The Szegö equation and its pertubations

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The cubic Szegö equation

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  (SE) $\qquad i\partial_t u = \Pi_+(|u|^2 u), \qquad u(t,x) \in \mathbb{C}, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$

  where Π_+ is the Szegö projector onto non-negative frequencies, was recently introduced by Gérard and Grellier who study it on $\mathbb{T}$

- mathematical model of a non-dispersive Hamiltonian non-linear PDE

- completely integrable $\Longrightarrow$ we find an explicit formula for the solution

- growth of high Sobolev norms $\|u(t)\|_{H^s} \to \infty$ if $t \to \infty$ and $s > 1/2$. More precisely, there are solutions such that

$$\|u(t)\|_{H^s} \sim t^{2s-1}.$$

Motivation: NLS on the sub-Riemannian manifolds

- The nonlinear Schrödinger equation

  (NLS) $$i\partial_t u + \Delta u = |u|^2 u, \quad u(t,x) \in \mathbb{C}, \quad x \in M$$

  where M is a sub-Riemannian manifold (e.g. the Heisenberg group).

- NLS on the Heisenberg group lacks dispersion

  $\Rightarrow$ classical tools break down

  $\Rightarrow$ even the problem of well-posedness is open.

- $\mathbb{H}^1 = \mathbb{C}_z \times \mathbb{R}_s$, $L^2_{rad}(\mathbb{H}^1) = \oplus_{\pm} \oplus_{m=0}^{\infty} V_m^{\pm}$ and $\Delta_{\big|V_m^{\pm}} = \pm i(2m+1)\frac{\partial}{\partial_s}$.

  Denote by $\Pi_m^{\pm}$ the projection onto $V_m^{\pm}$. NLS is equivalent to the system:

  $$i\partial_t u_m^{\pm} \pm i(2m+1)\partial_s u_m^{\pm} = \Pi_m^{\pm}(|u|^2 u).$$

- Interaction between the cubic nonlinearity and the projector $\Pi_m^{\pm}$:

  $$i\partial_t u = \Pi_m^{\pm}(|u|^2 u)$$

Motivation: A non-linear wave equation

(NLW) $$i\partial_t v - |D|v = |v|^2 v$$

- Apply the operator $i\partial_t + |D|$ to both sides:

$$-\partial_{tt}v + \Delta v = |v|^4 v + 2|v|^2(|D|v) - v^2(|D|\bar{v}) + |D|(|v|^2 v).$$

- No dispersion: NLW decouples into the system of transport equations:

$$\begin{cases} i(\partial_t v_+ + \partial_x v_+) = \Pi_+(|v|^2 v) \\ i(\partial_t v_- - \partial_x v_-) = \Pi_-(|v|^2 v). \end{cases}$$

- Dynamics dominated by v_+:
$v(0) = v_+(0)$, $\|v(0)\|_{H^{1/2}} = \varepsilon \Longrightarrow \|v_-(t)\|_{\dot{H}^{1/2}} = O(\varepsilon^2)$.

- $u(t,x) = v_+(t, x+t)$ almost satisfies

$$i\partial_t u = \Pi_+(|u|^2 u)$$

Plan of the talk

1. General properties of the Szegö equation on $\mathbb{R}$

2. Classification and orbital stability of solitons of the Szegö equation

3. Explicit formula for the solution of the Szegö equation and applications

(i) Soliton resolution

(ii) Example of a solution whose high Sobolev norms grow to infinity

4. The Szegö equation as the resonant dynamics of a non linear wave equation

(i) Growth of the high Sobolev norms of solutions of the nonlinear wave equation

(ii) Second order approximation of the non linear wave equation

5. The long-time stability of solitons when adding a small Toeplitz potential to the Szegö equation

The Hardy space and the Szegö projector

The Hardy space and the corresponding Sobolev spaces:

$$L^2_+(\mathbb{R}) = \left\{ f \text{ holomorphic on } \mathbb{C}_+ \middle| \|g\|_{L^2_+(\mathbb{R})} := \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^2 dx \right)^{1/2} < \infty \right\}$$
$$= \{ f \in L^2(\mathbb{R}) \,|\, \operatorname{supp} \hat{f} \subset [0, \infty) \}$$

$H^s_+(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$.

The Szegö projector on the Hardy space $\Pi_+ : L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$:

$$\mathcal{F}(\Pi_+ f)(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \xi \geq 0, \\ 0, & \text{if } \xi < 0. \end{cases}$$

Set $\Pi_- = I - \Pi_+$. The Szegö projector gives the name of the Szegö equation:

(SE) $$i\partial_t u = \Pi_+(|u|^2 u), \quad u(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}.$$

Conservation laws

Symplectic form on $L^2_+(\mathbb{R})$:

$$\omega(u, v) = 4\mathrm{Im} \int_{\mathbb{R}} u\bar{v}.$$

Hamiltonian:

$$E(u) = \int_{\mathbb{R}} |u|^4 dx,$$

Mass:

$$Q(u) = \int_{\mathbb{R}} |u|^2 dx,$$

Momentum:

$$M(u) = (Du, u)_{L^2} \geq 0, \quad \text{with } D = -i\partial_x.$$

The $H^{1/2}_+$-norm of the solution is conserved:

$$Q(u) + M(u) = \|u\|^2_{H^{1/2}_+}.$$

The Cauchy problem

Theorem

For all $u_0 \in H^{1/2}_+$, *there exists a unique global solution* $u \in C(\mathbb{R}, H^{1/2}_+)$ *of the equation*

$$\text{(SE)} \qquad i\partial_t u = \Pi_+(|u|^2 u)$$

such that $u(0) = u_0$.
Moreover, if $u_0 \in H^s_+$, $s > 1/2$, *then* $u \in C(\mathbb{R}, H^s_+)$.

Hankel and Toeplitz operators

- Hankel operator of symbol $u \in H_+^{1/2}$: $H_u : L_+^2 \to L_+^2$

$$H_u h = \Pi_+(u\bar{h})$$

Compact operator, $\mathbb{C}$-antilinear, in particular

$$(H_u h_1, h_2)_{L^2} = (H_u h_2, h_1)_{L^2}.$$

H_u^2 is a compact, self-adjoint linear operator.

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H_u^2 is a compact, self-adjoint linear operator.

- Toeplitz operator of symbol $b \in L^\infty(\mathbb{R})$: $T_b : L^2_+ \to L^2_+$

$$T_b h = \Pi_+(bh)$$

Bounded, linear operator, self-adjoint iff b is real-valued.

Lax pair structure

Theorem (Lax pair formulation)

$u \in C(\mathbb{R}, H^s_+)$, $s > 1/2$ is a solution of the Szegö equation iff

$$\partial_t H_u = [B_u, H_u],$$

where $B_u = \frac{i}{2} H_u^2 - i T_{|u|^2}$.

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Corollary

There exists an infinite sequence of conservation laws:

$$J_n(u) := (u, H_u^{n-2} u), n \geq 2$$

$$\partial_t J_{2n}(u(t)) = 0.$$

In particular, $J_2(u) = Q(u)$ and $J_4(u) = \frac{E(u)}{2}$.

Remark: The conservation law of the $H^{1/2}_+$-norm is stronger than that of J_{2n}

$$J_{2n}(u) \leq \|u\|^{2n}_{L^{2n}(\mathbb{R})} \leq \|u\|^{2n}_{H^{1/2}_+(\mathbb{R})}.$$

Consider the operator satisfying

$$\begin{cases} \frac{d}{dt}U(t) = B_{u(t)}U(t) \\ U(0) = I. \end{cases}$$

U(t) is unitary and

$$H_{u(t)} = U(t)H_{u_0}U(t)^*.$$

The eigenvalues of H_u are conserved by the flow of the Szegö equation:

$$\lambda_j(t) = \lambda_j(0).$$

If $e_j(t) \in \mathrm{Ran}(H_{u(t)})$ are the eigenvectors of $H_{u(t)}$ and $\nu(t) := |(u(t), e_j(t))|$, then we have

$$\nu_j(t) = \nu_j(0).$$

Remark: $J_{2n}(u) = \sum_j \lambda_j^{2n-2}\nu_j^2$.

Classification of solitons

Definition: A soliton for the Szegö equation is a solution u for which there exist $\omega, c \in \mathbb{R}$ such that

$$u(x,t) = e^{-i\omega t}\phi(x - ct),$$

Theorem (P'09)

The solitons of the Szegö equation are

$$u(x,t) = e^{-i\frac{\alpha^2\mu^2}{4}t}\phi_{C,p}(x - \frac{\alpha^2\mu}{2}t),$$

where $\alpha, \mu > 0$, $C = \alpha e^{i\phi}$, $p = a - \frac{i}{\mu}$, $a, \phi \in \mathbb{R}$ *and*

$$\phi_{C,p} = \frac{C}{x-p} = \frac{\alpha e^{i\phi}}{x - a + \frac{i}{\mu}}$$

Orbital stability of solitons

Theorem (P'09)

The solitons of the Szegö equation on $\mathbb{R}$ are orbitally stable.
More precisely, for $\alpha, \mu > 0$, consider the cylinder

$$C(\alpha, \mu) = \left\{ \frac{\alpha}{z - p}; |\alpha| = \alpha,\ \operatorname{Im} p = -\frac{1}{\mu} \right\}.$$

which is a submanifold in the manifold of solitons. If the sequence $\{u_0^n\} \subset H_+^{1/2}$ is close to the the cylinder $C(\alpha, \mu)$, then the corresponding sequence of solutions $\{u^n\}$, stays close to $C(\alpha, \mu)$ for all times $t \in \mathbb{R}$.

Proof includes:

- Gagliardo-Nirenberg inequality: $\|u\|_{L^4_+} \leq \frac{1}{\sqrt[4]{\pi}} \|u\|_{L^2_+}^{1/2} \|u\|_{\dot{H}_+^{1/2}}^{1/2}$
- Profile decomposition theorem (Gérard 1998, Hmidi, Keraani 2006)

Comparison with the solitons of the Szegö equation on $\mathbb{T}$

On $\mathbb{T}$ (Gérard, Grellier 2010, 2011):

- the solitons are rational functions $\frac{z^\ell}{z^N - p^N}$, where $|p| > 1$, $\ell = 0, 1, 2, \dots, N-1$
- for $N = 1$ and $\ell = 0$, we recover the analogues of the solitons on $\mathbb{R}$, $\frac{1}{z-p}$, and they are also orbitally stable
- the rest of solitons ($N > 1$) are unstable
- one exploits the compactness of the Sobolev embedding $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$
- in particular, in the case of $\mathbb{T}$, the operator $A_u := D - \frac{1}{c} T_{|u|^2}$ has compact resolvent and thus, only discrete spectrum. This is not the case on $\mathbb{R}$, where A_u has continuous spectrum as well.

Invariant finite dimensional submanifolds of L^2_+

$$\mathcal{M}(N) = \text{“rational functions of degree N”}$$
$$= \left\{ \frac{A}{B} \middle| A, B \in \mathbb{C}_N[z], 0 \le \deg(A) \le N-1, \deg(B) = N, \right.$$
$$\left. B(0) = 1, B(z) \neq 0, \text{ for all } z \in \mathbb{C}_+ \cup \mathbb{R}, (A,B) = 1 \right\}$$

Remarks: $\mathcal{M}(N)$ is 4N-dimensional real manifold
$\bigcup_{N\in\mathbb{N}^*} \mathcal{M}(N)$ is dense in L^2_+

Theorem (Kronecker type theorem)

$rk(H_u) = N$ if and only if $u \in \mathcal{M}(N)$.

Proposition

For all $N \in \mathbb{N}^*$, $\mathcal{M}(N)$ is invariant under the flow of the Szegö equation.

Infinitesimal shift operator

Property

Let $T_\lambda : L^2_+ \to L^2_+$ *be the shift operator* $T_\lambda(f) = e^{i\lambda x} f$, $\mathcal{F}(T_\lambda f)(\xi) = \hat{f}(\xi - \lambda)$.
Then, $H : L^2_+ \to L^2_+$ *is a Hankel operator if and only if*

$$T_\lambda^* H = H T_\lambda, \quad \forall \lambda > 0.$$

For $u \in \mathcal{M}(N)$ we have $\mathrm{Ran}(H_u) \subset \mathcal{M}(N)$. We define the infinitesimal shift operator on $\mathrm{Ran}(H_u)$ by:

$$T(f) = xf - \lim_{x \to \infty} x f(x)(1 - g),$$

where $H_u g = u$. Then, $T^* H_u = H_u T$.

Notations for $u_0 \in \mathcal{M}(N)$:

- There exists a unique $g_0 \in \mathrm{Ran}(H_{u_0})$ such that $u_0 = H_{u_0} g_0$.
- $0 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_N^2$ eigenvalues of $H_{u_0}^2$
- $\{e_j\}_{j=1}^N$ orthonormal basis of $\mathrm{Ran}(H_{u_0})$ such that $H_{u_0} e_j = \lambda_j e_j$
- $\beta_j = (g_0, e_j)$.

Explicit formula for the solution if $u_0 \in \mathcal{M}(N)$

Theorem (P '10 Explicit formula for rational function data)

Suppose $u_0 \in \mathcal{M}(N)$ *and let* $g_0 \in \mathrm{Ran}(H_{u_0})$ *be such that* $u_0 = H_{u_0} g_0$.
Let $M_j = \{k \in \{1, 2, \dots, N\} \big| H_{u_0} e_k = \lambda_j e_k\}$. *We define an operator* $S(t)$ *on* $\mathrm{Ran}(H_{u_0})$, *in the basis* $\{e_j\}_{j=1}^N$, *by*

$$S(t)_{k,j} = \begin{cases} \frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)} \left(\lambda_j e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)} \overline{\beta}_j \beta_k - \lambda_k e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)} \beta_j \overline{\beta}_k \right), & \text{if } k \notin M_j \\ \frac{\lambda_j^2}{2\pi} \overline{\beta}_j \beta_k t + (T e_j, e_k) + i \frac{|\beta_j|^2}{4\pi}, & \text{if } k \in M_j. \end{cases}$$

Then, the following explicit formula for the solution holds:

$$u(t,x) = \frac{i}{2\pi} \left(u_0, e^{i\frac{t}{2} H_{u_0}^2} (S(t) - xI)^{-1} e^{i\frac{t}{2} H_{u_0}^2} g_0 \right), \text{ for all } x \in \mathbb{R}.$$

Application to inverse problems for Hankel operators

Corollary

Suppose $u \in \mathcal{M}(N)$. If the eigenvalues λ_j^2 of H_u^2 are all simple and $(u, e_j) \neq 0$, then the symbol u can be written

$$u(x) = \frac{i}{2\pi}\Big(u, (T - xI)^{-1} g\Big) = \frac{i}{2\pi} \sum_{j,k=1}^{N} \lambda_j \overline{\beta}_j \overline{\beta}_k \overline{(T - xI)^{-1}_{jk}},$$

where

$$Te_j = \sum_{k \neq j} \frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)} \Big(\lambda_j \overline{\beta}_j \beta_k - \lambda_k \beta_j \overline{\beta}_k\Big) e_k + (\gamma_j + i\frac{|\beta_j|^2}{4\pi}) e_j.$$

Remark: The Corollary can be extended to functions that are not necessarily rational, satisfying $u \in H^s_+$, $s > 1/2$ and $xu(x) \in L^\infty(\mathbb{R})$.

Theorem (P '10)

Let $0 < \lambda_1 < \cdots < \lambda_N$ *and let* $(\nu_j)_{j=1}^N$ *be strictly positive.*

The set of all symbols $u \in H_+^{1/2}$ *such that the Hankel operator* H_u *is of finite rank and admits:*

- λ_j, $1 \leq j \leq N$, *as simple eigenvalues*
- ν_j, $1 \leq j \leq N$, *as length of the projections of* u *on the eigenvectors* $(\nu_j := |(u, e_j)| = \lambda_j |\beta_j|)$

is a toroidal cylinder $\mathbb{T}^N \times \mathbb{R}^N = (\arg \beta_j)_{j=1}^N \times (\gamma_j)_{j=1}^N$.

Open problem: Can one extend the above theorem to Hankel operators which are not of finite rank?

- Explicit formula in the spirit of the inverse scattering method, but one does not need to apply this method since the Hankel operator in the Lax pair is compact.

- One can find an explicit formula for solutions with general initial condition by using an approximation argument.

- Gérard and Grellier (2010) give a formula for solutions of the Szegö equation on the torus $\mathbb{T}$ (as a bias of introducing action-angle coordinates). They need **new spectral data** given by the operator $T_z H_u$.

Soliton resolution

$$\mathcal{M}(N)_{\mathrm{s}} = \{u \in \mathcal{M}(N) \,\big|\, 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N, (u, e_j) \neq 0, (u, e_j) \neq (u, e_k)\}.$$

Theorem (P '10)

If $u_0 \in \mathcal{M}(N)_{\mathrm{s}}$, *then the solution of the Szegö equation is*

$$u(t,x) = \sum_{j=1}^{N} e^{-it\lambda_j^2} \phi_{C_j,p_j}\left(x - \frac{\lambda_j^2 \nu_j^2}{2\pi} t\right) + \varepsilon(t,x)$$

where

$$\phi_{C_j,p_j}(x) = \frac{C_j}{x - p_j}, \quad C_j = \frac{i\lambda_j \nu_j^2 e^{-2i\phi_j(0)}}{2\pi}, \quad p_j = \mathrm{Re}(c_j(0)) - i\frac{\nu_j^2}{4\pi},$$

$$\lim_{t \to \pm\infty} \|\varepsilon(t,x)\|_{H_+^s} = 0 \text{ for all } s \geq 0.$$

Comparison with other completely integrable equations

- Soliton resolution holds for KdV (Echaus, Schuur 1983) in $L^\infty(\mathbb{R}_+)$:

$$\lim_{t\to\infty} \|\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} = 0,$$

but $\lim_{t\to\infty} \|\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}$ may not be zero.

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- Soliton resolution holds for one dimensional cubic NLS in $L^2(\mathbb{R})$

$$u(t,x) = \text{Solitons} + e^{it\Delta} f + \varepsilon(t,x),$$

where $\lim_{t\to\infty} \|\varepsilon(t,x)\|_{L^2} = 0$.

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where $\lim_{t\to\infty} \|\varepsilon(t,x)\|_{L^2} = 0$.

- No soliton resolution for the Szegö equation on $\mathbb{T}$ (Gérard, Grellier).

Growth of high Sobolev norms

Theorem (P '10)

Let $u_0 \in \mathcal{M}(2)$ *be such that* $H^2_{u_0}$ *has a double eigenvalue* $\lambda^2 > 0$. *Then*

$$u(t,x) = e^{-it\lambda^2}\phi_{C,p}\left(x - \frac{\|u_0\|^2_{L^2}}{2\pi}t\right) + \varepsilon(t,x).$$

The first term is a soliton and $\lim_{t\to\pm\infty}\|\varepsilon(t,x)\|_{H^s_+} = 0$ *for* $0 \le s < 1/2$.

However,

$$\lim_{t\to\pm\infty}\|\varepsilon(t,x)\|_{H^s_+} = \infty \text{ if } s > 1/2$$

and therefore

$$\|u(t)\|_{H^s_+} \to \infty \text{ as } t \to \pm\infty \text{ if } s > 1/2.$$

Example of such initial condition: $u_0 = \frac{1}{x+i} - \frac{2}{x+2i}$.

Open problem: Genericity of solutions whose high Sobolev norms grow to infinity?

- This phenomenon is due to the lack of dispersion. For dispersionless NLS

$$i\partial_t u = |u|^2 u,$$

we have $u(t) = u_0 e^{-i|u_0|^2 t}$ and thus $\|u(t)\|_{H^s} \sim |t|^s$ for $s \in \mathbb{N}$.

- More subtle situation for Szegö: the $H^{1/2}$-norm is conserved. Only the H^s-norms with $s > 1/2$ grow to ∞.

- This shows that the energy ($H^{1/2}$-norm) is supported on higher frequencies, while the mass is supported on lower frequencies: forward cascade.

- It agrees with the predictions of weak turbulence theory (Zakharov 2001,2004): the existence of an invariant state prescribing the power spectrum $|\hat{u}(n)| \sim n^\alpha$, such that reasonable classes of solutions approach this invariant state.

- For the Navier-Stokes equations: existence of turbulent flows: Kolmogorov's scaling law $|\hat{u}(n)| \sim n^{-5/3}$

Partial results regarding the growth of high Sobolev norms were obtained by:

- Gérard, Grellier (2010) for the Szegö equation on $\mathbb{T}$:

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \geq K(t^\varepsilon)^{2s-1}, \text{ for } s > 1/2 \text{ and } t^\varepsilon \to \infty.$$

- Bourgain (1993, 1995, 1995) for Hamiltonian PDEs with spectrally defined laplacian

- Kuksin (1997) for small dispersion NLS $-i\partial_t u + \varepsilon \Delta u = |u|^2 u$ with odd, periodic boundary condition on $\mathbb{T}^n$

- Colliander, Keel, Staffilani, Takaoka, and Tao (2010) for defocusing cubic NLS on $\mathbb{T}^2$

- Hani (2011) for defocusing truncated cubic NLS on $\mathbb{T}^2$

Proof: the case $u_0 \in \mathcal{M}(2)$ with a double eigenvalue

$$
\begin{aligned}
2\pi u(z,t) &= \frac{1}{i}\int_0^\infty \frac{u(x)}{x-z}dx = \frac{1}{2\pi i}\int_0^\infty \widehat{u}(\xi)\overline{\widehat{\frac{1}{x-\bar{z}}}}(\xi)d\xi \\
&= \int_0^\infty e^{iz\xi}\widehat{u}(\xi)d\xi = \int_0^\infty e^{iz\xi}(u, e^{i\xi x}g)d\xi \\
&= \int_0^\infty e^{iz\xi}(u, e^{i\xi T}g)d\xi = \Big(u, \big(\int_0^\infty e^{i\xi(T-\bar{z})}d\xi\big)g\Big) \\
&= i\Big(u, (T-\bar{z})^{-1}g\Big)
\end{aligned}
$$

Writing everything in the coordinates at $t=0$, we obtain

$$
u(t,x) = \frac{i}{2\pi}\Big(u_0, e^{i\frac{t}{2}H_{u_0}^2}(S(t)-xI)^{-1}e^{i\frac{t}{2}H_{u_0}^2}g_0\Big).
$$

The operator satisfying

$$
\frac{d}{dt}U(t) = B_{u(t)}U(t), \quad U(0) = I
$$

is unitary and $H_{u(t)} = U(t)H_{u_0}U(t)^*$. We have $S(t) = U^*(t)TU(t)$, and this definition depends on $u(t)$ through $U(t) \Longrightarrow$ Vicious circle

- $u_0 \in \mathcal{M}(2) \Longrightarrow \text{rang}(H_{u_0}) = 2 \Longrightarrow \text{Im}(H_{u_0}) = \text{vect}\{e_1, e_2\}$
- $H_{u_0} e_j = \lambda e_j$ for $j = 1, 2$
- $S(t) : \text{Im}(H_{u_0}) \to \text{Im}(H_{u_0})$ is a 2×2 matrix given by

$$S(t)_{jk} := (S(t)e_k, e_j), \quad j = 1, 2$$

- We determine $S(t)_{jk}$ by computing $\partial_t S(t)$:

$$\partial_t S(t) = U^* [T, B_u] U h + U^* (\partial_t T(t)) U$$
$$= \frac{1}{4\pi} \Big((h, H_{u_0}^2 \tilde{e}) \tilde{e} + (h, H_{u_0} \tilde{e}) H_{u_0} \tilde{e} \Big),$$

where $\tilde{e} = e^{i \frac{t}{2} H_{u_0}^2} g_0$ and thus

$$(\tilde{e}, e_j) = (e^{i \frac{t}{2} H_{u_0}^2} g_0, e_j) = (g_0, e^{-i \frac{t}{2} H_{u_0}^2} e_j) = e^{i \frac{t}{2} \lambda^2} (g_0, e_j) = e^{i \frac{t}{2} \lambda^2} \beta_j$$

- We obtain

$$\partial_t S(t)_{kj} = (\partial_t S(t) e_j, e_k) = \frac{\lambda^2}{4\pi} \Big((e_j, \tilde{e})(\tilde{e}, e_k) + (\tilde{e}, e_j)(e_k, \tilde{e}) \Big)$$
$$= \frac{\lambda^2}{4\pi} (\overline{\beta}_j \beta_k + \overline{\beta}_k \beta_j) = \frac{\lambda^2}{2\pi} \overline{\beta}_j \beta_k$$

since $\overline{\beta}_j \beta_k \in \mathbb{R}$ when e_j and e_k correspond to the same eigenvalue λ.

- Since $\overline{\beta}_1\beta_2 \in \mathbb{R}$, we have $\beta_1 = \nu_1 e^{i\theta}$, $\beta_2 = \nu_2 e^{i\theta}$, where $\nu_j = |\beta_j|$
- We make the change of basis

$$e_1 \mapsto \frac{1}{\sqrt{\nu_1^2 + \nu_2^2}}(\nu_1 e_1 + \nu_2 e_2)$$
$$e_2 \mapsto \frac{1}{\sqrt{\nu_1^2 + \nu_2^2}}(\nu_2 e_1 - \nu_1 e_2)$$

  We then replace β_2 by

$$\tilde{\beta}_2 = (g_0, \tilde{e}_2) = \frac{1}{\sqrt{\nu_1^2 + \nu_2^2}}(\nu_2\beta_1 - \nu_1\beta_2) = 0.$$

  We can therefore assume that $\beta_2 = 0$.
- Since $S(t)_{kj} = \frac{\lambda^2}{2\pi}\overline{\beta}_j\beta_k t + S(0)_{kj}$ and $\beta_2 = 0$, we obtain

$$S(t) = \begin{pmatrix} \frac{\lambda^2\nu_1^2}{2\pi}t + S_{11}(0) & S_{12}(0) \\ S_{21}(0) & S_{22}(0) \end{pmatrix}.$$

- The eigenvalues of this matrix are

$$\begin{cases} E_1(t) & = \frac{\lambda^2\nu_1^2}{2\pi}t + S_{11}(0) + F(t) \\ E_2(t) & = S_{22}(0) - F(t), \end{cases}$$

  where $F(t) = \frac{A}{t} + \frac{B}{t^2} + O(\frac{1}{t^3})$, $A \in \mathbb{R}$, $B \notin \mathbb{R}$.
- We have $\mathrm{Im}\, S_{jj}(0) = \frac{|\beta_j|^2}{4\pi}$. Then,

$$\begin{cases} \mathrm{Im}\, E_1(t) & > c > 0 \\ \mathrm{Im}\, E_2(t) & = O(\frac{1}{t^2}), \text{ quand } t \to \infty \end{cases}$$

- We have

$$(S(t) - xI)^{-1} = \frac{1}{(x - E_1)(x - E_2)} \begin{pmatrix} S_{22}(t) - x & -S_{21}(t) \\ -S_{12}(t) & S_{11}(t) - x \end{pmatrix}$$

- In conclusion

$$\begin{aligned} u(t,x) &= \frac{i}{2\pi}\left(u_0, e^{i\frac{t}{2}H_{u_0}^2}(S(t) - xI)^{-1}e^{i\frac{t}{2}H_{u_0}^2}g_0\right) \\ &= \frac{\frac{\lambda}{2\pi}\overline{\beta}_1^2 e^{-it\lambda^2}}{x - \bar{E}_1(t)} + R(t,x), \end{aligned}$$

  where the first term tends to a soliton.

We have

$$R(t,x) := \frac{\bar{F}(t)}{\bar{E}_1 - \bar{E}_2} \cdot \frac{\lambda}{2\pi} e^{-it\lambda^2} \overline{\beta}_1^2 \Big(\frac{1}{x - \bar{E}_1} - \frac{1}{x - \bar{E}_2} \Big)$$

We compute easily

$$\Big\| \frac{1}{x - \bar{E}_j} \Big\|_{\dot{H}^s} \sim \frac{1}{|\mathrm{Im}\, E_j|^{\frac{2s+1}{2}}}$$

In particular,

$$\Big\| \frac{1}{x - \bar{E}_1} \Big\|_{\dot{H}^s} \sim 1$$

$$\Big\| \frac{1}{x - \bar{E}_2} \Big\|_{\dot{H}^s} \sim t^{2s+1}, \text{ quand } t \to \infty$$

Then,

$$\|R(t,x)\|_{\dot{H}^s} \sim t^{2s-1}$$

In conclusion, if $u_0 \in \mathcal{M}(2)$ is such that $H_{u_0}^2$ has a double eigenvalue, we obtain

$$\|u(t)\|_{H^s} \sim t^{2s-1}$$

and thus $\|u(t)\|_{H^s} \to \infty$ when $s > \frac{1}{2}$.

The Szegö equation as the first approximation of NLW

Theorem (P '11)

Let $W_0 \in H^s_+(\mathbb{R})$, $s > \frac{1}{2}$. *Let* $v(t)$ *be the solution of the NLW on* $\mathbb{R}$

(NLW)
$$\begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \varepsilon W_0. \end{cases}$$

Denote by $u(t)$ *the solution of the Szegö equation*

$$\begin{cases} i\partial_t u = \Pi_+(|u|^2 u) \\ u(0) = \varepsilon W_0. \end{cases}$$

Assume that $\|u(t)\|_{H^s} \leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$ *for* $0 \leq \alpha \leq \frac{1}{2}$ *and* $\delta > 0$ *small.*
Then, if $0 \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$ *we have that*

$$\|v(t) - e^{-i|D|t}u(t)\|_{H^s} \leq C\varepsilon^{2-C_0\delta}.$$

Growth of high Sobolev norms for solutions of NLW

Corollary (P '11)

Let $0 < \varepsilon \ll 1$, $s > \frac{1}{2}$, *and* $\delta > 0$ *sufficiently small. Let* $W_0 \in H^s_+(\mathbb{R})$ *be the non-generic rational function* $W_0 = \frac{1}{x+i} - \frac{2}{x+2i}$. *Denote by* $v(t)$ *be the solution of the NLW equation on* $\mathbb{R}$

$$\text{(NLW)} \qquad \begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \varepsilon W_0. \end{cases}$$

Then, for $\frac{1}{2\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}} \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}}$, *we have that*

$$\frac{\|v(t)\|_{H^s(\mathbb{R})}}{\|v(0)\|_{H^s(\mathbb{R})}} \geq C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{4s-2}{4s-1}} \gg 1.$$

Remark: In order to show arbitrarily large growth of the solution, one needs an approximation at least for a time $0 \leq t \leq \frac{1}{\varepsilon^{2+\beta}}$, where $\beta > 0$.

The renormalization group (RG) method

- It is most often used to find a long-time approximate solution to a perturbed equation

- It was introduced by Chen, Goldenfeld, and Oono (1994) in theoretical physics

- The RG method was justified mathematically:

  (i) *for ODEs*: Ziane (2000); De Ville, Harkin, Holzer, Josic, Kaper (2008)

  (ii) *for PDEs: Navier-Stokes, Swift-Hohenberg, quadratic NLS*: Moise, Temam (2000); Moise, Ziane (2001); Petcu, Temam, Wirosoetisno (2005); Abou Salem (2010)

- Gérard and Grellier (2011) proved analogous results on the torus $\mathbb{T}$ using the theory of Birkhoff normal forms

- Change of variables $w(t) = \frac{1}{\varepsilon} e^{i|D|t} v(t)$ in NLW:

$$\text{(NLW')} \qquad \begin{cases} \partial_t w = -i\varepsilon^2 e^{i|D|t}(|e^{-i|D|t} w|^2 e^{-i|D|t} w) =: \varepsilon^2 f(w,t) \\ w(0) = W_0. \end{cases}$$

- Naive perturbation expansion:

$$w(t) = w^{(0)}(t) + \varepsilon^2 w^{(1)}(t) + \varepsilon^4 w^{(2)}(t) + \dots$$

- Taylor expansion:

$$\begin{aligned} f(w,t) &= f(w^{(0)},t) + f'(w^{(0)},t)(w(t) - w^{(0)}(t)) + \dots \\ &= f(w^{(0)},t) + \varepsilon^2 f'(w^{(0)},t) w^{(1)}(t) + \dots \end{aligned}$$

- Identifying the powers of ε:

$$\begin{cases} \partial_t w^{(0)} = 0 \\ \partial_t w^{(1)} = f(w^{(0)}(t),t) \\ \dots \end{cases}$$

- Then,

$$w(t) = W_0 + \varepsilon^2 w^{(1)}(t) + O(\varepsilon^4) = W_0 + \varepsilon^2 \int_0^t f(W_0,s)ds + O(\varepsilon^4).$$

$$\mathcal{F}\big(f(w,t)\big)(\xi) = -\,i \iint\limits_{\xi=\xi_1-\xi_2+\xi_3} e^{it\phi(\xi,\xi_1,\xi_2,\xi_3)}\hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

where $\phi(\xi,\xi_1,\xi_2,\xi_3) := |\xi| - |\xi_1| + |\xi_2| - |\xi_3|$.

$$f(w,t) = f_{\text{res}}(w) + f_{\text{osc}}(w,t),$$

$$f_{\text{res}}(w) := -i\mathcal{F}^{-1} \iint\limits_{\{\phi=0,\xi=\xi_1-\xi_2+\xi_3\}} \hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

$$f_{\text{osc}}(w,t) := -i\mathcal{F}^{-1} \iint\limits_{\{\phi\neq 0,\xi=\xi_1-\xi_2+\xi_3\}} e^{it\phi(\xi,\xi_1,\xi_2,\xi_3)}\hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3.$$

Then, $w(t) = W_0 + \varepsilon^2 t f_{\text{res}}(W_0) + \varepsilon^2 \int_0^t f_{\text{osc}}(W_0,s)ds + O(\varepsilon^4)$.

The term $W_0 + \varepsilon^2 t f_{\text{res}}(W_0)$ is a secular term. We consider the renormalization group equation:

$$\begin{cases} \partial_t W = \varepsilon^2 f_{\text{res}}(W) \\ W(0) = W_0 \end{cases}$$

An approximation for the solution will be:

$$w_{\text{app}}(t) = W(t) + \varepsilon^2 \underbrace{\int_0^t f_{\text{osc}}(W(t),s)ds}_{=:F_{\text{osc}}(W(t),t)}.$$

Special property of NLW: many resonances

The set $\{\phi(\xi,\xi_1,\xi_2,\xi_3)=0\}\subset\mathbb{R}^2$ has non-zero measure for fixed ξ.
It is the subset of $\mathbb{R}^2$ such that ξ_1, ξ_2, and ξ_3 have the same sign as ξ and $\xi=\xi_1-\xi_2+\xi_3$ (or $\xi_1=\xi$ or $\xi_3=\xi$).

$$\begin{aligned} f_{\text{res}}(w) &= -i\mathcal{F}^{-1} \iint_{\{\phi=0,\xi=\xi_1-\xi_2+\xi_3\}} \hat{w}(\xi_1)\overline{\hat{w}}(\xi_2)\hat{w}(\xi_3)d\xi_1 d\xi_2 d\xi_3 \\ &= -i\mathcal{F}^{-1}\mathbf{1}_{\xi\geq 0} \iint_{\xi=\xi_1-\xi_2+\xi_3} \hat{w}_+(\xi_1)\overline{\hat{w}_+}(\xi_2)\hat{w}_+(\xi_3)d\xi_1 d\xi_2 d\xi_3 \\ &\quad -i\mathcal{F}^{-1}\mathbf{1}_{\xi<0} \iint_{\xi=\xi_1-\xi_2+\xi_3} \hat{w}_-(\xi_1)\overline{\hat{w}_-}(\xi_2)\hat{w}_-(\xi_3)d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Thus, $f_{\text{res}}(w)=-i\big(\Pi_+(|w_+|^2w_+)+\Pi_-(|w_-|^2w_-)\big)$.
We choose W_0 such that $\Pi_-(W_0)=0$. Projecting onto the negative frequencies:

$$\begin{cases} i\partial_t W_- = \varepsilon^2\Pi_-(|W_-|^2W_-) \\ W_-(0)=0. \end{cases}$$

Then $W_-(t)=0$ for all $t\in\mathbb{R}$ and $W(t)=W_+(t)$ satisfies:

$$\begin{cases} i\partial_t W = \varepsilon^2\Pi_+(|W|^2W) \\ W(0)=W_0. \end{cases}$$

Theorem (Second order approximation)

Let $W_0 \in H^s_+(\mathbb{T})$, $s > 1/2$, *be such that the solution of the Szegö equation with initial condition* εW_0 *is bounded by* $\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$.

Denote by v *the solution of the NLW equation on* $\mathbb{T}$ *with initial condition* εW_0.

Let $\mathcal{W} \in C(\mathbb{R}, H^s_+(\mathbb{T}))$ *be the solution of the following equation on* $\mathbb{T}$*:*

$$\begin{cases} i\partial_t \mathcal{W} = \Pi_+(|\mathcal{W}|^2\mathcal{W}) - \Pi_+(|\mathcal{W}|^2 \frac{1}{D}\Pi_-(|\mathcal{W}|^2\mathcal{W})) - \frac{1}{2}\Pi_+(\mathcal{W}^2 \frac{1}{D}\overline{\Pi_-(|\mathcal{W}|^2\mathcal{W})}) \\ \mathcal{W}(0) = \mathcal{W}_0 = \varepsilon W_0. \end{cases}$$

Consider

$$v_{\text{app}}(t) = e^{-i|D|t}\big(\mathcal{W}(t) + F_{\text{osc}}(\mathcal{W}(t), t)\big).$$

Then, if $0 \le t \le \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$, *we have*

$$\|v(t) - v_{\text{app}}(t)\|_{H^s} \le \varepsilon^{5 - C_0\delta}.$$

The averaging method at order two

Temam and Wirosoetisno (2002)
For the equation

$$\begin{cases} \partial_t w = -i\varepsilon^2 e^{i|D|t}(|e^{-i|D|t}w|^2 e^{-i|D|t}w) =: \varepsilon^2 f(w,t) \\ w(0) = W_0. \end{cases}$$

we consider the averaging ansatz

$$w_{\text{app}}(t) = W(t) + \varepsilon^2 N_1(W,t) + \varepsilon^4 N_2(W,t) =: N(W,t,\varepsilon),$$

where W is a solution of the averaged equation:

$$\begin{cases} \partial_t W = \varepsilon^2 R_1(W) + \varepsilon^4 R_2(W) =: R(W,\varepsilon) \\ W(0) = W_0. \end{cases}$$

Replacing these expansions in the equation and identifying the powers of ε, we obtain:

$$\begin{cases} R_1(W) & = f_{\text{res}}(W) \\ N_1(W,t) & = F_{\text{osc}}(W,t) \\ R_2(W) & = \{f'(W,t) \cdot N_1(W,t)\}_{\text{res}} \\ \frac{\partial N_2}{\partial t}(W,t) & = \{f'(W,t) \cdot N_1(W,t)\}_{\text{osc}} - \{N_1'(W,t) \cdot R_1(W)\}_{\text{osc}}. \end{cases}$$

Stability of solitons when adding a small multiplicative potential/a slowly varying potential

- *cubic NLS*: Bronski, Jerrard 2000, Keraani 2002, 2006
- *Hartree, NLS with general non-linearity*: Fröchlich, Tsai, Yau 2002, Fröchlich, Gustafson, Jonsson, Sigal 2004, 2006
- *1D cubic NLS*: Holmer, Zworski 2007, 2008
- *mkdV with double soliton*: Holmer, Perelman, Zworski

For the Szegö equation on $\mathbb{R}$, the solitons can be written as:

$$u(t,x) = e^{i\phi(t)}\alpha_0\mu_0\eta(\mu_0(x-a(t))) = \frac{e^{i\phi(t)}\alpha_0}{x - a(t) + \frac{i}{\mu_0}},$$

where $\eta(x) := \frac{1}{x+i}$, $\alpha_0, \mu_0 \in (0,\infty)$, $\phi_0, a_0 \in \mathbb{R}$,

$$\phi(t) = -\frac{\alpha_0^2\mu_0^2}{4}t + \phi_0, \qquad a(t) = \frac{\alpha_0^2\mu_0}{2}t + a_0.$$

The Szegö equation with a small Toeplitz potential

Theorem (P. '10)

Let $b : \mathbb{R} \to \mathbb{R}$, $b \in L^\infty(\mathbb{R})$ *et* $b' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*. Let* $0 < \varepsilon \ll 1$ *and* $\frac{3}{10} < \delta < \frac{1}{2}$*. If* u *satisfies*

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) + \varepsilon T_b u \\ u(0,x) = \alpha_0 e^{i\phi_0} \mu_0 \eta(\mu_0(x - a_0)), \end{cases}$$

where $a_0, \phi_0 \in \mathbb{R}$ *and* $\alpha_0, \mu_0 \in (0, \infty)$*, then,*

$$\|u(t) - \bar{\alpha}(t) e^{i\bar{\phi}(t)} \bar{\mu}(t) \eta(\bar{\mu}(t)(x - \bar{a}(t)))\|_{H_+^{\frac{1}{2}}} \leq C\varepsilon^{\frac{1}{2} + \frac{\delta}{3}}$$

for a long time $0 \leq t \leq \frac{\delta}{6 \ln c_0} \cdot \frac{1}{\varepsilon^{\frac{1}{2} - \delta}} \ln(\frac{1}{\varepsilon})$*, where* $C = C(\alpha_0, \mu_0)$ *and* $\bar{a}, \bar{\alpha}, \bar{\phi}, \bar{\mu}$ *satisfy the ODEs*

$$\begin{cases} \dot{\bar{a}} = \frac{\bar{\alpha}^2 \bar{\mu}}{2} - \frac{2\varepsilon}{\pi \bar{\mu}} \int b'(\bar{a} + \frac{x}{\bar{\mu}}) \frac{x}{\bar{\mu}} |\eta(x)|^2 dx, \\ \dot{\bar{\alpha}} = \frac{\varepsilon \bar{\alpha}}{\pi \bar{\mu}} \int b'(\bar{a} + \frac{x}{\bar{\mu}}) |\eta(x)|^2 dx, \\ \dot{\bar{\phi}} = -\frac{\bar{\alpha}^2 \bar{\mu}^2}{4} - \frac{\varepsilon}{\pi} \int b(\bar{a} + \frac{x}{\bar{\mu}}) |\eta(x)|^2 dx - \frac{\varepsilon}{\pi} \int b'(\bar{a} + \frac{x}{\bar{\mu}}) \frac{x}{\bar{\mu}} |\eta(x)|^2 dx, \\ \dot{\bar{\mu}} = -\frac{2\varepsilon}{\pi} \int b'(\bar{a} + \frac{x}{\bar{\mu}}) |\eta(x)|^2 dx. \end{cases}$$