Problème de Cauchy pour l'équation de Schrödinger cubique à force aléatoire

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Stochastic cubic nonlinear Schrödinger equation on $\mathbb{R}^d$

We consider the stochastic cubic nonlinear Schrödinger equation with additive noise (SNLS) on $\mathbb{R}^d$ for $d \geq 3$:

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^2 u + \phi\,\xi = 0 \\ u|_{t=0} = u_0 \end{cases}, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R},$$

where $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$, $\xi = \partial_t W$ is a space-time white noise. Here W is a cylindrical Wiener process on $L^2(\mathbb{R}^d)$:

$$W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n \,,$$

- $\{\beta_n\}_{n\in\mathbb{N}}$ independent complex-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R}^d)$

and ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$, $s > 0$, $\phi \in \mathrm{HS}^s := \mathrm{HS}(L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))$.

We consider solutions of SNLS in the mild sense, i.e. satisfying Duhamel's formulation:

$$u(t) = S(t)u_0 \pm i \int_0^t S(t-t')|u|^2 u(t')\, dt' + \Psi(t)\,.$$

where $S(t) = e^{it\Delta}$ denotes the linear Schrödinger propagator and

$$\Psi(t) := -i \int_0^t S(t-t')\phi\, dW(t') = -i \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\phi(e_n) d\beta_n(t')$$

is the stochastic convolution.

Remark: $\phi \in HS^s \implies \Psi \in C(\mathbb{R}; H^s(\mathbb{R}^d))$ almost surely.

Goal: To construct **a unique local-in-time solution** of SNLS in $C([0,T]; H^s(\mathbb{R}^d))$ in the case of very rough stochastic forcing (i.e. for low s)

The deterministic cubic nonlinear Schrödinger equation

The deterministic cubic nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u \pm |u|^2 u = 0.$$

- Scaling symmetry:
  $u(t,x)$ solution of NLS $\Longrightarrow$ $u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x)$ also solution of NLS.
- This scaling symmetry induces the scaling-critical Sobolev index:

$$s_{\text{crit}} = \frac{d-2}{2},$$

  the unique Sobolev exponent such that $\|u_\lambda(0)\|_{\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)} = \|u(0)\|_{\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)}$.
- Regularity threshold for local well-posedness:
  - $s \geq s_{\text{crit}}$: NLS is **locally well-posed** in $H^s(\mathbb{R}^d)$ (Cazenave-Weissler 1990)
  - $s < s_{\text{crit}}$: NLS is **ill-posed** in $H^s(\mathbb{R}^d)$(Christ-Colliander-Tao 2003, Alazard-Carles 2009, Carles-Kappeler 2015, Kishimoto 2018, Oh 2017)

Main result

- De Bouard and Debussche (2003)(essentially) proved that: SNLS with a general power-type nonlinearity is locally well-posed in $H^s(\mathbb{R}^d)$ for $\phi \in HS^s$, $s \geq \max(s_{\text{crit}}, 0)$.
- Here we focus our attention on the cubic SNLS with initial data $u_0 \in H^{s_{\text{crit}}}(\mathbb{R}^d)$ and $\phi \in HS^s$ for $s < s_{\text{crit}} \implies \Psi(t) \notin H^{s_{\text{crit}}}(\mathbb{R}^d), \forall t > 0$
- We set

$$s_d := \begin{cases} \dfrac{1}{4}\,, & \text{if} \quad d = 3\,, \\ s_{\text{crit}} - \dfrac{2}{5}\,, & \text{if} \quad d \geq 4\,. \end{cases}$$

Theorem (Cheung, P. 2019)

Let $d \geq 3$, $u_0 \in H^{s_{\text{crit}}}(\mathbb{R}^d)$, and $\phi \in HS^s$ with $s \in (s_d, s_{\text{crit}})$.

Then there exists a **unique local-in-time solution of the cubic SNLS** in $u \in C([0,T]; H^s(\mathbb{R}^d))$, where $T = T_\omega$ is almost surely positive.

Ideas of the proof

We use the 'Da Prato-Debussche decomposition': $u = \Psi + v$, where

$$\begin{cases} i\partial_t v + \Delta v = |\Psi + v|^2(\Psi + v) \\ v(0) = u_0, \end{cases}$$

and then we **solve the fixed point problem for** v.

Main tools:

- the Fourier restriction norm method of Klainerman-Machedon (1993) and Bourgain (1993) adapted to the spaces V^p of functions of bounded p-variation and their preduals U^p (spaces introduced by Koch, Tataru and collaborators (2009, 2011))
- the bilinear refinement of the Strichartz estimate
- a case-by-case analysis for estimating the terms $v\bar{v}v$, $v\bar{v}\Psi$, $v\bar{\Psi}\Psi$, $\Psi\bar{\Psi}\Psi$

A glimpse at function spaces

- $V^p = V^p(\mathbb{R}; L^2)$ is the space of functions $u : \mathbb{R} \to L^2$ such that

$$\|u\|_{V^p(\mathbb{R};L^2)} := \sup_{\{t_k\}_{k=0}^K} \left(\sum_{k=1}^{K} \|u(t_k) - u(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} < \infty,$$

  where the sup is taken over all finite partitions $-\infty < t_0 < \cdots < t_K \leq \infty$ of $\mathbb{R}$.

- $V_\Delta^p L^2$ is the space of functions $u : \mathbb{R} \to L^2$ such that

$$\|u\|_{V_\Delta^p L^2} := \|S(-t)u\|_{V^p(\mathbb{R};L^2)} < \infty,$$

  where $S(t) = e^{it\Delta}$ denotes the linear Schrödinger propagator.

- $Y_p^s(\mathbb{R})$ is the closure of $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V_{\mathrm{rc},\Delta}^p L^2$ with respect to the norm

$$\|u\|_{Y_p^s(\mathbb{R})} := \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|_{V_\Delta^p L^2}^2 \right)^{\frac{1}{2}},$$

  where $\mathbf{P}_N$ is the Littlewood-Paley projection onto frequencies of size N.

‘Toate’s vechi si nouă toate’ ‘Tout est ancien, tout est nouveau’

Our tools are similar to those in Bényi-Oh-P. (2015), where we consider (deterministic) cubic NLS with random initial data u_0^ω and prove almost sure local well-posedness in $H^s(\mathbb{R}^d)$ for some $s < s_{\text{crit}}$.

Two new aspects:

1. The Brownian motion does **not** belong to V^2 a.s., so we cannot measure the stochastic convolution Ψ in Y_2^s.

   The Brownian motion, however, does belong to V^p for any $p > 2$, a.s.. So, $\Psi \in Y_p^s$ a.s. and **we use the Y_p^s-spaces in our analysis for Ψ, instead of Y_2^s spaces**.

2. There is a **limitation in the space-time integrability of** Ψ (in fact, implicit in the work of de Bouard-Debussche (2003)). In comparison to this, the random linear solution $z^\omega := S(t)u_0^\omega$ enjoys more space-time integrability

Space-time integrability of the stochastic convolution

Proposition (Space-time integrability of Ψ, Oh, P., Wang 2018)

Let $d \geq 3$, $T > 0$. Given any $q \in [1, \infty)$ and $2 \leq r \leq \frac{2d}{d-2}$, we have that $\Psi \in L^q_t([0,T]; W^{s,r}(\mathbb{R}^d))$ a.s.. Moreover, for $\sigma \geq \max\{q, r\}$,

$$\mathbb{E}\left[\|\langle\nabla\rangle^s \Psi\|^\sigma_{L^q_t([0,T],L^r_x(\mathbb{R}^d))}\right]^{\frac{1}{\sigma}} \leq C\sqrt{\sigma}\,T^\theta \|\phi\|_{\mathrm{HS}^s},$$

for some $\theta > 0$.

Proof: There exists $C > 0$ such that for any $\sigma \geq 2$ and any Gaussian g, we have

$$\|g\|_{L^\sigma(\Omega)} \leq C\sqrt{\sigma}\|g\|_{L^2(\Omega)}.$$

By Minkowski's inequality (with $\sigma \geq \max\{q, r\}$) and since Ψ is Gaussian:

$$\begin{aligned}\mathbb{E}\left[\|\langle\nabla\rangle^s \Psi\|^\sigma_{L^q([0,T),L^r(\mathbb{R}^d))}\right]^{\frac{1}{\sigma}} &= \Big\|\|\langle\nabla\rangle^s \Psi\|_{L^q([0,T),L^r(\mathbb{R}^d))}\Big\|_{L^\sigma(\Omega)} \\ &\leq \big\|\|\langle\nabla\rangle^s \Psi\|_{L^\sigma(\Omega)}\big\|_{L^q([0,T),L^r(\mathbb{R}^d))} \\ &\leq C\sqrt{\sigma}\big\|\|\langle\nabla\rangle^s \Psi\|_{L^2(\Omega)}\big\|_{L^q([0,T),L^r(\mathbb{R}^d))}\end{aligned}$$

The Wiener integral $I : L^2((a,b)) \to L^2(\Omega)$, $I(f) := \int_a^b f(t)d\beta(t)$ with β real-valued Brownian motion, satisfies:

$$\|I(f)\|_{L^2(\Omega)} = \|f\|_{L^2((a,b))}$$

and $\mathbb{E}(I(f)) = 0$ for all $f \in L^2((a,b))$. Since

$$\langle\nabla\rangle^s\Psi = -i\sum_{n\in\mathbb{N}}\int_0^t S(t-t')\langle\nabla\rangle^s\phi(e_n)d\beta_n(t') = -i\sum_{n\in\mathbb{N}} I_n\big(S(t-\cdot)\langle\nabla\rangle^s\phi(e_n)\big),$$

we have

$$\mathbb{E}\left[\|\langle\nabla\rangle^s\Psi\|^\sigma_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)}\right]^{\frac{1}{\sigma}} \le C\sqrt{\sigma}\Big\|\Big(\|\langle\nabla\rangle^s\Psi\|^2_{L^2(\Omega)}\Big)^{\frac12}\Big\|_{L^q_t([0,T),L^r_x)}$$

$$= C\sqrt{\sigma}\Big\|\Big(\Big\|\sum_{n\in\mathbb{N}} I_n\big(S(t-\cdot)\langle\nabla\rangle^s\phi(e_n)\Big\|^2_{L^2(\Omega)}\Big)^{\frac12}\Big\|_{L^q_t([0,T),L^r_x)}$$

$$= C\sqrt{\sigma}\Big\|\Big(\mathbb{E}\Big(\Big(\sum_{n\in\mathbb{N}} I_n\big(S(t-\cdot)\langle\nabla\rangle^s\phi(e_n)\big)\Big)\cdot\overline{\Big(\sum_{m\in\mathbb{N}} I_m\big(S(t-\cdot)\langle\nabla\rangle^s\phi(e_m)\big)\Big)}\Big)^{\frac12}\Big\|_{L^q_t([0,T),L^r_x)}$$

$$= C\sqrt{\sigma}\Big\|\Big(\sum_{n\in\mathbb{N}}\Big\|I_n\big(S(t-\cdot)\langle\nabla\rangle^s\phi(e_n)\Big\|^2_{L^2(\Omega)}\Big)^{\frac12}\Big\|_{L^q_t([0,T),L^r_x)}$$

$$= C\sqrt{\sigma}\Big\|\Big\|S(\tau)\langle\nabla\rangle^s\phi(e_n)\Big\|_{\ell^2_n L^2_\tau([0,t])}\Big\|_{L^q_t([0,T),L^r_x)}$$

So far, we obtained

$$\mathbb{E}\left[\|\langle\nabla\rangle^s\Psi\|^\sigma_{L^q_tL^r_x([0,T)\times\mathbb{R}^d)}\right]^{\frac{1}{\sigma}} \le C\sqrt{\sigma}\left\|\left\|S(\tau)\langle\nabla\rangle^s\phi(e_n)\right\|_{\ell^2_nL^2_\tau([0,t])}\right\|_{L^q_t([0,T),L^r_x)}$$

Applying Minkowski's inequality once more (using $r \ge 2$) and Hölder's inequality in time:

$$\le C\sqrt{\sigma}T^{\frac{1}{q}}\|\|S(\tau)\langle\nabla\rangle^s\phi(e_n)\|_{L^2_\tau([0,T),L^r_x)}\|_{\ell^2_n}$$
$$\le C\sqrt{\sigma}T^{\frac{1}{q}+\frac{1}{2}-\frac{1}{\tilde{q}}}\sqrt{\sigma}\|\|S(\tau)\langle\nabla\rangle^s\phi(e_n)\|_{L^{\tilde{q}}_\tau([0,T),L^r{}_x)}\|_{\ell^2_n}$$

We choose $\tilde{q}$ such that $\tilde{q} \ge 2$ and such that it is the unique index making the pair $(\tilde{q}, r)$ Schrödinger admissible, i.e. $\frac{2}{\tilde{q}} + \frac{d}{r} = \frac{d}{2}$.

$$\tilde{q} = \frac{4r}{d(r-2)} \ge 2 \Longrightarrow r \le \frac{2d}{d-2}.$$

We can then apply the Strichartz estimate

$$\|S(t)u_0\|_{L^{\tilde{q}}(\mathbb{R};L^r(\mathbb{R}^d))} \le C\|u_0\|_{L^2(\mathbb{R})}, \text{for all } u_0 \in L^2(\mathbb{R}^d),$$

to obtain

$$\begin{aligned}\mathbb{E}\left[\|\langle\nabla\rangle^s\Psi\|^\sigma_{L^q_tL^r_x([0,T)\times\mathbb{R}^d)}\right]^{\frac{1}{\sigma}} &\le C\sqrt{\sigma}T^{\frac{1}{q}+\frac{1}{2}-\frac{1}{\tilde{q}}}\|\|\langle\nabla\rangle^s\phi(e_n)\|_{L^2_x}\|_{\ell^2_n}\\ &= C\sqrt{\sigma}T^{\frac{1}{q}+\frac{1}{2}-\frac{1}{\tilde{q}}}\|\phi\|_{\mathrm{HS}^s}.\end{aligned}$$

Remark:

In Oh, P., Wang (2018), we considered the SNLS with a generic power-type nonlinearity $|u|^{p-1}u$, $p > 1$. In the energy-(super)critical regime $p \geq \frac{4}{d-2}$ with $u_0 \in H^{s_0}(\mathbb{R}^d)$ and $\phi \in HS^s(\mathbb{R}^d)$,

$$s_0 > s_{\text{crit}} \qquad \text{and} \qquad s > s_{\text{crit}} - 1,$$

we proved that there exists a unique solution

$$u \in \Psi + C([0,T]; W^{\min(s_0-1,s), \frac{2d}{d-2}-}(\mathbb{R}^d)) \cap C([0,T]; H^{\min(s_0-1,s)}(\mathbb{R}^d)).$$

The noise here can be rougher than s_d, but in the main result:

- the initial data lies in the critical space $H^{s_{\text{crit}}}(\mathbb{R}^d)$
- we construct $u \in \Psi + C([0,T]; H^{s_{\text{crit}}}(\mathbb{R}^d))$

The proof is very simple and is based on the dispersive estimate

$$\|S(t)u_0\|_{L^r_x(\mathbb{R}^d)} \leq \frac{C_r}{|t|^{\frac{d}{2}-\frac{d}{r}}} \|u_0\|_{L^{r'}_x(\mathbb{R}^d)}$$

for any $2 \leq r \leq \infty$ and $\frac{1}{r} + \frac{1}{r'} = 1$.