Explicit formula for the solution of the Szegö equation on the real line and applications

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Nonlinear Dispersive PDEs and Related Topics
Institut Henri Poincaré

- The cubic Szegö equation

$$\text{(SE)} \qquad i\partial_t u = \Pi_+(|u|^2 u), \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where Π_+ is the Szegö projector onto positive frequencies, was recently introduced by Gérard and Grellier who study it on $\mathbb{T}$

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- It is completely integrable $\Longrightarrow$ we find an explicit solution for the solution

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- Applications of the explicit formula:

1. **Soliton resolution** for generic rational function solutions

2. Example of a **non-generic solution whose high Sobolev norms grow to infinity**

3. **Growth of the high Sobolev norms of a solution of a nonlinear wave equation** (NLW), whose resonant dynamics are given by the Szegö equation.

Motivation: NLS on the sub-Riemannian manifolds

- The first motivation is the study of the nonlinear Schrödinger equation

  (NLS) $$i\partial_t u + \Delta u = |u|^2 u,$$

  on sub-Riemannian manifolds (e.g. the Heisenberg group)
- NLS on the Heisenberg group lacks dispersion

  $\Rightarrow$ classical tools break down

  $\Rightarrow$ even the problem of well-posedness is open.
- $\mathbb{H}^1 = \mathbb{C}_z \times \mathbb{R}_s$, $L^2_{rad}(\mathbb{H}^1) = \oplus_\pm \oplus_{m=0}^\infty V_m^\pm$ and $\Delta_{\big|V_m^\pm} = \pm i(2m+1)\frac{\partial}{\partial_s}$.

  Then, NLS is equivalent to the system:

  $$i\partial_t u_m^\pm \pm i(2m+1)\partial_s u_m^\pm = \Pi_m^\pm(|u|^2 u),$$

  where $\Pi_m^\pm$ is the projection onto $V_m^\pm$.
- One needs to study the interaction between the cubic nonlinearity and the projectors $\Pi_m^\pm$, $i\partial_t u = \Pi_m^\pm(|u|^2 u)$, which leads to the Szegö equation.

Motivation: A non-linear wave equation

(NLW) $$i\partial_t u - |D|u = |u|^2 u$$

- Applying the operator $i\partial_t + |D|$ to both sides, we obtain the wave equation:

$$-\partial_{tt} v + \Delta v = |v|^4 v + 2|v|^2(|D|v) - v^2(|D|\bar{v}) + |D|(|v|^2 v).$$

- It decouples into the system of transport equations

$$\begin{cases} i(\partial_t u_+ + \partial_x u_+) = \Pi_+(|u|^2 u) \\ i(\partial_t u_- - \partial_x u_-) = \Pi_-(|u|^2 u). \end{cases}$$

- Dynamics dominated by u_+: if $u(0) = u_+(0)$ and $\|u(0)\|_{H^{1/2}} = \varepsilon$, then $\|u_-(t)\|_{\dot{H}^{1/2}} = O(\varepsilon^2)$.
- Forgetting the small terms in u_- in the nonlinearity, $v(t,x) = u_+(t, x+t)$ almost satisfies

$$i\partial_t v = \Pi_+(|v|^2 v)$$

The Hardy space and the Szegö projector

Consider the Hardy space:

$$
\begin{aligned}
L^2_+(\mathbb{R}) =& \left\{ f \text{ holomorphic on } \mathbb{C}_+ \,\middle|\, \|g\|_{L^2_+(\mathbb{R})} := \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^2 dx \right)^{1/2} < \infty \right\} \\
=& \{ f \in L^2(\mathbb{R}) \,|\, \operatorname{supp} \hat{f} \subset [0,\infty) \}
\end{aligned}
$$

and the corresponding Sobolev spaces $H^s_+(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$.

The Szegö projector on the Hardy space is $\Pi_+ : L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$:

$$
\mathcal{F}(\Pi_+ f)(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \xi \geq 0, \\ 0, & \text{if } \xi < 0. \end{cases}
$$

We also set $\Pi_- = I - \Pi_+$. The Szegö projector gives the name of the Szegö equation:

(SE) $$i\partial_t u = \Pi_+(|u|^2 u), \qquad x \in \mathbb{R}.$$

Conservation laws

Symplectic form on $L^2_+(\mathbb{R})$:

$$\omega(u,v) = 4\mathrm{Im}\int_{\mathbb{R}} u\bar{v}.$$

Hamiltonian:

$$E(u) = \int_{\mathbb{R}} |u|^4 dx,$$

Mass:

$$Q(u) = \int_{\mathbb{R}} |u|^2 dx,$$

Momentum:

$$M(u) = (Du, u)_{L^2} \geq 0, \quad \text{with } D = -i\partial_x.$$

The $H^{1/2}_+$-norm of the solution is conserved:

$$Q(u) + M(u) = \|u\|^2_{H^{1/2}_+}.$$

The Cauchy problem

Theorem

For all $u_0 \in H^{1/2}_+$, *there exists a unique global solution* $u \in C(\mathbb{R}, H^{1/2}_+)$ *of the equation*

$$\text{(SE)} \qquad i\partial_t u = \Pi_+(|u|^2 u)$$

such that $u(0) = u_0$.
Moreover, if $u_0 \in H^s_+$, $s > 1/2$, *then* $u \in C(\mathbb{R}, H^s_+)$.

Hankel and Toeplitz operators

- Hankel operator of symbol $u \in H^{1/2}_+$: $H_u : L^2_+ \to L^2_+$

$$H_u h = \Pi_+(u\bar{h})$$

Compact operator, $\mathbb{C}$-antilinear, in particular

$$(H_u h_1, h_2)_{L^2} = (H_u h_2, h_1)_{L^2}.$$

H_u^2 is a self-adjoint linear operator.

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H_u^2 is a self-adjoint linear operator.

- Toeplitz operator of symbol $b \in L^\infty(\mathbb{R})$: $T_b : L^2_+ \to L^2_+$

$$T_b h = \Pi_+(bh)$$

Bounded, linear operator, self-adjoint iff b is real-valued.

Lax pair structure

Theorem (Lax pair formulation)

$u \in C(\mathbb{R}, H^s_+)$, $s > 1/2$ is a solution of the Szegö equation iff

$$\partial_t H_u = [B_u, H_u],$$

where $B_u = \frac{i}{2} H_u^2 - i T_{|u|^2}$.

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Corollary

There exists an infinite sequence of conservation laws:

$$J_n(u) := (u, H_u^{n-2} u), n \geq 2$$

$$\partial_t J_{2n}(u(t)) = 0.$$

In particular, $J_2(u) = Q(u)$ and $J_4(u) = \frac{E(u)}{2}$.

Remark: The conservation law of the $H^{1/2}_+$-norm is stronger than that of J_{2n}

$$J_{2n}(u) \leq \|u\|^{2n}_{L^{2n}(\mathbb{R})} \leq \|u\|^{2n}_{H^{1/2}_+(\mathbb{R})}.$$

Invariant finite dimensional submanifolds of L^2_+

$$\mathcal{M}(N) = \text{"rational functions of degree N"}$$
$$= \left\{ \frac{A(z)}{B(z)} \middle| A, B \in \mathbb{C}_N[z], 0 \le \deg(A) \le N-1, \deg(B) = N, \right.$$
$$\left. B(0) = 1, B(z) \neq 0, \text{ for all } z \in \mathbb{C}_+ \cup \mathbb{R}, (A, B) = 1 \right\}$$

Remarks: $\mathcal{M}(N)$ is 4N-dimensional real manifold
$\bigcup_{N \in \mathbb{N}^*} \mathcal{M}(N)$ is dense in L^2_+

Theorem (Kronecker type theorem)

$rk(H_u) = N$ if and only if $u \in \mathcal{M}(N)$.
Moreover, if $u \in \mathcal{M}(N)$, then $u \in Ran(H_u)$,
i.e. there exists a unique $g \in \mathrm{Ran}(H_u)$ such that $u = H_u g$.

Proposition

For all $N \in \mathbb{N}^*$, $\mathcal{M}(N)$ is invariant under the flow of the Szegö equation.

Explicit formula for the solution if $u_0 \in \mathcal{M}(N)$

Notations for $u \in \mathcal{M}(N)$:

- $0 < \lambda_1^2 \le \lambda_2^2 \le \cdots \le \lambda_N^2$ eigenvalues of H_u^2
- $\{e_j\}_{j=1}^N$ orthonormal basis of $\mathrm{Ran}(H_u)$ such that $H_u e_j = \lambda_j e_j$
- $\beta_j = (g, e_j)$, where g is such that $u = H_u g$.

Theorem (P '10 Explicit formula for rational function data)

Suppose that $u_0 \in \mathcal{M}(N)$ *and* $g_0 \in \mathrm{Ran}(H_{u_0})$ *is such that* $u_0 = H_{u_0} g_0$. *Let* $M_j = \{k \in \{1, 2, \dots, N\} \,|\, H_{u_0} e_k = \lambda_j e_k\}$. *We define an operator* $S(t)$ *on* $\mathrm{Ran}(H_{u_0})$, *in the basis* $\{e_j\}_{j=1}^N$, *by*

$$S(t)_{k,j} = \begin{cases} \frac{\lambda_j}{2\pi i(\lambda_k^2 - \lambda_j^2)} \left(\lambda_j e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)} \overline{\beta}_j \beta_k - \lambda_k e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)} \beta_j \overline{\beta}_k \right), & \text{if } k \notin M_j \\ \frac{\lambda_j^2}{2\pi} \overline{\beta}_j \beta_k t + (T e_j, e_k), & \text{if } k \in M_j. \end{cases}$$

Then, the following explicit formula for the solution holds:

$$u(t, x) = \frac{i}{2\pi} \left(u_0, e^{i\frac{t}{2} H_{u_0}^2} (S(t) - xI)^{-1} e^{i\frac{t}{2} H_{u_0}^2} g_0 \right), \text{ for all } x \in \mathbb{R}.$$

Infinitesimal shift operator

Property

Let $T_\lambda : L^2_+ \to L^2_+$ be the shift operator $T_\lambda(f) = e^{i\lambda x} f$, $\mathcal{F}(T_\lambda f)(\xi) = \hat{f}(\xi - \lambda)$. Then, $H : L^2_+ \to L^2_+$ is a Hankel operator if and only if

$$T_\lambda^* H = H T_\lambda, \quad \forall \lambda > 0.$$

The adjoint $T_\lambda^* : L^2_+ \to L^2_+$, defined by

$$T_\lambda^* f(x) = e^{-i\lambda x}(f * \mathcal{F}^{-1}(\chi_{[\lambda,\infty)}))(x),$$

is inconvenient to use.
For $u \in \mathcal{M}(N)$ we define the **infinitesimal shift operator** on $\mathrm{Ran}(H_u)$ by:

$$T(f) = xf - \lim_{x\to\infty} xf(x)(1-g),$$

for all $f \in \mathrm{Ran}(H_u)$, where $H_u g = u$. Then, $T^* H_u = H_u T$.

Explicit formula for solutions with general initial data

Theorem (P '10 Explicit formula for general data)

Let $u_0 \in H^s_+$, $s \geq 1$, $xu_0 \in L^\infty(\mathbb{R})$. Let $M_j = \{k \in \mathbb{N}^* \big| H_{u_0} e_k = \lambda_j e_k\}$. We define $S^*(t)$ on $\mathrm{Ran}(H_{u_0})$, in the basis $\{e_j\}_{j=1}^\infty$ by

$$(S^*(t)e_j, e_k) = \begin{cases} \frac{\lambda_k}{2\pi i(\lambda_k^2 - \lambda_j^2)} \left(\lambda_k e^{i\frac{t}{2}(\lambda_k^2 - \lambda_j^2)} \overline{\beta}_j \beta_k - \lambda_j e^{i\frac{t}{2}(\lambda_j^2 - \lambda_k^2)} \beta_j \overline{\beta}_k \right), & \text{if } k \in \mathbb{N} \setminus M_j \\ \frac{\lambda_k^2 \overline{\beta}_j \beta_k}{2\pi} t + (T^* e_j, e_k), & \text{if } k \in M_j. \end{cases}$$

Let A be the closure of S^*. Then, for $\mathrm{Im} z > 0$, the solution writes

$$u(t,z) = \lim_{\varepsilon \to 0} \frac{i}{2\pi} \left(e^{-i\frac{t}{2} H_{u_0}^2} (A - zI)^{-1} e^{-i\frac{t}{2} H_{u_0}^2} u_0, \frac{1}{1 - i\varepsilon z} \right).$$

Remark: S^* acts on an infinite dimensional space. Explicitly computing $(A - zI)^{-1}$ comes down to solving an infinite system of linear differential equations. The above theorem actually states that we can transform our nonlinear infinite dimensional dynamical system into a linear one.

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- Gérard and Grellier (2010) give a formula for *generic* solutions of the Szegö equation on the torus $\mathbb{T}$ (as a bias of introducing action-angle coordinates). They need **new spectral data** given by the operator $T_z H_u$.

Proof

Main ingredient: Lax pair structure

The proof in the case $u_0 \in \mathcal{M}(N)$ is based on:

- $u_0 \in \mathrm{Ran}(H_{u_0})$, i.e. $u_0 = H_{u_0} g_0$
- $\lim_{x\to\infty} xu(x) \in \mathbb{R}$ if $u \in \mathcal{M}(N)$
- Infinitesimal shift operator T and the commutation relation

$$T^* H_u = H_u T.$$

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The proof in the case of general initial data is based on:

- $u_0 \in \overline{\mathrm{Ran}(H_{u_0})}$, i.e. $u_0 = \lim_{\varepsilon\to 0} H_{u_0}(\frac{1}{1-i\varepsilon x})$ (approximation method)
- $xu_0(x) \in L^\infty \Longrightarrow xu(t,x) \in L^\infty$ for all $t \in \mathbb{R}$
- Definition of the “adjoint of the infinitesimal shift operator“ on $\mathrm{Ran}(H_u)$:

$$T^*(H_u f) = \Pi_+(xu\bar{f}).$$

Solitons

Theorem (P'09 Classification of solitons)

The solitons of the Szegö equation are

$$u(x,t) = e^{-i\omega t}\phi_{C,p}(x-ct), \tag{0.1}$$

where $\phi_{C,p}(x) = \frac{C}{x-p}$, $C, p \in \mathbb{C}$, $\mathrm{Im}p < 0$,

$$\omega = \frac{|C|^2}{4(\mathrm{Im}p)^2}, \qquad c = \frac{|C|^2}{-2\mathrm{Im}p}.$$

A soliton has therefore the form

$$u(t,x) = \frac{e^{-i\omega t}C}{x-ct-p}.$$

Soliton resolution

Strongly generic rational functions:

$\mathcal{M}(N)_{\text{sgen}} = \{u \in \mathcal{M}(N) \,\big|\, 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N, (u, e_j) \neq 0, (u, e_j) \neq (u, e_k)\}.$

Theorem (P '10)

If $u_0 \in \mathcal{M}(N)_{\text{sgen}}$, then the solution of the Szegö equation is

$$u(t,x) = \sum_{j=1}^{N} e^{-it\lambda_j^2} \phi_{C_j,p_j}(x - \frac{\lambda_j^2 \nu_j^2}{2\pi} t) + \varepsilon(t,x)$$

where

$$\phi_{C_j,p_j}(x) = \frac{C_j}{x - p_j}, \;\; C_j = \frac{i\lambda_j \nu_j^2 e^{-2i\phi_j(0)}}{2\pi}, \;\; p_j = \mathrm{Re}(c_j(0)) - i\frac{\nu_j^2}{4\pi},$$

$$\lim_{t\to\pm\infty} \|\varepsilon(t,x)\|_{H^s_+} = 0 \text{ for all } s \geq 0.$$

Comparison with other completely integrable equations

- Soliton resolution holds for KdV (Echaus, Schuur 1983) in $L^\infty(\mathbb{R}_+)$:

$$\lim_{t\to\infty} \|\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} = 0,$$

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- Soliton resolution holds for one dimensional cubic NLS in $L^2(\mathbb{R})$

$$u(t,x) = \text{Solitons} + e^{it\Delta} f + \varepsilon(t,x),$$

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- No soliton resolution for the Szegö equation on $\mathbb{T}$ (Gérard, Grellier).

Growth of high Sobolev norms for non-generic solutions

Theorem (P '10)

Let $u_0 \in \mathcal{M}(2)$ *be such that* $H^2_{u_0}$ *has a double eigenvalue* $\lambda^2 > 0$. *Then*

$$u(t,x) = e^{-it\lambda^2}\phi_{C,p}\Big(x - \frac{\|u_0\|^2_{L^2}}{2\pi}t\Big) + \varepsilon(t,x),$$

where

$$|C| = \frac{\|u_0\|^2_{L^2}}{\sqrt{\pi}\|u_0\|_{\dot{H}^{1/2}_+}}, \quad \mathrm{Im}(p) = -\Big(\frac{\|u_0\|_{L^2}}{\|u_0\|_{\dot{H}^1}}\Big)^2,$$

$$\lim_{t\to\pm\infty} \|\varepsilon(t,x)\|_{H^s_+} = 0 \text{ for } 0 \le s < 1/2.$$

The first term is a soliton. However,

$$\lim_{t\to\pm\infty} \|\varepsilon(t,x)\|_{H^s_+} = \infty \text{ if } s > 1/2.$$

Therefore,

$$\|u(t)\|_{H^s_+} \to \infty \text{ as } t \to \pm\infty \text{ if } s > 1/2.$$

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$$i\partial_t u = |u|^2 u,$$

its solutions are $u(t) = \phi e^{-i|\phi|^2 t}$ and thus $\|u(t)\|_{H^s} \sim |t|^s$ for $s \in \mathbb{N}$.

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- More subtle situation for Szegö: the $H^{1/2}$-norm is conserved. Only the H^s-norms with $s > 1/2$ grow to ∞.

- This shows that the energy ($H^{1/2}$-norm) is supported on higher frequencies, while the mass is supported on lower frequencies: forward cascade. It agrees with the predictions of weak turbulence theory.

Partial results regarding the growth of high Sobolev norms were obtained by:

- Gérard, Grellier (2010) for the Szegö equation on $\mathbb{T}$:

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \geq K(t^\varepsilon)^{2s-1}, \text{ for } s > 1/2 \text{ and } t^\varepsilon \to \infty.$$

- Bourgain (1993, 1995, 1995) for Hamiltonian PDEs with spectrally defined laplacian

- Kuksin (1997) for small dispersion NLS $-i\partial_t u + \varepsilon\Delta u = |u|^2 u$ with odd, periodic boundary condition on $\mathbb{T}^n$

- Colliander, Keel, Staffilani, Takaoka, and Tao (2010) for defocusing cubic NLS on $\mathbb{T}^2$

- Hani (2011) for defocusing truncated cubic NLS on $\mathbb{T}^2$

Proof

If $u_0 \in \mathcal{M}(N)$, we have that $u(t) \in \mathcal{M}(N)$ for all t. We use the explicit formula to decompose $u(t)$ as a sum of simple fractions $\sum_{j=1}^{N} \frac{C_j(t)}{x-p_j(t)} + O(\frac{1}{t})$.

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Notice that

- $u_0 \in \mathcal{M}(N)_{\text{sgen}} \Longrightarrow p_j(t) = a_j t + b_j + O(\frac{1}{t})$ as $t \to \pm\infty$, where $a_j \neq 0$, $a_j \neq a_k$ for $j \neq k$, $\text{Im}(b_j) \neq 0$
$\Longrightarrow$ soliton resolution in H^s for all $s \geq 0$

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  where $a_j \neq 0$, $a_j \neq a_k$ for $j \neq k$, $\text{Im}(b_j) \neq 0$
  $\Longrightarrow$ soliton resolution in H^s for all $s \geq 0$
- u_0 is such that $H_{u_0}^2$ has a double eigenvalue
  $\Longrightarrow$ there exists p_{j_0} with $p_{j_0}(t) = b_{j_0} + O(\frac{1}{t})$ as $t \to \pm\infty$ and $\text{Im}(b_{j_0}) = 0$
  $\Longrightarrow$ one of the poles of $u(t)$ approaches the real line
  $\Longrightarrow \|u(t)\|_{H^s} \to \infty$ as $t \to \pm\infty$ and $s > 1/2$

The Szegö equation as the first approximation of NLW

Theorem (P '11)

Let $W_0 \in H^s_+(\mathbb{R})$, $s > \frac{1}{2}$. *Let* $v(t)$ *be the solution of the NLW on* $\mathbb{R}$

$$\text{(NLW)} \qquad \begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \mathcal{W}_0 = \varepsilon W_0. \end{cases}$$

Denote by $\mathcal{W}(t)$ *the solution of the Szegö equation*

$$\begin{cases} i\partial_t \mathcal{W} = \Pi_+(|\mathcal{W}|^2 \mathcal{W}) \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases}$$

Assume that $\|\mathcal{W}(t)\|_{H^s} \leq C\varepsilon\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^\alpha$ *for* $0 \leq \alpha \leq \frac{1}{2}$ *and* $\delta > 0$ *small.*
Then, if $0 \leq t \leq \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{1-2\alpha}$, *we have that*

$$\|v(t) - e^{-i|D|t}\mathcal{W}(t)\|_{H^s} \leq C\varepsilon^{2-C_0\delta}.$$

Growth of high Sobolev norms for solutions of NLW

Corollary (P '11)

Let $0 < \varepsilon \ll 1$, $s > \frac{1}{2}$, *and* $\delta > 0$ *sufficiently small. Let* $W_0 \in H^s_+(\mathbb{R})$ *be the non-generic rational function* $W_0 = \frac{1}{x+i} - \frac{2}{x+2i}$. *Denote by* $v(t)$ *be the solution of the NLW equation on* $\mathbb{R}$

$$\text{(NLW)} \qquad \begin{cases} i\partial_t v - |D|v = |v|^2 v \\ v(0) = \varepsilon W_0. \end{cases}$$

Then, for $\frac{1}{2\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}} \le t \le \frac{1}{\varepsilon^2}\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{1}{4s-1}}$, *we have that*

$$\frac{\|v(t)\|_{H^s(\mathbb{R})}}{\|v(0)\|_{H^s(\mathbb{R})}} \ge C\Big(\log(\frac{1}{\varepsilon^\delta})\Big)^{\frac{4s-2}{4s-1}} \gg 1.$$

Remark: In order to show arbitrarily large growth of the solution, one needs an approximation at least for a time $0 \le t \le \frac{1}{\varepsilon^{2+\beta}}$, where $\beta > 0$.

Idea of the proof: Resonant dynamics

- IDEA: The dynamics of a nonlinear equation is governed by the resonant part of the nonlinearity
- We use the renormalization group (RG) method introduced by Chen, Goldfend, and Oono (1994) in theoretical physics.
- Gérard and Grellier (2011) proved analogous results on the torus $\mathbb{T}$ using the theory of Birkhoff normal forms
- The resonant dynamics was also used by Colliander, Keel, Stafillani, Takaoka, and Tao (2010) to give an example of solution of cubic NLS on $\mathbb{T}^2$ whose high Sobolev norms grow arbitrarily large
- Grébert and Thomann (preprint 2011) determine the resonant dynamics of the quintic NLS on the torus $\mathbb{T}$

Proof

With the change of variables $u(t) = \frac{1}{\varepsilon} e^{i|D|t} v(t)$ in NLW, we have that u satisfies the equation:

$$\text{(NLW')} \qquad \begin{cases} \partial_t u = -i\varepsilon^2 e^{i|D|t}(|e^{-i|D|t}u|^2 e^{-i|D|t}u) =: \varepsilon^2 f(u,t) \\ u(0) = W_0. \end{cases}$$

We write the nonlinearity in the Fourier space:

$$\mathcal{F}\big(f(u,t)\big)(\xi) = -\, i \iint e^{it\phi(\xi,\eta,\zeta)} \hat{u}(\eta-\zeta)\hat{u}(\zeta)\overline{\hat{u}}(\eta-\xi)d\zeta d\eta,$$

where $\phi(\xi,\eta,\zeta) := |\xi| - |\zeta| + |\eta-\xi| - |\eta-\zeta|$. Then

$$f(u,t) = f_{\text{res}}(u) + f_{\text{osc}}(u,t),$$

where

$$f_{\text{res}}(u) : = -i\mathcal{F}^{-1} \iint_{\phi=0} \hat{u}(\eta-\zeta)\hat{u}(\zeta)\overline{\hat{u}}(\eta-\xi)d\zeta d\eta,$$

$$f_{\text{osc}}(u,t) : = -i\mathcal{F}^{-1} \iint_{\phi\neq 0} e^{it\phi(\xi,\eta,\zeta))} \hat{u}(\eta-\zeta)\hat{u}(\zeta)\overline{\hat{u}}(\eta-\xi)d\zeta d\eta.$$

Specificity of NLW: many resonances

The set $\{\phi(\xi,\eta,\zeta)=0\}\subset\mathbb{R}^2$ has non-zero measure for fixed ξ.
It is the set of $(\eta,\zeta)\in\mathbb{R}^2$ such that $\eta-\zeta$, ζ, and $\eta-\xi$ have the same sign as ξ (or $\zeta=\xi$ or $\eta-\zeta=\xi$).

$$\begin{aligned} f_{\mathrm{res}}(u) &= -\,i\mathcal{F}^{-1}\iint_{\phi=0}\hat{u}(\eta-\zeta)\hat{u}(\zeta)\overline{\hat{u}}(\eta-\xi)d\zeta d\eta \\ &= -\,i\mathcal{F}^{-1}\mathbf{1}_{\xi\geq 0}\iint \hat{u}_+(\eta-\zeta)\hat{u}_+(\zeta)\overline{\hat{u}_+}(\eta-\xi)d\zeta d\eta \\ &\quad -\,i\mathcal{F}^{-1}\mathbf{1}_{\xi<0}\iint \hat{u}_-(\eta-\zeta)\hat{u}_-(\zeta)\overline{\hat{u}_-}(\eta-\xi)d\zeta d\eta. \end{aligned}$$

Thus,

$$f_{\mathrm{res}}(u) = -i\big(\Pi_+(|u_+|^2u_+)+\Pi_-(|u_-|^2u_-)\big).$$

The idea of the renormalization group method is that the dynamics of

(NLW')
$$\begin{cases} \partial_t u = \varepsilon^2 f(u,t) \\ u(0) = W_0. \end{cases}$$

can be approximated by the resonant dynamics

(RD)
$$\begin{cases} \partial_t W = \varepsilon^2 f_{\text{res}}(W) \\ W(0) = W_0. \end{cases}$$

We choose W_0 such that $\Pi_-(W_0) = 0$. Using

$$f_{\text{res}}(u) = -i\big(\Pi_+(|u_+|^2 u_+) + \Pi_-(|u_-|^2 u_-)\big)$$

and projecting the RD equation onto the negative frequencies, we have

$$\begin{cases} i\partial_t W_- = \Pi_-(|W_-|^2 W_-) \\ W_-(0) = 0. \end{cases}$$

$W_-(t) = 0$ for all $t \in \mathbb{R}$. Then $W(t) = W_+(t)$ satisfies the Szegö equation:

$$\begin{cases} i\partial_t W_+ = \Pi_+(|W_+|^2 W_+) \\ W_+(0) = W_0. \end{cases}$$