Scattering for cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$

Oana Pocovnicu

Princeton University

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Joint work with R. Killip (UCLA), T. Oh (Princeton), M. Vişan (UCLA)

SCAPDE
UCLA

- Cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$:

$$(\mathrm{CQ}) \qquad \begin{cases} i\partial_t u + \Delta u = |u|^4 u - |u|^2 u \\ u(0) = u_0 \in H^1(\mathbb{R}^3) \end{cases}$$

- Model in various physical problems in :
  nonlinear optics, plasma physics, Bose-Einstein condensation

- Conserved quantities : mass, energy, and momentum :

$$M(u) = \int |u|^2 dx, \quad E(u) = \int \frac{|\nabla u|^2}{2} + \frac{|u|^6}{6} - \frac{|u|^4}{4} dx, \quad P(u) = \mathrm{Im} \int \overline{u} \nabla u dx$$

- Globally well-posed in $H^1(\mathbb{R}^3)$

- Scattering : there exist u_+ and u_- in $H^1(\mathbb{R}^3)$ such that

$$\lim_{t \to \pm\infty} \| u(t) - e^{it\Delta} u_\pm \|_{H^1(\mathbb{R}^3)} = 0$$

- (CQ) admits solitons $e^{it\omega} P_\omega(x)$: solutions which do not scatter

- Goal : Scattering in a large region

Nonlinear Schrödinger equations

- Nonlinear Schrödinger equation (NLS) on $\mathbb{R}^d$:

(NLS) $$i\partial_t u + \Delta u = \pm |u|^{p-1} u$$

- defocusing with the sign "+" and focusing with the sign "-"
- Scaling invariance : $u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$.
  - NLS is energy-critical ($\dot{H}^1$-critical) if

$$\|u_\lambda(0)\|_{\dot{H}^1(\mathbb{R}^d)} = \|u(0)\|_{\dot{H}^1(\mathbb{R}^d)} \iff \lambda^{\frac{2}{p-1}+1-\frac{d}{2}} \|u(0)\|_{\dot{H}^1(\mathbb{R}^d)} = \|u(0)\|_{\dot{H}^1(\mathbb{R}^d)}$$
$$\iff p = 1 + \frac{4}{d-2}$$

  - NLS is energy-subcritical if $p < 1 + \frac{4}{d-2}$
  - NLS is energy-supercritical if $p > 1 + \frac{4}{d-2}$
- (CQ) on $\mathbb{R}^3$: quintic nonlinearity is energy-critical and defocusing
  cubic nonlinearity is subcritical and focusing

Energy-critical NLS

Defocusing energy-critical NLS : $i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}} u$ on $\mathbb{R}^d$, $d \geq 3$

- Global well-posedeness (GWP) and scattering
- Bourgain (1999) $d = 3, 4$ radial case
- Grillakis (2000) $d = 3$, radial case, no scattering
- Colliander-Keel-Staffilani-Takaoka-Tao (2008) $d = 3$
- Ryckman-Vişan, Vişan (2007, 2012) $d \geq 4$

Focusing energy-critical NLS : $i\partial_t u + \Delta u = -|u|^{\frac{4}{d-2}} u$ on $\mathbb{R}^d$, $d = 3, 4, 5$:

- GWP and scattering versus finite time blowup solutions
- Kenig-Merle (2006) radial case
- Threshold for scattering : (kinetic) energy of the stationary solution W

$$\Delta W + |W|^4 W = 0$$

Specific features of the cubic-quintic NLS on $\mathbb{R}^3$:

- GWP for all data in $H^1(\mathbb{R}^3)$
- Scattering versus soliton behavior $(e^{it\omega} P_\omega(x))$
- Threshold for scattering : the energies of a branch of (rescaled) solitons

Cubic-Quintic NLS on $\mathbb{R}^3$

- Tao, Vişan, Zhang (2007) :
  - global well-posedness in $H^1(\mathbb{R}^3)$:
    treat (CQ) as a perturbation of defocusing energy-critical NLS
  - scattering for small mass : $M(u_0) \leq c(\|\nabla u_0\|_{L^2})$
- Crucial space-time norm for scattering : $L^{10}_{t,x}(\mathbb{R} \times \mathbb{R}^3)$

$$\|u\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty \Longrightarrow \text{ solution } u \text{ scatters}$$

- Virial $V(u) := \int |\nabla u|^2 - \frac{3}{4}|u|^4 + |u|^6 dx$
- $E_{\min}$ defined by

$$E_{\min}(m) := \inf\left\{E(f) : f \in H^1(\mathbb{R}^3), M(f) = m, V(f) = 0\right\}$$

Theorem (R. Killip, T. Oh, O.P., M. Vişan 2012)

If $u_0 \in H^1(\mathbb{R}^3)$ is such that $0 < E(u_0) < E_{\min}(M(u_0))$, then the corresponding solution scatters both forward and backward in time.

Strategy of proof

- Variational part
  - describe the region $\mathcal{R}$ given by $0 \leq E(f) < E_{\min}(M(f))$
  - find the minimizers of $E_{\min}$ when they exist : (rescaled) solitons
- Dispersive part
  - to specify the region $\mathcal{R}$, introduce

$$D(f) := E(f) + \frac{M(f) + E(f)}{\operatorname{dist}((M(f), E(f)), \overline{\text{epigraph } E_{\min}})}$$

$D \ll 1$ near the origin and $D \to \infty$ near the graph of $E_{\min}$

  - $L(D) := \sup \left\{ \|u\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} : u \text{ solution, } D(u) \leq D \right\}$
  - Scattering in the region $\mathcal{R} \iff L(D) < \infty$ for all $0 < D < \infty$
  - By small data theory : $D \ll 1 \implies L(D) < \infty$
  - Argue by contradiction : there exists a critical $0 < D_c < \infty$ such that

$$L(D) < \infty \text{ if } D < D_c$$
$$L(D) = \infty \text{ if } D > D_c$$

Variational part : Solitons

- Soliton solutions of the form $u(t,x) = e^{i\omega t} P_\omega(x)$, where P_ω satisfies

$$\Delta P_\omega - |P_\omega|^4 P_\omega + |P_\omega|^2 P_\omega - \omega P_\omega = 0$$

- Berestycki-Lions (1983) : radial, positive solitons P_ω exist $\iff \omega \in (0, \frac{3}{16})$
- Serrin-Tang (2000) : uniqueness of non-negative radial solitons P_ω
- Vanishing virial : $V(P_\omega) = 0$ for all $\omega \in (0, \frac{3}{16})$.
- Desyatnikov et al. (Phys. Rev. E, 2000) and Mihalache et al. (Phys. Rev. Lett., 2002) show the existence of two branches of solitons
- We can prove analytically the asymptotic behavior of $M(P_\omega)$ and $E(P_\omega)$ as $\omega \to 0$ and as $\omega \to \frac{3}{16}$, but NOT their monotonicity

β-Gagliardo-Nirenberg inequality on $\mathbb{R}^3$

$$\|u\|_{L^4}^4 \leq C_\beta \|u\|_{L^2} \|u\|_{L^6}^{\frac{3\beta}{1+\beta}} \|\nabla u\|_{L^2}^{\frac{3}{1+\beta}}$$

<u>Proof</u> : $\|u\|_{L^4}^4 \leq \|u\|_{L^2} \|u\|_{L^6}^3 \leq \|u\|_{L^2} \|u\|_{L^6}^{\frac{3\beta}{1+\beta}} \|\nabla u\|_{L^2}^{\frac{3}{1+\beta}}$

- The optimal constant is achieved modulo scaling by positive, radial Q_β satisfying
$$\Delta Q_\beta - Q_\beta^5 + Q_\beta^3 - \omega_\beta Q_\beta = 0$$
- In particular, Q_β is equal to the soliton P_{ω_β} for some $\omega_\beta \in (0, \frac{3}{16})$

β-Gagliardo-Nirenberg inequality on $\mathbb{R}^3$

$$\|u\|_{L^4}^4 \le C_\beta \|u\|_{L^2} \|u\|_{L^6}^{\frac{3\beta}{1+\beta}} \|\nabla u\|_{L^2}^{\frac{3}{1+\beta}}$$

Proof : $\|u\|_{L^4}^4 \le \|u\|_{L^2}\|u\|_{L^6}^3 \le \|u\|_{L^2} \|u\|_{L^6}^{\frac{3\beta}{1+\beta}} \|\nabla u\|_{L^2}^{\frac{3}{1+\beta}}$

- The optimal constant is achieved modulo scaling by positive, radial Q_β satisfying

$$\Delta Q_\beta - Q_\beta^5 + Q_\beta^3 - \omega_\beta Q_\beta = 0$$

- In particular, Q_β is equal to the soliton P_{ω_β} for some $\omega_\beta \in (0, \frac{3}{16})$
- Uniqueness of Q_β is NOT known
- The optimal constant can be expressed as

$$C_\beta = \frac{4(1+\beta)}{3\beta^{\frac{\beta}{2(1+\beta)}}} \cdot \frac{1}{\|Q_\beta\|_{L^2} \|\nabla Q_\beta\|_{L^2}^{\frac{1-\beta}{1+\beta}}}$$

- For $\beta = 1$ and any Q_1, we have unique mass $M(Q_1) = \left(\frac{8}{3C_1}\right)^2$
- Fix an optimizer Q_1 and define $R(x) := \frac{1}{\sqrt{2}} Q_1(\frac{\sqrt{3}}{2}x)$

Feasible pairs $(M(u), E(u))$ for $u \in H^1(\mathbb{R}^3)$

All the feasible pairs $(M(u), E(u))$ lie above or on the graph of $\mathcal{E}$

$$\mathcal{E}(m) := \inf \left\{ E(f) : f \in H^1(\mathbb{R}^3), M(f) = m \right\},$$

with the following properties :

- $\mathcal{E}$ is continuous, concave, non-increasing
- We have that
  - $\mathcal{E}(m) = 0$ if $0 \leq m \leq M(Q_1)$
  - $\mathcal{E}(m) < 0$ if $m > M(Q_1)$
- $m \geq M(Q_1)$: the infimum $\mathcal{E}(m)$ is achieved by a soliton P_ω
  Indeed, by the Euler-Lagrange equation :

$$\frac{dE}{du} = -\frac{\omega}{2}\frac{dM}{du} \Longleftrightarrow \Delta u - u^5 + u^3 - \omega u = 0$$

- $m < M(Q_1)$: the infimum $\mathcal{E}(m) = 0$ is not achieved

Positive virial for mass less than $M(R)$

Virial $V(u) := \int |\nabla u|^2 - \frac{3}{4}|u|^4 + |u|^6 dx$

Lemma

If $M(u) < M(R)$, *then* $V(u) > 0$.

Proof : Since $C_1 = \frac{8}{3\|Q_1\|_{L^2}}$, Gagliardo-Nirenberg inequality for $\beta = 1$ and Young's inequality yield :

$$\begin{aligned}
\frac{3}{4}\int |u|^4 dx &\leq 2\frac{\|u\|_{L^2}}{\|Q_1\|_{L^2}}\Big(\int |u|^6 dx\Big)^{1/4}\Big(\int |\nabla u|^2 dx\Big)^{3/4} \\
&\leq \frac{3^{3/4}}{2}\frac{\|u\|_{L^2}}{\|Q_1\|_{L^2}}\Big(4\int |u|^6 dx\Big)^{1/4}\Big(\frac{4}{3}\int |\nabla u|^2 dx\Big)^{3/4} \\
&\leq \frac{3^{3/4}}{2}\frac{\|u\|_{L^2}}{\|Q_1\|_{L^2}}\int (|u|^6 + |\nabla u|^2) dx.
\end{aligned}$$

Thus, if $\|u\|_{L^2} < \frac{2}{3^{3/4}}\|Q_1\|_{L^2} = \|R\|_{L^2}$, then $V(u) > 0$.

Remark : R has the smallest positive mass with vanishing virial.

The region $\mathcal{R} : 0 \leq E(f) < E_{\min}(M(f))$

$$E_{\min}(m) := \inf \left\{ E(f) : f \in H^1(\mathbb{R}^3), M(f) = m, V(f) = 0 \right\}$$

- $E_{\min}(m) = \infty$ if $m < M(R)$
- $E_{\min}(m) = \mathcal{E}(m)$ if $m \geq M(Q_1)$
- For $M(R) \leq m < M(Q_1)$, $E_{\min}(m)$ is
  - finite
  - strictly decreasing
  - attained by solitons P_ω or rescaled solitons $aP_\omega(\lambda x)$
- Indeed, the Euler-Lagrange equation yields

$$\frac{dE}{du} = -\mu \frac{dV}{du} - \nu \frac{dM}{du} \iff -\Delta u + u^5 - u^3 = -\mu(-2\Delta u + 6u^5 - 3u^3) - 2\nu u$$

  - u is a soliton if $\mu = 0$
  - u is a rescaled soliton for $\mu \in (0, \infty] : aP_\omega(\lambda x)$

$$a = \sqrt{\frac{1+3\mu}{1+6\mu}}, \quad \lambda = \frac{1+3\mu}{\sqrt{(1+2\mu)(1+6\mu)}}, \quad \omega = \frac{\nu(1+6\mu)}{(1+3\mu)^2}$$

Positive virial in the region $\mathcal{R} : 0 \leq E(f) < E_{\min}(M(f))$

Lemma

We have $V(u) > 0$ for all $(M(u), E(u)) \in \mathcal{R}$.

Proof :

- By the definition of $E_{\min}$, $V(u) \neq 0$ for all $u \in \mathcal{R}$.
- Suppose there is $u \in \mathcal{R}$ such that $V(u) < 0$. Then, there is $\lambda_0 > 1$ such that $u^{\lambda_0}(x) := \lambda_0^{\frac{3}{2}} u(\lambda_0 x)$ satisfies

$$V(u^{\lambda_0}) = 0, \quad M(u^{\lambda_0}) = M(u), \quad 0 < E(u^{\lambda_0}) < E(u) < E_{\min}(M(u)),$$

$\implies$ contradiction

Positive virial in the region $\mathcal{R} : 0 \le E(f) < E_{\min}(M(f))$

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$$V(u^{\lambda_0}) = 0, \quad M(u^{\lambda_0}) = M(u), \quad 0 < E(u^{\lambda_0}) < E(u) < E_{\min}(M(u)),$$

$\Longrightarrow$ contradiction

- $\frac{d}{d\lambda} E(u^\lambda) = \frac{V(u^\lambda)}{\lambda} = \lambda \int |\nabla u|^2 dx - \frac{3}{4}\lambda^2 \int |u|^4 dx + \lambda^5 \int |u|^6 dx$
- $V(u) < 0$ and $\lim_{\lambda\to\infty} V(u^\lambda) = \infty \Longrightarrow$ there exists $\lambda_0 > 1$ with $V(u^{\lambda_0}) = 0$
- take the first such $\lambda_0 \Longrightarrow V(u^\lambda) < 0$ for $\lambda \in [1, \lambda_0) \Longrightarrow E(u^{\lambda_0}) < E(u)$

Dispersive part

We use the **concentration-compactness and rigidity method** developed by Kenig-Merle.

1. Concentration-compactness :
   - Using a profile decomposition theorem for bounded sequences in $H^1(\mathbb{R}^3)$, we prove the existence of a minimal blowup solution u : $D(u) = D_c < \infty$ and
$$\|u\|_{L^{10}_{t,x}((-\infty,0]\times\mathbb{R}^3)} = \|u\|_{L^{10}_{t,x}([0,\infty)\times\mathbb{R}^3)} = \infty$$
   - u is well-localized in physical space (almost periodic modulo translations) : for all $\eta > 0$ there exists $C(\eta)$ and $|x(t)| = o(t)$ as $t \to \infty$ such that
$$\sup_{t\in\mathbb{R}} \int_{|x-x(t)|\geq C(\eta)} \big(|\nabla u|^2 + |u|^6 + |u|^4 + |u|^2\big)dx \leq \eta$$
2. Rigidity :
   - such an almost periodic blowup solution cannot occur
   - key element : $V(f) > 0$ for all f in the region $\mathcal{R}$

Dispersive part : Profile decomposition

- Strichartz admissible pair (p, q) : $\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \implies (10, \frac{30}{13})$ is admissible
- Strichartz + Sobolev inequality :

$$\|e^{it\Delta} f\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq \|\nabla e^{it\Delta} f\|_{L^{10}_t L^{\frac{30}{13}}_x} \leq \|\nabla f\|_{L^2}$$

Theorem (Profile decomposition theorem)

Let $\{f_n\}_{n\in\mathbb{N}}$ be bounded in $H^1(\mathbb{R}^3)$. Then, there exist $J^* \in \{0, 1, 2, \ldots\} \cup \infty$,

$$\phi^j \in \dot{H}^1(\mathbb{R}^3) \setminus \{0\}, \quad \{\lambda^j_n\}_{n\in\mathbb{N}} \subset (0, 1], \quad \{t^j_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \quad \text{and} \quad \{x^j_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$$

$$\lambda^j_n \equiv 1 \;\; \text{or} \;\; \lambda^j_n \to 0 \text{ as } n \to \infty, \quad \text{and} \quad t^j_n \equiv 0 \;\; \text{or} \;\; t^j_n \to \pm\infty \text{ as } n \to \infty.$$

Moreover, with

$$P^j_n \phi^j := \begin{cases} \phi^j, & \text{if } \lambda^j_n \equiv 1, \\ P_{\geq (\lambda^j_n)^\theta} \phi^j, & \text{if } \lambda^j_n \to 0, \end{cases} \quad 0 < \theta < 1,$$

we have

$$f_n(x) = \sum_{j=1}^{J} (\lambda^j_n)^{-\frac{1}{2}} \left(e^{it^j_n \Delta} P^j_n \phi^j \right) \left(\frac{x - x^j_n}{\lambda^j_n} \right) + w^J_n \quad \text{in } H^1(\mathbb{R}^3),$$

with

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta} w^J_n\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = 0.$$

Decoupling of mass and energy

$$e^{-it_n^j\Delta}(\lambda_n^j)^{\frac{1}{2}} w_n^J(\lambda_n^j \cdot + x_n^j) \rightharpoonup 0 \quad \text{in } \dot{H}^1(\mathbb{R}^3), \text{ for all } 1 \le j \le J$$

$$\sup_J \lim_{n\to\infty} \Big(M(f_n) - \sum_{j=1}^{J} M(g_n^j) - M(w_n^J) \Big) = 0$$

$$\sup_J \lim_{n\to\infty} \Big(E(f_n) - \sum_{j=1}^{J} E(g_n^j) - E(w_n^J) \Big) = 0$$

$$\lim_{n\to\infty} \left[\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|x_n^j - x_n^{j'}|^2}{\lambda_n^j \lambda_n^{j'}} + \frac{\left| t_n^j (\lambda_n^j)^2 - t_n^{j'} (\lambda_n^{j'})^2 \right|}{\lambda_n^j \lambda_n^{j'}} \right] = \infty, \quad \text{for all } j \neq j',$$

where g_n^j is defined by

$$g_n^j(x) := (\lambda_n^j)^{-\frac{1}{2}} \left(e^{it_n^j\Delta} P_n^j \phi^j \right) \left(\frac{x - x_n^j}{\lambda_n^j} \right)$$

Compactness of minimizing sequences

Theorem

If $D(u_n) \to D_c$ *and* $\lim_{n\to\infty} \|u_n\|_{L^{10}_{t,x}([t_n,\infty)\times\mathbb{R}^3)} = \lim_{n\to\infty} \|u_n\|_{L^{10}_{t,x}((-\infty,t_n]\times\mathbb{R}^3)} = \infty$, *then there exists* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ *such that, on a subsequence,*

$$u_n(t_n, \cdot + x_n) \to \phi \text{ in } H^1(\mathbb{R}^3).$$

Proof : $t_n \equiv 0$, $D(u_n(0)) \le D_c + 1 < \infty \implies \{u_n(0)\}_{n\in\mathbb{N}}$ bounded in $H^1(\mathbb{R}^3)$:

$$M(u_n(0)) \to M_0 < M(Q_1), \quad E(u_n(0)) \to E_0, \quad \|\nabla u_n(0)\|_{L^2}^2 \sim E(u_n(0))$$

Profile decomposition : $u_n(0) = \sum_{j=1}^J (\lambda_n^j)^{-\frac{1}{2}} \left(e^{it_n^j\Delta} P_n^j \phi^j\right)\left(\frac{x - x_n^j}{\lambda_n^j}\right) + w_n^J$

Nonlinear profiles for each j :

- if $\lambda_n^j \equiv 1$, $t_n^j \equiv 0 \implies v^j(0) = \phi^j$ and $v_n^j(t,x) = v^j(t + t_n^j, x - x_n^j)$
- if $\lambda_n^j \equiv 1$, $t_n^j \to \pm\infty \implies v^j$ scatters forward/backward to $e^{it\Delta}\phi^j$
- if $\lambda_n^j \to 0 \implies v_n^j(0) = (\lambda_n^j)^{-\frac{1}{2}} \left(e^{it_n^j\Delta} P_n^j \phi^j\right)\left(\frac{x - x_n^j}{\lambda_n^j}\right)$:

$$v_n^j(t,x) \sim (\lambda_n^j)^{-\frac{1}{2}} w^j\left(\frac{t}{(\lambda_n^j)^2} + t_n^j, \frac{x - x_n^j}{\lambda_n^j}\right),$$

w^j is the solution of quintic NLS with $w^j(0) = \phi^j$

By mass and energy decoupling :

$$\limsup_{n\to\infty} \sum_{j=1}^{J^*} \left(M(v_n^j(0)) + M(w_n^{J^*}) \right) \leq M_0$$

$$\limsup_{n\to\infty} \sum_{j=1}^{J^*} E(v_n^j(0)) + E(w_n^{J^*}) \leq E_0$$

1. Case 1 : $\limsup_{n\to\infty} \sup_j M(v_n^j(0)) = M_0$ and $\limsup_{n\to\infty} \sup_j E(v_n^j(0)) = E_0$ $\Longrightarrow$ only one profile ϕ^1 and $\lambda_n^1 \equiv 1$, $t_n^1 \equiv 0$:

$$u_n(0,x) = \phi^1(x - x_n^1) + w_n^1 \text{ and } w_n^1 \to 0 \text{ in } H^1(\mathbb{R}^3)$$

2. Case 2 : $\limsup_{n\to\infty} \sup_j M(v_n^j(0)) < M_0$ or $\limsup_{n\to\infty} \sup_j E(v_n^j(0)) < E_0$
   - $D(v_n^j) \leq D_c - \varepsilon$ for all j
   - Introduce : $$u_n^J(t) := \sum_{j=1}^{J} v_n^j(t) + e^{it\Delta} w_n^J, \quad J \text{ large, finite}$$

By mass and energy decoupling :

$$\limsup_{n\to\infty} \sum_{j=1}^{J^*} \left(M(v_n^j(0)) + M(w_n^{J^*}) \right) \le M_0$$

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$$u_n(0,x) = \phi^1(x - x_n^1) + w_n^1 \text{ and } w_n^1 \to 0 \text{ in } H^1(\mathbb{R}^3)$$

2. Case 2 : $\limsup_{n\to\infty} \sup_j M(v_n^j(0)) < M_0$ or $\limsup_{n\to\infty} \sup_j E(v_n^j(0)) < E_0$
   - $D(v_n^j) \le D_c - \varepsilon$ for all j
   - Introduce :
   $$u_n^J(t) := \sum_{j=1}^{J} \underbrace{v_n^j(t)}_{\text{scatter}} + \underbrace{e^{it\Delta} w_n^J}_{\to 0}, \quad J \text{ large, finite}$$
   - u_n^J approximates well u_n (in the $L^{10}_{t,x}$-norm)
   - By perturbation theory $\Longrightarrow$ u_n scatters $\Longrightarrow$ contradiction

By mass and energy decoupling :

$$\limsup_{n\to\infty} \sum_{j=1}^{J^*} \Big(M(v_n^j(0)) + M(w_n^{J^*}) \Big) \le M_0$$

$$\limsup_{n\to\infty} \sum_{j=1}^{J^*} E(v_n^j(0)) + E(w_n^{J^*}) \le E_0$$

1. Case 1 : $\limsup_{n\to\infty} \sup_j M(v_n^j(0)) = M_0$ and $\limsup_{n\to\infty} \sup_j E(v_n^j(0)) = E_0$
$\Longrightarrow$ only one profile ϕ^1 and $\lambda_n^1 \equiv 1$, $t_n^1 \equiv 0$:

$$u_n(0,x) = \phi^1(x - x_n^1) + w_n^1 \text{ and } w_n^1 \to 0 \text{ in } H^1(\mathbb{R}^3)$$

2. Case 2 : $\limsup_{n\to\infty} \sup_j M(v_n^j(0)) < M_0$ or $\limsup_{n\to\infty} \sup_j E(v_n^j(0)) < E_0$
   - $D(v_n^j) \le D_c - \varepsilon$ for all j
   - Introduce :

$$\underbrace{u_n^J(t)}_{\text{scatters}} := \sum_{j=1}^{J} \underbrace{v_n^j(t)}_{\text{scatter}} + \underbrace{e^{it\Delta} w_n^J}_{\to 0}, \quad J \text{ large, finite}$$

   - u_n^J approximates well u_n (in the $L^{10}_{t,x}$-norm)
   - By perturbation theory $\Longrightarrow$ u_n scatters $\Longrightarrow$ contradiction

Extraction of the minimal blowup solution

- Consider solutions $\{u_n\}_{n\in\mathbb{N}}$ of (CQ) such that

$$D(u_n) \to D_c \text{ and } \lim_{n\to\infty} \|u_n\|_{L^{10}_{t,x}((-\infty,0]\times\mathbb{R}^3)} = \lim_{n\to\infty} \|u_n\|_{L^{10}_{t,x}([0,\infty)\times\mathbb{R}^3)} = \infty$$

- By the compactness property of minimizing sequences :

$$u_n(0,\cdot + x_n) \to u_0 \text{ in } H^1(\mathbb{R}^3).$$

- Consider u the solution of (CQ) with $u(0) = u_0$. Then :

$$D(u_n) \to D(u) = D_c \text{ and } \|u\|_{L^{10}_{t,x}((-\infty,0]\times\mathbb{R}^3)} = \|u\|_{L^{10}_{t,x}([0,\infty)\times\mathbb{R}^3)} = \infty$$

$\implies u$ is a minimal blowup solution

- u is almost periodic modulo translations : for any $\{t_n\}_{n\in\mathbb{N}}$, there exists $\{x(t_n)\}_{n\in\mathbb{N}}$ such that

$$u(t_n,\cdot + x(t_n)) \to u_\infty \text{ in } H^1(\mathbb{R}^3)$$

Virial bounded away from zero for the minimal blowup solution

- For the minimal blowup solution $u : D(u) = D_c < \infty$
- $(M(u), E(u)) \in \mathcal{R} \Longrightarrow V(u(t)) > 0$ for all $t \in \mathbb{R}$

Virial bounded away from zero for the minimal blowup solution

- For the minimal blowup solution $u : D(u) = D_c < \infty$
- $(M(u), E(u)) \in \mathcal{R} \Longrightarrow V(u(t)) > 0$ for all $t \in \mathbb{R}$
- Actually, there is $\delta > 0$ such that $V(u(t)) > \delta$ for all $t \in \mathbb{R}$.
- Indeed, assume $V(u(t_n)) \to 0$. By compactness of the minimal blowup solution u, there exists $u_\infty \in H^1$ such that

$$u(t_n, \cdot + x(t_n)) \to u_\infty \text{ in } H^1(\mathbb{R}^3).$$

- By Sobolev embeddings, it follows that

$$M(u(t_n)) = M(u_\infty), \quad E(u(t_n)) = E(u_\infty) \Longrightarrow u_\infty \in \mathcal{R}$$
$$V(u(t_n)) \to V(u_\infty) = 0,$$

$\Longrightarrow$ contradiction

Rigidity argument

- Almost periodic minimal blowup solution : for all $\eta > 0$ there exists $C(\eta)$ and $|x(t)| = o(t)$ as $t \to \infty$ s.t.

$$\sup_{t\in\mathbb{R}} \int_{|x-x(t)|\geq C(\eta)} (|\nabla u|^2 + |u|^6 + |u|^4 + |u|^2)dx \leq \eta$$

- Consider $\phi(r) = 1$ if $r \leq 1$ and $\phi(r) = 0$ if $r \geq 2$ and

$$U_R(t) := 2\mathrm{Im} \int \phi(\frac{x}{R})\overline{u(t)}x \cdot \nabla u(t)dx$$

- $|U_R(t)| \leq C'R$ and

$$\partial_t U_R(t) = 4V(u(t)) + O\Big(\int_{|x|\geq R}(|\nabla u|^2 + |u|^6 + |u|^4 + |u|^2)dx\Big)$$

- Taking $R = C(\eta) + \sup_{t\in[T_0,T_1]} |x(t)|$, we get $\partial_t U_R(t) > V(u(t)) > \delta$

- Integrating from T_0 to T_1 :

$$\delta(T_1 - T_0) < U_R(T_1) - U_R(T_0) \leq 2C'R \leq C'(\eta) + \frac{\delta}{2}T_1$$

- Taking $T_1 \to \infty \implies$ contradiction

Maximality of the connected region $\mathcal{R}$

Lemma

The region $\mathcal{R}$ is a maximal connected component of the origin with positive virial.

Proof : For any u minimizer of $E_{\min}(m)$, there exist two branches of continuous curves emanating from u, situated above the graph of $E_{\min}$ such that :

- for $u^\lambda(x) := \lambda^{\frac{3}{2}} u(\lambda x)$ with $1 < \lambda \leq 1 + \varepsilon$, we have $V(u^\lambda) > 0$
- for $u_\lambda(x) := \lambda^{\frac{1}{2}} u(\lambda x)$ with $1 - \varepsilon \leq \lambda < 1$, we have $V(u_\lambda) < 0$

Comparison to other NLS with combined-power nonlinearity

- Miao, Xu, Zhao (2011) :
  - focusing quintic, defocusing cubic NLS on $\mathbb{R}^3$, radial data
  - $e := \inf\{E(u) : \mathcal{K}(u) = 0\}$ is not achieved
  - $e = E^c(W)$ where E^c is a modified energy and W is the stationary solution for focusing energy-critical NLS
  - GWP and scattering if $E(u_0) < e$ and $\mathcal{K}(u_0) \geq 0$
- Akahori, Ibrahim, Hikuchi, Nawa (2012) :
  - both nonlinearities are focusing, one energy critical, the other energy-subcritical, on $\mathbb{R}^d$ for $d \geq 5$
  - $s := \inf\{S_\omega(u) = E(u) + \omega M(u) : \mathcal{K}(u) = 0\}$
  - s is achieved by solitons
  - GWP and scattering if $S_\omega(u_0) < s$ and $\mathcal{K}(u_0) > 0$
- In both the above cases :
  - there are finite time blowup solutions : the equations have a different nature than (CQ)
  - less precise understanding of the scattering region

Open issues

- Uniqueness of the optimizers Q_β of the β-Gagliardo-Nirenberg inequality
- Any optimizer Q_β is a soliton P_ω. Is any soliton P_ω an optimizer $Q_{\beta(\omega)}$?
- On the curve $E_{\min}$, $M(\text{rescaled solitons}) < M(P_{\omega^*}) \leq M(\text{solitons})$
- Scattering above the graph of $E_{\min}(m)$ for $M(R) \leq m < M(P_{\omega^*})$
- Continuity of $m \mapsto E_{\min}(m)$ for $M(R) \leq m < M(Q_1)$
- Stability/instability of solitons : related to the monotonicity of $\omega \mapsto M(P_\omega)$
  - lower branch is conjectured to be stable
  - upper branch is conjectured to be unstable
- Behavior near unstable solitons
  - after a transitory period of time, perturbations of unstable solitons are expected to approach stable solitons
  - Buslaev, Grikurov (2001), LeMesurier, Papanicolaou, Sulem, Sulem (1988)