model $\Longrightarrow$ significant information is available
- Infinitely many conservation laws, all controlled by the $H^{\frac{1}{2}}$-norm
  $\Longrightarrow$ no information about higher Sobolev norms

Theorem (P. 2013, Approximation of HW by the Szegő equation)

For well-prepared initial data $u(0) \in H_+^s$ with $s \geq 1$, $\|u(0)\|_{H^s} = \varepsilon \ll 1$, HW is approximated in $H^s(\mathbb{R})$ by the Szegő equation **for a long time**.

- By Duhamel's formula

$$u(t) = e^{-it|D|}u_0 - i\int_0^t e^{-i(t-t')|D|}(|u|^2u)(t')dt'$$

- With $z(t) := e^{it|D|}u(t)$ the interaction representation:

$$z(t) = u_0 - i\int_0^t e^{it'|D|}|e^{-it'|D|}z(t')|e^{-it'|D|}z(t')dt'$$

- Taking the Fourier transform of both sides:

$$\hat{z}(\xi,t) = \hat{u}_0(\xi) - i\int_0^t \iint_{\xi_1-\xi_2+\xi_3=\xi} e^{it'(|\xi|-|\xi_1|+|\xi_2|-|\xi_3|)}\hat{z}(\xi_1,t')\overline{\hat{z}(\xi_2,t')}\hat{z}(\xi_3,t')d\xi_2 d\xi$$

$$= \hat{u}_0(\xi) + i\int_0^t \iint_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi|>0}\frac{e^{it'\Phi}}{i\Phi}\partial_t\left[\hat{z}(\xi_1,t')\overline{\hat{z}(\xi_2,t')}\hat{z}(\xi_3,t')\right]d\xi_2 d\xi_3 dt'$$

$$- i\int_0^t \iint_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{\Phi=0}\hat{z}(\xi_1,t')\overline{\hat{z}(\xi_2,t')}\hat{z}(\xi_3,t')d\xi_2 d\xi_3 dt'$$

- Resonant frequencies for HW:

$$\Phi := |\xi| - |\xi_1| + |\xi_2| - |\xi_3| = 0 \quad \text{and} \quad \xi - \xi_1 + \xi_2 - \xi_3 = 0$$

$\Longrightarrow$ ξ, ξ_1, ξ_2, ξ_3 have the same sign

- The cubic Szegő equation is the resonant equation corresponding to HW
- Hence, heuristically speaking, the dynamics of HW is dictated for a long time by that of the Szegő equation

Weak turbulence – Growth of high Sobolev norms

- weak turbulence: out-of-equilibrium statistics of random nonlinear waves
- it appeared in plasma physics, water waves: Zakharov 1960s
- similar to the hydrodynamical turbulence of Kolmogorov
- in the physical space: dynamics moves to smaller and smaller scales causing a chaotic behaviour
- "forward energy cascade": energy moves from lower frequencies to higher and higher frequencies
- the energy cascade implies growth of high Sobolev norms

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = \limsup_{t\to\infty} \left\| \langle \xi \rangle^s \hat{u}(t,\xi) \right\|_{L^2} = \infty \text{ for } s \text{ large}$$

Results on growth of high Sobolev norms

Defocusing nonlinear Schrödinger equations on $\mathbb{T}^d$:

(NLS) $$i\partial_t u + \Delta u = |u|^{p-1}u.$$

- Conservation laws $\Longrightarrow$ the H^1-norm is bounded in time
- What happens to H^s-norms for $s > 1$?
- **Upper bounds**: Bourgain 1996, Staffilani 1997, Sohinger 2010, Colliander-Kwon-Oh 2012
$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{c(s-1)}$$
- **Examples of growing solutions**: Bourgain 1995, 1996, 2004, Kuksin 1997
- **Colliander-Keel-Staffilani-Takaoka-Tao** 2010: cubic NLS on $\mathbb{T}^2$: arbitrarily large growth in finite time:
  - Fix $s > 1$. For any $\varepsilon \ll 1$ and any $N \gg 1$, there exists $T > 0$ and a solution u of NLS such that
$$\|u(0)\|_{H^s} \leq \varepsilon, \quad \|u(T)\|_{H^s} \geq N.$$
  - Hani 2011, Guardia-Kaloshin 2012, Guardia 2012, Hani-Pausader-Tzvetkov-Visciglia 2013, Haus-Procesi 2014, Guardia-Haus-Procesi 2015
  - The behaviour of the solution for $t > T$ remains unknown

Solitons are special solutions of the form $u(t,x) = u_0(x-ct)e^{-i\omega t}$.

Theorem (P. 2011, Classification of solitons on Szegő equation)

Solitons for the Szegő equation on $\mathbb{R}$:

$$u(t,x) = \alpha Q^+\Big(\frac{x - X - ct}{\lambda}\Big)e^{-i(\gamma + \frac{2c^2}{\lambda}t)},$$

where $(\alpha, X, \lambda, c, \gamma) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}$ and $Q^+(x) = \frac{1}{x+\frac{i}{2}}$ satisfies

$$DQ^+ + Q^+ = \Pi_+(|Q^+|^2 Q^+).$$

- All the solitons of the Szegő equation on $\mathbb{R}$ are **rational functions with one simple pole in the lower half-plane**

Growth of high Sobolev norms for Szegő equation on $\mathbb{R}$

Theorem (P. 2011, Infinite growth for Szegő equation on $\mathbb{R}$)

There exists a modulated two-soliton solution of the cubic Szegő equation on $\mathbb{R}$:

$$u(t,x) := \alpha_1(t) Q^+\left(\frac{x - x_1(t)}{\lambda_1(t)}\right) e^{-i\gamma_1(t)} + \alpha_2(t) Q^+\left(\frac{x - x_2(t)}{\lambda_2(t)}\right) e^{-i\gamma_2(t)} + \varepsilon(t,x),$$

with $\lim_{t\to\infty} \|\varepsilon(t,\cdot)\|_{H^s} = 0$, such that

$$\|u(t,\cdot)\|_{H^s} \sim t^{2s-1} \to \infty \quad \text{as} \quad t \to \infty, \quad s > 1/2.$$

In particular,

$$\alpha_1(t) \sim 1, \quad \lambda_1(t) \sim 1, \quad x_1(t) \sim t$$
$$\alpha_2(t) \sim 1, \quad \lambda_2(t) \sim \frac{1}{t^2}, \quad x_2(t) = O(1).$$

- consequence of the complete integrability of the Szegő equation
- due to **multiplicity of eigenvalues** of a Hankel operator
- Gérard-Grellier 2010-2015: growth for the Szegő equation on $\mathbb{T}$ is generic

Finite time growth of high Sobolev norms for HW on $\mathbb{R}$

- Upper bound, Thirouin 2015: $\|u(t)\|_{H^1} \leq Ce^{Ct^2}$.

Theorem (P. 2013, CKSTT-type result)

Let $0 < \varepsilon \ll 1$. There exists a solution of HW such that

$$\|u(0)\|_{H^1} = \varepsilon \ll 1 \quad \text{and} \quad \|u(T)\|_{H^1} \geq \frac{1}{\varepsilon} \gg 1,$$

where $T \sim e^{\frac{c}{\varepsilon^3}}$.

Proof: Combines:

- growth of high Sobolev norms for the Szegő equation,
- long time approximation of HW by the Szegő equation,
- HW is a L^2-critical equation

Remarks:

- Gérard-Grellier: Analogous growth result for HW on $\mathbb{T}$
- The behaviour of u for $t > T$ remains unknown

Main result

Theorem (Gérard, Lenzmann, P., Raphaël, 2016)

Let $0 < \varepsilon \ll 1$. There exist $T > 0$ and a solution of HW such that

$$\|u(0)\|_{H^1} = \varepsilon \ll 1 \quad \text{and} \quad \|u(t)\|_{H^1} \geq \frac{1}{\varepsilon} \gg 1 \quad \text{for all } t \geq T.$$

Main result restated

Theorem (Gérard, Lenzmann, P., Raphaël, 2016)

There exist $0 < \delta, \eta_* \ll 1$ universal constants such that the following hold. For all $0 < \eta < \eta_*$, we define the times:

$$1 \ll T_{\text{in}} = \frac{1}{\eta^{2\delta}} \ll T_\eta^- = \frac{\delta}{\eta}.$$

Then, there exists a modulated two-soliton solution $u \in \mathcal{C}([T_{\text{in}}, +\infty), H^1)$ of HW with:

- *turbulent regime*: for $t \in [T_{\text{in}}, T_\eta^-]$ the H^1-norm grows:

$$\|u(t)\|_{H^1} = \frac{t^2}{\eta}(1 + O(\sqrt{\delta}))$$

- *saturation regime*: $\|u(t)\|_{H^1} = \frac{1}{\eta^3} e^{O(\frac{1}{\delta})}$ for all $t \in [T_\eta^-, \infty)$.

Mass-subcritical solitons for HW

- Krieger-Lenzmann-Raphaël 2012: for $\beta \in (0,1)$ there exists a soliton for HW:
$$u_\beta(t,x) = Q_\beta\Big(\frac{x-\beta t}{1-\beta}\Big)e^{-it}, \qquad \text{where} \quad \frac{|D|-\beta D}{1-\beta}Q_\beta + Q_\beta - |Q_\beta|^2 Q_\beta = 0$$

- in the singular relativistic limit $\beta \to 1$, the equation for Q_β reduces to the equation of the Szegő profile Q^+:
$$DQ + Q - \Pi_+(|Q|^2 Q) = 0, \qquad Q = \Pi_+(Q)$$

- there exists a unique family of solitons u_β with $\lim_{\beta \nearrow 1} Q_\beta = Q^+$ in H^s, $s \geq 0$

- solitons u_β have arbitrarily small mass as $\beta \to 1$:
$$\|u_\beta\|_{L^2} \sim \sqrt{1-\beta}\|Q^+\|_{L^2} \to 0$$

- focusing L^2-critical NLS does not admit mass-subcritical solitons: all solutions with a subcritical mass scatter (Dodson 2011, Killip-Tao-Vişan 2009, Killip-Vişan-Zhang 2008)

Growth mechanism

- the modulated two-soliton solution:

$$u(t,x) = \frac{e^{-i\gamma_1(t)}}{\lambda_1^{\frac{1}{2}}(t)} Q_{\beta_1(t)}\left(\frac{x-x_1(t)}{\lambda_1(t)(1-\beta_1(t))}\right) + \frac{e^{-i\gamma_2(t)}}{\lambda_2^{\frac{1}{2}}(t)} Q_{\beta_2(t)}\left(\frac{x-x_2(t)}{\lambda_2(t)(1-\beta_2(t))}\right) + \varepsilon(t,x)$$

  with $\|\varepsilon(t,\cdot)\|_{H^1} \lesssim \frac{1}{\sqrt{N}t^{\frac{N}{8}}}$, for some $N \gg 1$.

- the mechanism for turbulence is the concentration of the second soliton:

$$1-\beta_2(t) = \frac{\eta}{t^2}(1+O(\sqrt{\delta})) \quad \text{for} \quad T_{\text{in}} \le t \le T_\eta^-,$$

- stabilization for large times: $1-\beta_2(t) = \eta^3 e^{O(\frac{1}{\delta})}$ for $t \ge T_\eta^-$

- the first soliton remains unchanged under the evolution

Comments

- The rate of concentration is explicit and consistent with that for the Szegő eqn.

- **No infinite growth as in the case of the Szegő equation !**

- The two solitons interact strongly in the turbulent regime:
$$|x_2 - x_1| \ll 1,$$
but drift away from each other over time, in the saturation regime:
$$|x_2 - x_1| \sim \eta t \geq \delta.$$

- For the Szegő equation: $Q^+(x) \sim \frac{1}{\langle x \rangle}$

- For HW, Q_β decays faster:
$$Q_\beta(x) \sim \frac{1}{\langle x \rangle (1 + (1-\beta)\langle x \rangle)}$$
$\Longrightarrow$ the interaction between the two waves weakens from $\frac{1}{R}$ to $\frac{1}{R^2}$ in the stationary regime, where $R := \frac{x_2 - x_1}{\lambda_1(1-\beta_1)} \gg 1$.

Method of proof: modulation analysis

- used by many authors: Buslaev-Perelman, Merle, Raphaël, Martel, Rodnianski, Krieger, Schlag, Tataru, Chiron-Rousset, ...
- successfully used to *construct finite time blowup solutions* for NLS: Merle-Raphaël 2004, 2005
- also used to construct multi-soliton solutions: Merle 1990, Martel 2005, Martel-Merle 2006, ...
- here we generalize the strategy developed by Krieger, Martel, and Raphaël (2009) to build a nondispersive two-soliton for the Hartree equation
- **first instance** when modulation analysis is used to prove growth of high Sobolev norms

Strategy of proof

- **Step I**: Construction of an approximate solution, using modulation analysis

$$u^{\text{app}} = u_1 + u_2,$$

where

$$u_j(t,x) := \frac{1}{\lambda_j^{\frac{1}{2}}(t)} V_j^{(N)}\Big(y_j := \frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))}, \mathcal{P}(t)\Big) e^{i\gamma_j(t)}$$

with $\mathcal{P}(t) = (\lambda_1, \lambda_2, \beta_1, \beta_2, \Gamma := \gamma_2 - \gamma_1, x_2 - x_1)$, such that:

$$i\partial_t u_j - |D|u_j + u_j|u_j|^2 = O\Big(\frac{1}{t^N \langle y_j \rangle}\Big), \quad N \gg 1$$

- **Step II**: Study of the finite system of ODEs satisfied by the modulation parameters: $\lambda_j(t), \beta_j(t), x_j(t), \gamma_j(t)$, $j = 1, 2$

- **Step III**: Construction of the exact solution
  - write the exact solution as "u^{app} + remainder"
  - control of remainder: energy estimate for a localized energy functional around two-solitons

Proof: I. Construction of an approximate solution

$$u^{\text{app}}(t,x) := u_1(t,x) + u_2(t,x) = \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{1}{2}}(t)} V_j^{(N)}\Big(y_j := \frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))}, \mathcal{P}(t)\Big) e^{i\gamma_j(t)},$$

where $\mathcal{P}(t) = \Big(\lambda_1, \lambda_2, \beta_1, \beta_2, \Gamma := \gamma_2 - \gamma_1, R := \frac{x_2 - x_1}{\lambda_1(1-\beta_1)}\Big)$.

$$u|u|^2 \quad = \quad u_1(|u_1|^2 + 2|u_2|^2 + u_1\bar{u_2}) + u_2(|u_2|^2 + 2|u_1|^2 + u_2\bar{u_1})$$

Using a cut off: $\chi(x) = 1$ for $|x| \leq \frac{1}{4}$, support$(\chi) \subset [-\frac{1}{2}, \frac{1}{2}]$

$$\chi_R(x) = \chi\left(\frac{y_1}{R}\right) = \chi(1 + \frac{\mu b}{R} y_2),$$

where $\mu = \frac{\lambda_2}{\lambda_1}$ and $b = \frac{1-\beta_2}{1-\beta_1}$.

On supp (χ_R): $|\frac{y_1}{R}| \leq \frac{1}{2} \Longleftrightarrow |x - x_1| \leq \frac{|x_2 - x_1|}{2} \Longrightarrow$ we are "close" to the first soliton

$$i\partial_t u^{\text{app}} - |D|u^{\text{app}} - u^{\text{app}}|u^{\text{app}}|^2 = \sum_{j=1}^{2} \frac{1}{\lambda_j(t)^{\frac{3}{2}}} \mathcal{E}_j^{(N)}\big(y_j(t), \mathcal{P}(t)\big) e^{i\gamma_j(t)}.$$

$$
\begin{aligned}
\mathcal{E}_1^{(N)} &= i\partial_t \mathcal{P}\cdot\nabla_{\mathcal{P}} V_1^{(N)} - \frac{(|D|-\beta_1 D)V_1^{(N)}}{1-\beta_1} - V_1^{(N)} + V_1^{(N)}|V_1^{(N)}|^2 - iM_1^{(N)}\Lambda V_1^{(N)} \\
&- \frac{i}{1-\beta_1}[(x_1)_t - \beta_1]\partial_{y_1} V_1^{(N)} + iB_1^{(N)} y_1 \partial_{y_1} V_1^{(N)} - [\lambda_1(\gamma_1)_t - 1]V_1^{(N)} \\
&+ \chi_R \left[\frac{2}{\mu} V_1^{(N)}|V_2^{(N)}|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}}(V_1^{(N)})^2\overline{V_2^{(N)}} + \frac{2e^{i\Gamma}}{\sqrt{\mu}}|V_1^{(N)}|^2 V_2^{(N)} + \frac{e^{2i\Gamma}}{\mu}\overline{V_1^{(N)}}(V_2^{(N
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_2^{(N)} &= i\partial_t \mathcal{P}\cdot\nabla_{\mathcal{P}} V_2^{(N)} - \frac{(|D|-\beta_2 D)V_2^{(N)}}{1-\beta_2} - V_2^{(N)} + V_2^{(N)}|V_2^{(N)}|^2 - iM_2^{(N)}\Lambda V_2^{(N)} \\
&- \frac{i}{1-\beta_2}[(x_2)_t - \beta_2]\partial_{y_2} V_2^{(N)} + iB_2^{(N)} y_2 \partial_{y_2} V_2^{(N)} - [\lambda_2(\gamma_2)_t - 1]V_2^{(N)} \\
&+ (1-\chi_R)\left[2\sqrt{\mu}e^{-i\Gamma} V_1^{(N)}|V_2^{(N)}|^2 + \mu e^{-2i\Gamma}(V_1^{(N)})^2\overline{V_2^{(N)}} + 2\mu V_2^{(N)}|V_1^{(N)}|^2 + \sqrt{\mu
\end{aligned}
$$

where $\Lambda_x = x\partial_x$ and we set

$$
(\lambda_j)_t =: M_j^{(N)}(\mathcal{P}), \qquad \frac{(\beta_j)_t}{1-\beta_j} =: \frac{B_j^{(N)}(\mathcal{P})}{\lambda_j}
$$

We look for solutions of

$$
\mathcal{E}_1 = \mathcal{E}_2 = 0
$$

The time dependence of the parameters of translation and phase is frozen:

$$
(x_j)_t = \beta_j, \qquad (\gamma_j)_t = \frac{1}{\lambda_j}
$$

Expansion of the approximate solution

Equation $\mathcal{E}_j = 0$ writes:

$$\frac{(|D| - \beta_j D)V_j^{(N)}}{1-\beta_j} + V_j^{(N)} - V_j^{(N)}|V_j^{(N)}|^2 = i\partial_t \mathcal{P} \cdot \nabla_{\mathcal{P}} V_j^{(N)} + \dots$$

Expansion:

$$V_j^{(N)} = Q_{\beta_j} + \sum_{n=1}^{N} T_{j,n}(y_j, \mathcal{P}), \quad M_j^{(N)}(\mathcal{P}) = \sum_{n=0}^{N} M_{j,n}(\mathcal{P}), \quad B_j^{(N)} = \sum_{n=0}^{N} B_{j,n}(\mathcal{P})$$

Case $n=0$: $T_{j,0} = Q_{\beta_j}(y_j)$, $\quad M_{j,0} = B_{j,0} = 0, \quad j = 1,2$

$$|\mathcal{E}_1^{(0)}(y_1)| = \chi_R \left| \frac{2}{\mu} Q_{\beta_1}(y_1)|Q_{\beta_2}(y_2)|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}} Q_{\beta_1}^2 \overline{Q_{\beta_2}} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}} |Q_{\beta_1}|^2 Q_{\beta_2} + \frac{e^{2i\Gamma}}{\mu} \overline{Q_{\beta_1}} Q_{\beta_2}^2 \right|$$

$$\lesssim \chi_R \frac{1}{\langle y_1 \rangle \langle y_2 \rangle} \lesssim \frac{1}{R\langle y_1 \rangle} \sim \frac{1}{t\langle y_1 \rangle}, \quad \text{since } |y_2| \gg R \text{ on the support of } \chi_R$$

Case $n=1$: Elliptic equation for $T_{j,1}$:

$$\mathcal{L}_{\beta_j} T_{j,1} = -iM_{j,1} \Lambda Q_{\beta_j} + iB_{j,1}\left[y_j \partial_y Q_{\beta_j} + (1-\beta_j)\partial_{\beta_j} Q_{\beta_j}\right] + \text{ remainder}$$

where $\Lambda f = \frac{f}{2} + y\partial_y f$ and $\mathcal{L}_{\beta_j}$ is the **linearized operator around** Q_{β_j}:

$$\mathcal{L}_{\beta_j} f := \frac{|D| - \beta D}{1-\beta_j} f + f - 2|Q_{\beta_j}|^2 f - Q_{\beta_j}^2 \overline{f}$$

- based on explicit computations for the Szegő equation (P. 2012):
  - $\ker \mathcal{L}_{\beta_j} = \text{span}\{iQ_{\beta_j}, \partial_y Q_{\beta_j}\}$
  - $\mathcal{L}_{\beta_j}$ is coercive on the orthogonal complement of $\ker \mathcal{L}_{\beta_j}$

- solvability condition = RHS of the elliptic equation orthogonal to $\ker \mathcal{L}_{\beta_j}$

- The two solvability conditions determine $M_{j,1}$ and $B_{j,1}$

Case $n \geq 2$: Plugging in $V_j^{(n)} = V_j^{(n-1)} + T_{j,n}$ for $2 \leq n \leq N$:

$$\begin{aligned}\mathcal{E}_j^{(n)} =& -\mathcal{L}_{\beta_j} T_{j,n} + \mathcal{E}_j^{(n-1)} - iM_{j,n}\Lambda Q_{\beta_j} \\ & + iB_{1,n}\left(y_j \partial_y Q_{\beta_j} + (1-\beta_j)\partial_{\beta_j} Q_{\beta_j}\right) + i\frac{1-\mu}{\mu}\partial_\Gamma T_{j,n} \\ & + \mathcal{E}rr_j^{(n)}(V_j^{(n-1)}, M_j^{(n-1)}, B_j^{(n-1)}, T_{j,n}, M_{j,n}, B_{j,n})\end{aligned}$$

where $\mathcal{E}rr_j^{(n)}$ encodes the interaction terms of $T_{j,n}, M_{j,n}, B_{j,n}$ with functions of decay at least $\frac{1}{R}$. We solve the elliptic equation

$$\mathcal{L}_{\beta_j} T_{j,n} - i\frac{1-\mu}{\mu}\partial_\Gamma T_{j,n} = \mathcal{E}_j^{(n-1)} - iM_{j,n}\Lambda Q_{\beta_j} + iB_{1,n}\left[y_j \partial_y Q_{\beta_j} + (1-\beta_j)\partial_{\beta_j} Q_{\beta_j}\right]$$

- at each step we need to solve for $T_{j,n}$ and determine $M_{j,n}, B_{j,n}$ and "get control" on these as well as on a high number of their derivatives
- we also need to show that $\mathcal{E}_j^{(n)}$ is "smaller" than $T_{j,n}$
- we introduce a notion of admissibility to keep track of these:

A function f is admissible with respect to the bubble j if $\forall \alpha \in \mathbb{N}^7 \; \exists A_\alpha > 0$

$$\|\langle y\rangle(1-(1-\beta_j)\langle y\rangle)\Lambda_y^{\alpha_1}\Lambda_R^{\alpha_2}\partial_{\lambda_1}^{\alpha_3}\partial_{\lambda_2}^{\alpha_4}\partial_\Gamma^{\alpha_5}\tilde{\Lambda}_{\beta_1}^{\alpha_6}\tilde{\Lambda}_{\beta_2}^{\alpha_7} f(\cdot,\mathcal{P})\|_{L^\infty} \leq A_\alpha,$$

where $\Lambda_x = x\partial_x$ and $\tilde{\Lambda}_{\beta_k} = (1-\beta_k)\partial_{\beta_k}$.

- we show that $b^{-1}R^n T_{1,n}$ is 1-admissible and $R^n T_{2,n}$ is 2-admissible (and similar statements hold for $M_{j,n}$ and $B_{j,n}$)
- also $b^{-1}R^{n+1}\mathcal{E}_1^{(n)}$ is 1-admissible and $R^{n+1}\mathcal{E}_2^{(n)}$ is 2-admissible
- we develop a stability theory (under multiplication, change of variables, convolution...) for admissible functions
- we invert $\mathcal{L}_\beta$ in the class of invertible functions. This follows using multiplier estimates and the coercivity of $\mathcal{L}_{\beta_j}$:

$$\|f\|_{H^{\frac{1}{2}}} \leq C\left(\|\mathcal{L}_\beta f\|_{H^{-\frac{1}{2}}} + |(f, iQ_\beta)| + |(f,\partial_x Q_\beta)|\right).$$

II. Study of the finite system of ODEs

$$(S)^{(N)} \quad \begin{cases} (x_j^{(N)})_t = \beta_j^{(N)}, \quad (\gamma_j^{(N)})_t = \frac{1}{\lambda_j^{(N)}}, \\ (\lambda_j^{(N)})_t = M_j^{(N)}(\mathcal{P}^{(N)}), \quad \frac{(\beta_j^{(N)})_t}{1-\beta_j^{(N)}} = \frac{B_j^{(N)}(\mathcal{P}^{(N)})}{\lambda_j^{(N)}}, \\ \Gamma^{(N)} = \gamma_2^{(N)} - \gamma_1^{(N)}, \quad R^{(N)} = \frac{x_2^{(N)} - x_1^{(N)}}{\lambda_1^{(N)}(1-\beta_1^{(N)})} \end{cases} \qquad j = 1, 2,$$

For $0 < \delta, \eta_* \ll 1$ and $0 < \eta < \eta_*$, we define the times

$$T_{\text{in}} = \frac{1}{\eta^{2\delta}} < T_\eta^- = \frac{\delta}{\eta}$$

We solve the system with data at $t = T_\eta^-$:

$$\begin{cases} \lambda_1^{(N)} = 1, \quad \lambda_2^{(N)} = 1 \quad \text{i.e.} \quad \mu = 1 \\ \gamma_2^{(N)} = 0, \quad \Gamma^{(N)} = 0 \quad \text{i.e.} \quad \gamma_1^{(N)} = 0 \\ 1 - \beta_1^{(N)} = \eta, \quad b^{(N)} = \frac{1}{(T_\eta^-)^2} \quad \text{i.e.} \quad 1 - \beta_2^{(N)} = \frac{\eta}{(T_\eta^-)^2} \\ x_1^{(N)} = 0, \quad R^{(N)} = T_\eta^- \quad \text{i.e.} \quad x_2^{(N)} = T_\eta^- \eta = \delta \end{cases}$$

Decay of Q_β

To study this system of ODEs, we need to have a precise description of the decay properties of Q_β. First, with $Q^+(x) := \frac{2}{2x+i}$,

$$\|Q_\beta - Q^+\|_{H^1} = O\left((1-\beta)^{\frac{1}{2}}|\log(1-\beta)|^{\frac{1}{2}}\right).$$

Secondly, as $x \to \infty$, we have the asymptotics:

$$Q_\beta(x) = \frac{c_\beta}{x} F\left(-\frac{1-\beta}{1+\beta}x\right) + O\left(\frac{1}{x^2}\right),$$

where $F(x) = \int_0^\infty \frac{\alpha e^{-\alpha}}{\alpha - ix} d\alpha$ and $c_\beta := \frac{i}{2\pi}\int_{\mathbb{R}} |Q_\beta(x)|^2 Q_\beta(x) dx$. Now,

$$c_\beta = 1 + O((1-\beta)|\log(1-\beta)|).$$

- $F(x) = 1 + O(x|\log x|)$ as $x \to 0$, so for $(1-\beta)|x| \ll 1$:

  $$Q_\beta(x) = \frac{1}{x}\left(1 + O((1-\beta)|\log(1-\beta)|)\right)\left(1 + O((1-\beta)x|\log(1-\beta)x|\right) + O\left(\frac{1}{x^2}\right)$$

- In general, we have $|F(x)| \lesssim \frac{1}{|x|}$, so for $(1-\beta)|x| \gtrsim 1$:

  $$|Q_\beta(x)| \lesssim \frac{1}{(1-\beta)|x|^2}.$$

We refine $B_2^{(N)} := B_{2,1} + \dots B_{2,N} = O(\frac{1}{R})$ to

$$B_2^{(N)} = 2\mathrm{Re}\left(\overline{Q_{\beta_1}(R)}e^{i\Gamma}\right) + O\Big(\frac{|1-\mu| + R^{-1}}{R(1+(1-\beta_1)R)}\Big)$$

Step 1: $t \in [T_{\text{in}}, T_\eta^-]$ (turbulent regime)

We prove by a bootstrap argument that:

$$\begin{cases} |\lambda_j^{(N)}(t) - 1| \lesssim \frac{\eta^\delta}{t}, \quad j = 1, 2, \\ |1 - \beta_1^{(N)}(t) - \eta| \lesssim \eta^{1+\delta}, \\ 1 - \beta_2^{(N)}(t) = \eta\frac{1+O(\sqrt{\delta})}{t^2} \\ \frac{|R^{(N)}(t) - t|}{t} \lesssim \eta^\delta \\ |\Gamma^{(N)}(t)| \lesssim \eta^\delta + \eta t |\log \eta t| \end{cases}$$

By bootstrap assumption: $R \sim t \leq \frac{\delta}{\eta}$. Then, we have

$$0 < (1-\beta_1)R \lesssim \eta t \lesssim \delta \ll 1$$

Thus, for $t \in [T_{\text{in}}, T_\eta^-]$, $Q_{\beta_1}(R) = \frac{1}{t} + O\left(\frac{\eta^\delta}{t} + \eta|\log \eta t|\right)$

Control of the speed $1-\beta_2$ in the turbulent regime

$$B_2^{(N)} = \frac{2\cos\Gamma}{t} + O\left(\frac{\eta^\delta}{t} + \eta|\log \eta t|\right)$$

By bootstrap assumption we have

$$\cos\Gamma = 1 + O(\Gamma^2) = 1 + O\Big((\eta t|\log \eta t|)^2\Big),$$

and thus,

$$\frac{(\beta_2)_t}{1-\beta_2} = \frac{B_2}{\lambda_2} = \frac{2}{t} + O\left(\frac{\eta^\delta}{t} + \eta|\log \eta t|\right)$$

Integrating (backward) from T_η^- to t:

$$-\log\left(\frac{1-\beta_2(T_\eta^-)}{1-\beta_2(t)}\right) = 2\log\left(\frac{T_\eta^-}{t}\right) + O(\sqrt{\delta}),$$

and so

$$1-\beta_2(t) = \frac{\eta}{t^2}\left(1 + O(\sqrt{\delta})\right).$$

Control of the phase in the turbulent regime

Main difficulty: keep the phase shift $\Gamma(t)$ small for $t \in [T_{\text{in}}, T_\eta^-]$

$$\Gamma_t = \frac{1}{\lambda_2} - \frac{1}{\lambda_1}, \quad (\lambda_1)_t = M_1^{(N)}, \quad (\lambda_2)_t = M_2^{(N)}$$

Thus,

$$\Gamma_{tt} = \frac{(\lambda_1)_t}{\lambda_1^2} - \frac{(\lambda_2)_t}{\lambda_2^2} = \frac{M_1^{(N)}}{\lambda_1^2} - \frac{M_2^{(N)}}{\lambda_2^2} = O\left(\frac{1}{t^2}\right)$$

We need to integrate twice to recover Γ in the presence of $\frac{1}{t^2}$ decay only
$\Longrightarrow$ sharp estimates for $M_1^{(N)}$ and $M_2^{(N)}$ required to avoid logarithmic losses

Setting $v := 1 - \mu = \frac{\lambda_1 - \lambda_2}{\lambda_1}$, we get

$$\begin{cases} \Gamma_t = v + R_\Gamma(t) \\ v_t = \frac{2v}{t} - \frac{2\Gamma}{t^2} + \frac{\eta}{t} + R_v(t) \end{cases}$$

with $|R_\Gamma(t)| + |R_v(t)| \lesssim \frac{\eta^\delta}{t^2} + K^2\eta^2|\log \eta t|^2$. We now solve this and get

$$|\Gamma(t)| \lesssim \eta^\delta + \eta t|\log \eta t|.$$

Remark: the construction of a two-soliton solution *without growth of high Sobolev norms* is easier: we don't need to control the phase on $[T_{\text{in}}, T_\eta^-]$, we simply prescribe asymptotic conditions at ∞ and integrate backward in time

Step 2: $t \in [T_\eta^-, \infty)$ (saturation regime)

By the bootstrap assumption,

$$R \sim t \geq T_\eta^- = \frac{\delta}{\eta} \sim \frac{\delta}{1-\beta_1}.$$

Thus, for $t \geq T_\eta^-$ we have $R(1-\beta_1) \gtrsim \delta$ and therefore

$$|Q_{\beta_1}(R)| \lesssim \frac{1}{(1-\beta_1)R^2} \sim \frac{1}{\eta t^2}.$$

Then,

$$B_2^{(N)} = 2\mathrm{Re}\left(\overline{Q_{\beta_1}(R)}e^{i\Gamma}\right) + O\left(\frac{|1-\mu| + R^{-1}}{R(1+(1-\beta_1)R)}\right) = O\left(\frac{1}{\eta t^2}\right)$$

Integrating (forward) from T_η^- to t the equation

$$\frac{(\beta_2)_t}{1-\beta_2} = \frac{B_2}{\lambda_2}$$

we obtain

$$\left|\log\left(\frac{(1-\beta_2)(t)}{(1-\beta_2)(T_\eta^-)}\right)\right| \lesssim \frac{1}{\eta T_\eta^-} \lesssim \frac{1}{\delta}$$

which shows that

$$1-\beta_2(t) = \eta^3 e^{O(\frac{1}{\delta})}$$

<u>Remark</u>: for $t \geq T_\eta^-$, the phase shift Γ grows, but this does not affect the dynamics of β_2

III. Construction of the exact solution

Let a sequence $T_n \to +\infty$ and consider $u_n(t)$ the solution of:

$$\begin{cases} i\partial_t u_n - |D|u_n = -|u_n|^2 u_n, \\ u_n(T_n) = \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{1}{2}}(t)} V_j^{(N)}\Big(y_j := \frac{x-x_j(t)}{\lambda_j(t)(1-\beta_j(t))}, \mathcal{P}(t)\Big) e^{i\gamma_j(t)} =: \Phi^{(N)}_{\tilde{\mathcal{P}}^{(N)}(T_n)}(x) \end{cases}$$

Decompose

$$u_n(t,x) = \Phi^{(N)}_{\tilde{\mathcal{P}}(t)}(x) + \varepsilon(t,x),$$

where ε satisfies suitable orthogonality conditions. Main goal:

$$\forall n \geq 1, \quad \forall t \in [T_{\text{in}}, T_n], \quad \|\varepsilon(t,\cdot)\|_{H^1} \leq \frac{1}{t^{\frac{N}{8}}}.$$

With $b := \frac{1-\beta_2}{1-\beta_1}$, consider cutoff functions ζ and θ such that

$$\zeta(t,x) = \begin{cases} \beta_1 & \text{for} \quad y_1 \leq \frac{(1-b)R}{2} \\ \beta_2 & \text{for} \quad y_1 \geq (1-b)R \end{cases},$$

$$\theta(t,x) = \begin{cases} \frac{1}{\lambda_1} & \text{for} \quad y_1 \leq \frac{(1-b)R}{2} \\ \frac{1}{\lambda_2} & \text{for} \quad y_1 \geq (1-b)R \end{cases}$$

and define the localized energy functional:

$$\mathcal{G}(t)\varepsilon \quad := \quad \frac{1}{2}(|D|\varepsilon - \zeta D\varepsilon, \varepsilon) + \frac{1}{2}(\theta\varepsilon, \varepsilon) + \frac{1}{4}\Big[\int_{\mathbb{R}} (|\varepsilon + \Phi|^4 - |\Phi|^4)dx - (4\varepsilon, \Phi|\Phi|^2)\Big].$$

Coercivity of the energy functional

Precise definition of ζ. Consider a smooth nonincreasing function

$$\Psi_1(z_1) = \begin{cases} 1 & \text{for } z_1 \leq \frac{1}{4} \\ (1-z_1)^{10} & \text{for } \frac{1}{2} \leq z_1 \leq 1 \\ 0 & \text{for } z_1 \geq 1 \end{cases},$$

$\Phi_1(z_1) := \psi_1 + b(t)(1-\Psi_1)$ and, with $y_1 := \frac{x-x_1}{\lambda_1(1-\beta_1)}$,

$$\phi(t,x) := \phi_1(t,y_1) = \Psi_1\left(z_1 = \frac{y_1}{R(t)(1-b(t))}\right).$$

Then, we set $\zeta(t,x) := \beta_1(t) + (1-\beta_1)(1-\phi(t,x))$. With $\varepsilon^+ := \Pi_+\varepsilon$, $\varepsilon^- := \varepsilon - \varepsilon^+$:

$$\mathcal{G}(t)\varepsilon \gtrsim (1-\beta_1)\int_{\mathbb{R}} \phi \left||D|^{\frac{1}{2}}\varepsilon^+\right|^2 dx + \|\varepsilon^-\|^2_{\dot{H}^{\frac{1}{2}}} + \|\varepsilon\|^2_{L^2}$$

- relies on a careful localization of the kinetic energy
- the coercivity of the limiting Szegő quadratic form (P. 2012) is also key $\mathcal{L}_+u := Du + u - \Pi_+(2|Q^+|^2u + (Q^+)^2\bar{u})$ for all $u \in H_+^{\frac{1}{2}}$:

$$(\mathcal{L}_+u, u) \geq c_0\|u\|^2_{H_+^{\frac{1}{2}}} - \frac{1}{c_0}[(u, \partial_y Q^+)^2 + (u, iQ^+)^2]$$

- one looses control of $\|\varepsilon^+\|_{\dot{H}^{\frac{1}{2}}}$ as $\beta_1 \to 1$: singular bifurcation $Q_\beta \mapsto Q^+$

Energy estimates

We prove using a bootstrap argument that

$$\mathcal{G}(t) \lesssim \frac{1}{Nt^{\frac{N}{2}}} \text{ for } t \in [T_{\text{in}}, T_n]$$

Using coercivity this implies the bound

$$\|\varepsilon(t)\|_{H^{\frac{1}{2}}} \lesssim \frac{1}{\sqrt{N}t^{\frac{N}{4}}}$$

To prove the bound on $\mathcal{G}$, we use the following energy estimate

$$\left|\frac{d}{dt}\mathcal{G}(t)\right| \lesssim \frac{1}{t}\mathcal{G}(t) + \frac{C}{t^N}$$

Remarks:

- the localization creates large errors that we need to control $\Longrightarrow$ our cutoff functions need to be carefully chosen
- also, we need to exploit some subtle cancellations, for example when treating terms such as $((\partial_t\zeta + \partial_x\zeta)D\varepsilon^+, \varepsilon^+)$
- we rely heavily on commutator estimates involving nonlocal operators, Π_+, and cutoff functions, for eg.

$$\|[|D|^{\frac{1}{2}}, \chi]f\|_{L^2} \lesssim \|\partial_x\chi\|_{L^1}^{\frac{1}{2}} \|\partial_x^2\chi\|_{L^1}^{\frac{1}{2}} \|f\|_{L^2}$$

- we need an approximation of high order $N \gg 1$ to close the bootstrap for $\mathcal{G}(t)$

Conclusions

- **Difficulties**:
  - detailed study of the decay properties of Q_β
  - dramatic influence of the phase shift Γ
  - nonlocal nature of the problem and slow decay of Q_β
    $\Longrightarrow$ the two solitons are strongly coupled
  - the limiting Szegő problem arises in the form of various estimates for $\Pi_\pm \varepsilon$
- **Open problems**:
  - existence of a solution with $\lim_{t\to\infty} \|u(t)\|_{H^1} = \infty$
  - existence of solutions with different growth rates, genericity
  - growth of high Sobolev norms for other problems with nonlocal dispersion