3. Propagating fronts : speeds and linear determinacy

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LMS Research School : PDEs in Mathematical Biology

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Background: consider reaction-diffusion equations

$$u_t = du_{xx} + f(u)$$

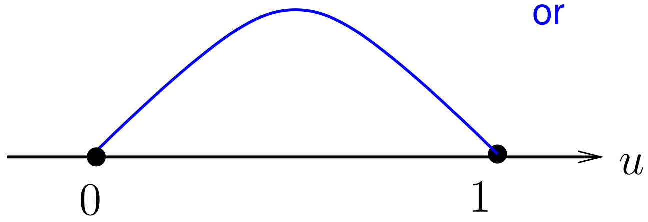
with 'monostable' reaction f satisfying

$f(0) = f(1) = 0$

$f'(0) > 0, f'(1) < 0$

$f(u) > 0$ when $u \in (0, 1)$

$e.g.$ $f(u) = u(1-u)$



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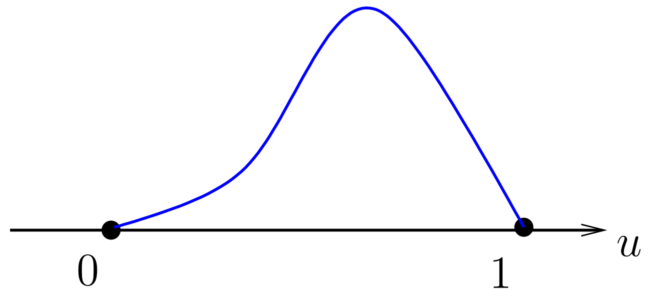

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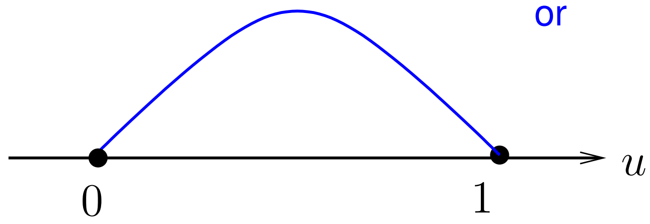

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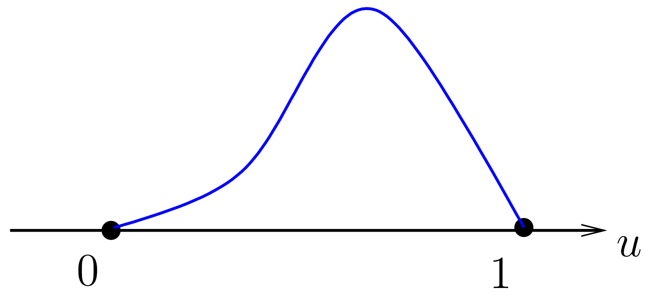
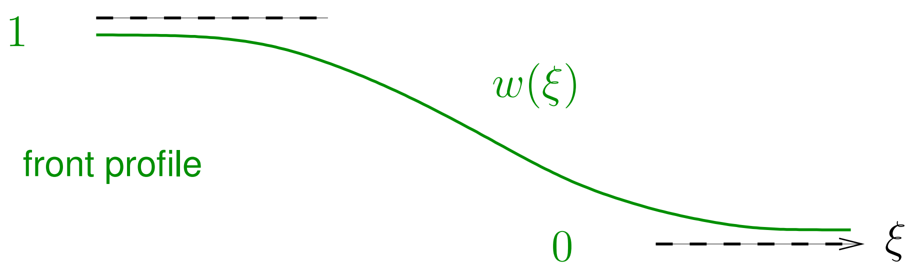


• there exist decreasing front solutions $u(x,t) = w(x - ct)$



for all speeds

$$c \geq c^*$$

for some minimal front speed $c^* > 0$ (Fisher, KPP '37)

Some key facts about minimal front speed c^*

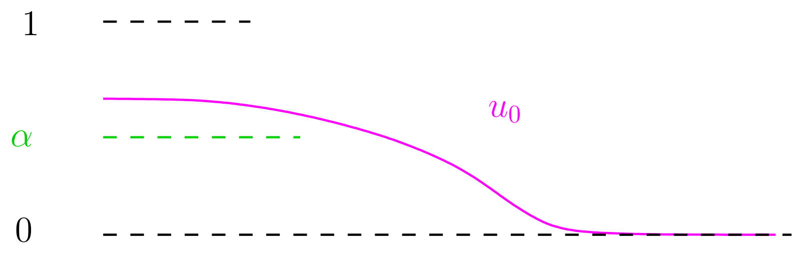
(i) (Aronson-Weinberger '78) c^* can be characterised as a

spreading speed

namely, for an initial condition $u(x,0) = u_0(x) \in [0,1]$ with

$$u_0(x) = 0 \text{ for } x >> 0, \quad u_0(x) \in (\alpha, 1) \text{ for } x << 0,$$

for some $\alpha > 0$,



the solution u of $u_t = du_{xx} + f(u)$ satisfies

- if $c > c^*$, then

$$\lim_{t\to\infty} \sup_{x \geq ct} u(x,t) = 0$$

- if $c < c^*$, then

$$\lim_{t\to\infty} \inf_{x \leq ct} u(x,t) = 1$$

(ii) define the linear value $\bar{c}$ to be the minimal c for which the linear equation

$$dw'' + cw' + f'(0)w = 0$$

has a solution $w(\xi) = \exp(-\mu\xi)$ with real $\mu > 0$, and hence

$$d\mu^2 - c\mu + f'(0) = 0$$

so that

$$\bar{c} = 2\sqrt{d\,f'(0)}$$

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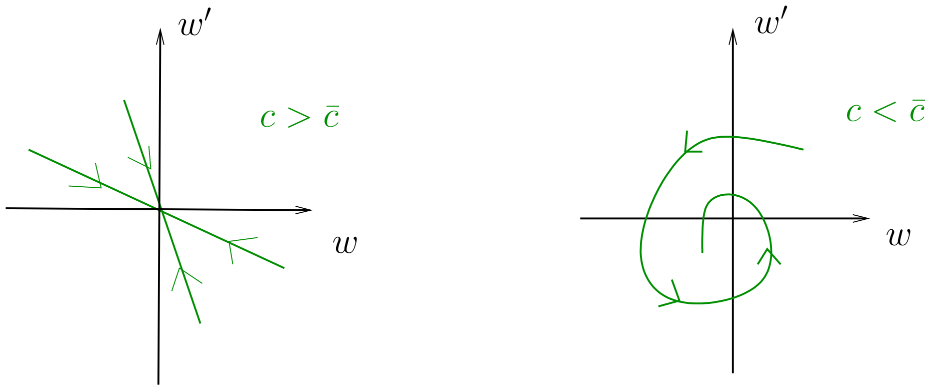
- then

$$c^* \geq \bar{c}$$

since the existence of $-\mu < 0$ is *necessary* for front $w(x - ct)$ to exist

Local phase portraits at $(w, w') = (0, 0)$



- in general, may have

$$c^* > \bar{c} \quad \text{or} \quad c^* = \bar{c}$$

- say equation is linearly determinate if

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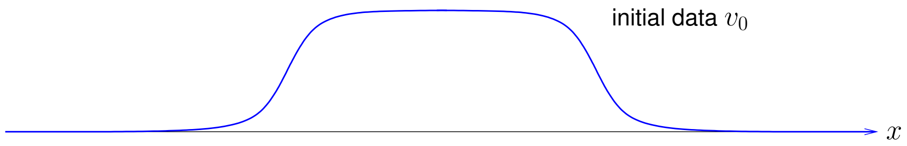
Important because

- want to know value of minimal speed c_*

  - especially since c_* = spreading speed of solutions of the initial-value problem of $v_t = v_{xx} + f(v)$ with compactly supported data v_0



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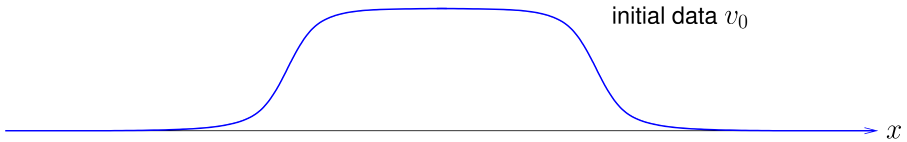

- stability of fronts, in terms of the IVP of $v_t = v_{xx} + f(v)$, depends on
  - whether a front has minimal speed: $c = c_*$ or $c > c_*$
  - whether the minimal speed front is linearly determinate: $c_* = \bar{c}$?

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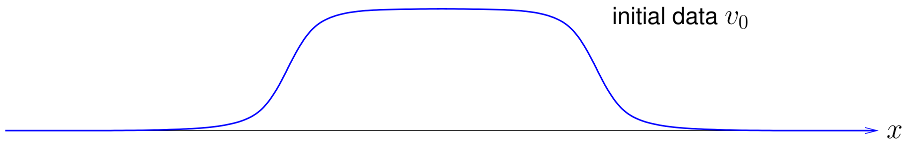


- stability of fronts, in terms of the IVP of $v_t = v_{xx} + f(v)$, depends on
  - whether a front has minimal speed: $c = c_*$ or $c > c_*$
  - whether the minimal speed wave is linearly determinate: $c_* = \bar{c}$?

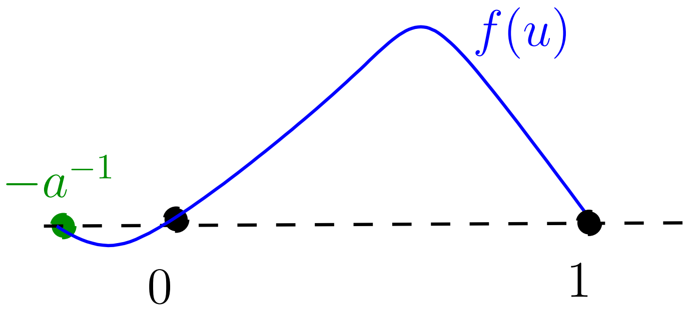
i.e. for which initial conditions? in what sense? solution v of IVP converges to front(s) as $t \to \infty$

Example (Hadeler-Rothe '75)

The equation

$$u_t = u_{xx} + u(1-u)(1+au)$$

with



has

$$\bar{c} = 2 \quad \text{for all } a > 0$$

and

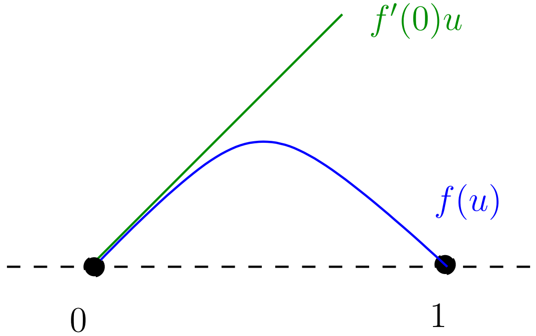
$$c^* = \begin{cases} 2 & \text{if } 0 \le a \le 2, \\ \frac{a+2}{\sqrt{2a}} & \text{if } a \ge 2 \end{cases}$$

$\therefore$ linearly determinate if and only if $a \in (0, 2]$

A famous sufficient condition for linear determinacy (Stokes, Hadeler-Rothe)

If

$$f(u) \leq f'(0)u \quad \text{for all} \quad u \in (0,1)$$



i.e. no Allee effect

then

$$c^* = \bar{c}$$

Note: this condition is not necessary - in the previous example, this condition only holds when $0 < a \leq 1$, whereas $c^* = \bar{c}$ whenever $0 < a \leq 2$

Useful change of variables:

- for a strictly monotone scalar front profile w, we can think of

w' as a function of w

$\therefore$ write

$$w' = -\rho(w)$$

so that

$$w'' = \rho'(w)\rho(w)$$

and hence

$$dw''(\xi) + cw'(\xi) + f(w(\xi)) = 0$$

if and only if

$$d\rho'(w)\rho(w) - c\rho(w) + f(w) = 0$$

Proof : use the min-max formula

$$c^* = \inf_{\rho \in \Lambda} \sup_{u \in (0,1)} \left\{ d\rho'(u) + \frac{f(u)}{\rho(u)} \right\},$$

where

$$\Lambda = \{\rho \in C^1([0,1]) : \rho(u) > 0 \text{ if } u \in (0,1),\ \rho(0) = 0,\ \rho'(0) > 0\},$$

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together with the family of test functions $\rho_\kappa(u) = \kappa u$ to get

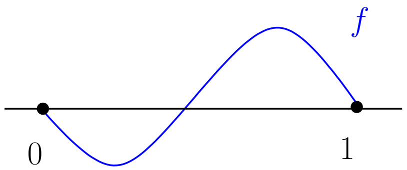
$$\begin{aligned} c^* &\le \inf_{\kappa>0} \sup_{u\in(0,1)} \left\{ d\kappa + \frac{f(u)}{\kappa u} \right\} \\ &\le \inf_{\kappa>0} \left\{ d\kappa + \frac{f'(0)}{\kappa} \right\} \qquad \text{since } f(u) \le f'(0)u \\ &= 2\sqrt{df'(0)} = \bar{c} \end{aligned}$$

so that

$$\bar{c} \le c^* \le \bar{c} \quad \Rightarrow \quad c^* = \bar{c}$$

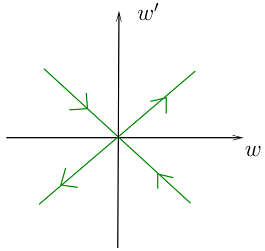
□

Aside: if f is bistable, *e.g.*,



there exists *unique* $c_0 \in \mathbb{R}$ for which a decreasing travelling front exists

- this speed c_0 is *not* controlled by a linearisation: for every $c \in \mathbb{R}$



- difficult to determine c_0: for scalar case, can find sign c via

$$c \int_{\mathbb{R}} (w')^2 \, d\xi = \int_0^1 f(w) \, dw \quad \Rightarrow \quad \text{sign } c = \text{ sign } \int_0^1 f(w) \, dw$$

- but for systems, need other ideas,

  e.g. strong competition limit (Girardin-Nadin 2015)

Some references on travelling fronts, spreading speeds, and linear determinacy for monostable equations/systems....

- classical: Fisher, Kolmorogorov-Petrovskii-Piskounov, Aronson-Weinberger, Hadeler-Rothe....

- linear determinacy in the scalar case: Weinberger, Berestycki-Nirenberg, Lucia-Muratov-Novaga, Malaguti-Marcelli, Gilding-Kersner, Benguria-Depassier-Mendez....

- linear determinacy for co-operative or competition systems: Lui, Hosono, Weinberger-Lewis-Li, Liang-Zhao, Holzer-Scheel, Huang-Han, Roques-Hosono-Bonnefon-Boivin....

  .....and for non-co-operative systems Weinberger-Kawasaki-Shigesada, Haiyan Wang, Wang-Castillo-Chavez, Dunbar, Griette-Raoul, Girardin....

\+ many other papers e.g. on equations with delay, spatial heterogeneity, nonlocal terms, etc

Example 1: Scalar equation with (nonlinear) convection

$$u_t + h'(u)u_x = du_{xx} + f(u)$$

(with A. Al Kiffai, Kufa University, Iraq)

Motivating example: a one-dimensional model of motion of chemotactic cells (Benguria-Depassier-Mendez, based on Keller-Segel model)

- suppose

  $\rho(x,t) =$ density of bacteria chemotactic to a chemical of concentration $s(x,t)$,

$$\rho_t = [d\rho_x - \rho\chi s_x]_x + f(\rho),$$

- if rate of chemical consumption is mainly due to ability of bacteria to consume chemical, then

$$s_t = -k\rho$$

  and for travelling fronts $s(x-ct)$, $\rho(x-ct)$, we have

$$s_x = -\frac{1}{c}s_t = \frac{k\rho}{c},$$

  so that

$$\rho_t + \frac{\chi k}{c}\rho\rho_x = d\rho_{xx} + f(\rho)$$

By similar arguments to above....

- there exist decreasing fronts for all speeds $c \geq c^*$, where

$$c^* = \inf_{\rho \in \Lambda} \sup_{u \in (0,1)} \left\{ d\rho'(u) + h'(u) + \frac{f(u)}{\rho(u)} \right\},$$

where

$$\Lambda = \{\rho \in C^1([0,1]) : \rho(u) > 0 \text{ if } u \in (0,1),\ \rho(0) = 0,\ \rho'(0) > 0\},$$

- sufficient condition for 'right' linear determinacy: if

$$h'(u) + \sqrt{d}\frac{f(u)}{\sqrt{f'(0)}\,u} \leq h'(0) + \sqrt{d}\sqrt{f'(0)} \quad \text{for all } u \in (0,1),$$

then

$$c^* = \bar{c} = h'(0) + 2\sqrt{df'(0)}$$

Lack of symmetry in the presence of convection

- if $h'(u) \not\equiv 0$, the function $\hat{u}(x,t) = u(-x,t)$ instead solves

$$\hat{u}_t - h'(\hat{u})\hat{u}_x = d\hat{u}_{xx} + f(\hat{u}) \qquad (*)$$

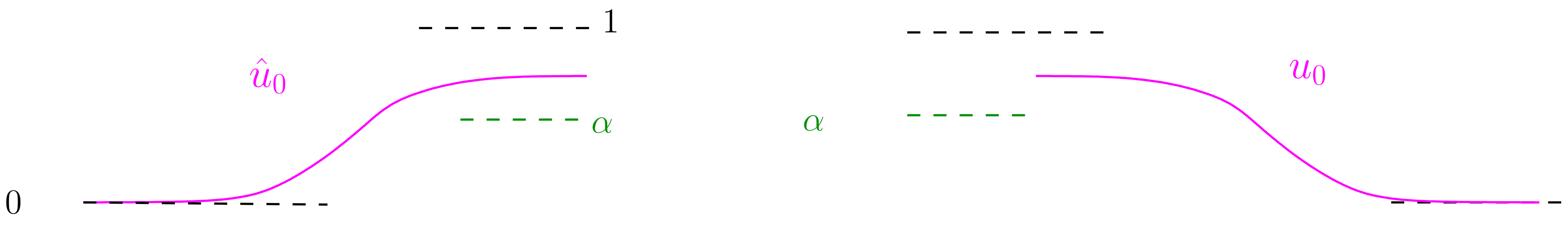
$\therefore$ no symmetry between increasing/decreasing fronts

- there exist increasing fronts $\hat{w}(x-ct)$ for all

$$c \ \le \ c_l^* \ \le \ \bar{c}_l = h'(0) - 2\sqrt{df'(0)},$$

and there exist decreasing fronts $w(x-ct)$ for all

$$c \ \ge \ c_r^* \ \ge \ \bar{c}_r = h'(0) + 2\sqrt{df'(0)}$$



spreading speed for initial condition $\hat{u}_0$ $= c_l^*$

spreading speed for initial condition u_0 $= c_r^*$

An example that is right, but not left, linearly determinate

- given $a > 2$, consider

$$u_t + (1-a)uu_x = u_{xx} + u(1-u)(1+au) \qquad (\dagger)$$

so that

$$h'(u) = (1-a)u, \quad f(u) = u(1-u)(1+au)$$

(i) $c_r^* = \bar{c}_r$

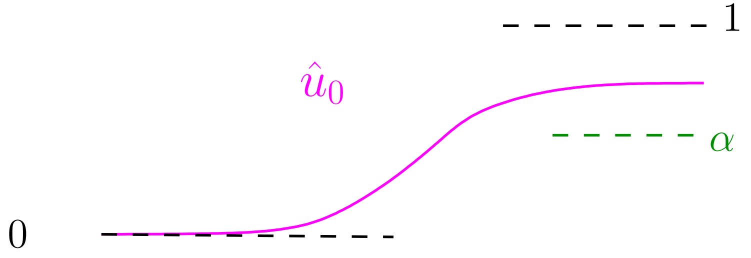
- check sufficient condition: $h'(u) + \frac{f(u)}{\sqrt{f'(0)}\, u} \leq h'(0) + \sqrt{f'(0)}$

(ii) $c_l^* < \bar{c}_l$

- the solution u of the convectionless equation

$$u_t = u_{xx} + f(u)$$

with non-decreasing initial condition



is a subsolution for $(\dagger)$

Linear determinacy and the diffusion constant d

- if no convection, *i.e.* $h'(u) \equiv 0$, then

  either (right or left) linearly determinate for all $d > 0$, or for no $d > 0$

.... because wave speeds and linear values are proportional to $\sqrt{d}$

- *i.e.*,

$$w(\xi) \text{ is a wave profile, speed } c, \text{ for } d = 1$$

$$\Leftrightarrow \quad v(\xi) = w\left(\frac{\xi}{\sqrt{d}}\right) \text{ is a wave profile, speed } \sqrt{d}\, c, \text{ for general } d$$

and linear values satisfy

$$\bar{c}_d = \sqrt{d}\, \bar{c}$$

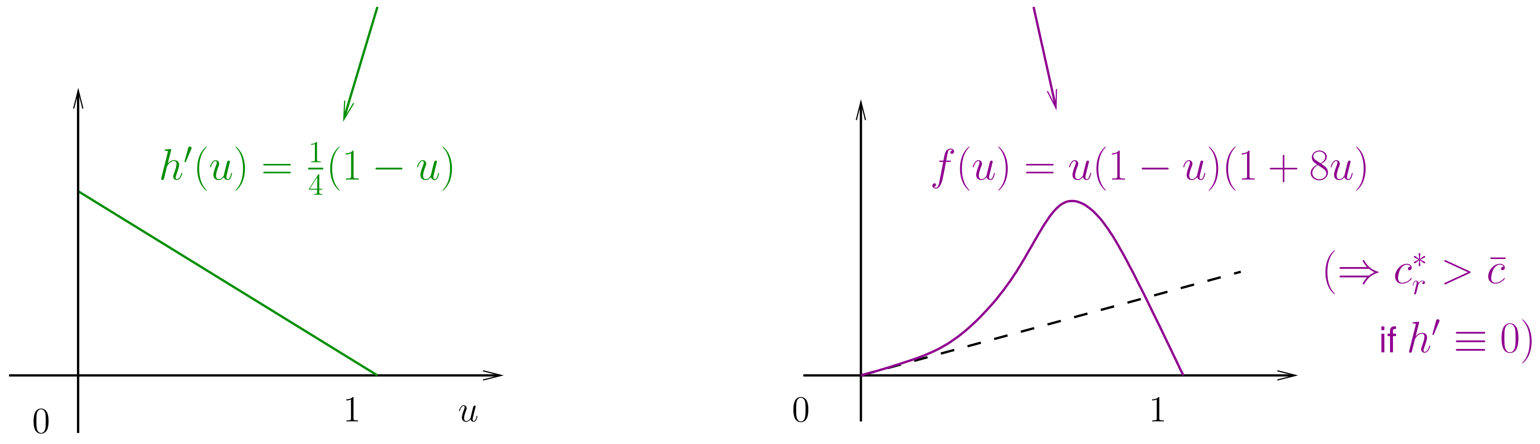
- but if $h'(u) \not\equiv 0$, linear determinacy may depend on diffusion d

An example where (right) linear determinacy depends on diffusion d

- consider

$$u_t + \frac{1}{4}(1-u)u_x = d\,u_{xx} + u(1-u)(1+8u) \qquad (\dagger)$$

so that



(i) $c_r^* = \bar{c}_r$ if $\sqrt{d} \leq \frac{1}{28}$, by sufficient condition

(ii) $c_r^* > \bar{c}_r$ if $\sqrt{d} > \frac{1}{2}$, by using suitable subsolution

Example 2: Travelling fronts in anisotropic smectic C* liquid crystals

$$v_t = \sqrt{1+\xi\cos(2\pi v)}(\sqrt{1+\xi\cos(2\pi v)}v_x)_x + f(v)$$

(with M. Grinfeld and G. McKay, Strathclyde)

Problem

- consider the quasilinear equation

$$v_t = \sqrt{1 + \xi \cos(2\pi v)}(\sqrt{1 + \xi \cos(2\pi v)} v_x)_x + f(v), \quad x \in \mathbb{R}, t > 0$$

where

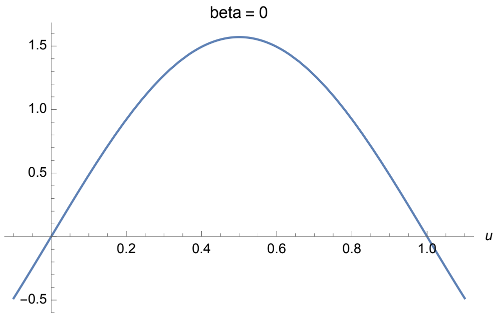
$$f(v) := \frac{\sin(\pi v)}{2\pi} [1 - \beta \cos(\pi v)],$$

and $\xi \in (-1, 1)$ and $\beta \in [0, 1)$ are constants.

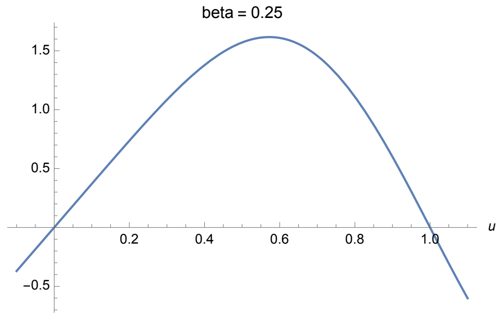
Here

- ξ is a measure of anisotropy
- β controls the shape of the nonlinearity f

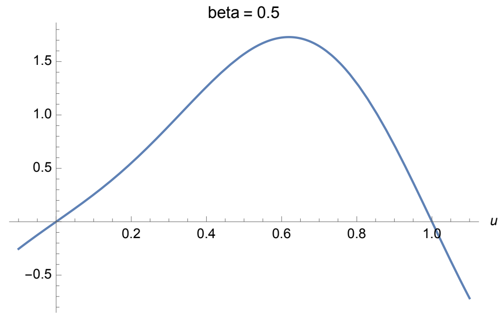
- form of nonlinearity $f(v) = \frac{\sin(\pi v)}{2\pi}\left[1 - \beta\cos(\pi v)\right]$
  - equilibria at all $k \in \mathbb{Z}$, in particular, $v = 0$, $v = 1$
  - if $\beta \in [0, 1)$, $f'(0) > 0$, $f'(1) < 0 \quad \Rightarrow \quad$ "monostable"



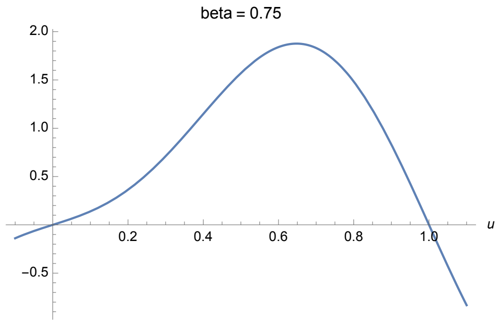
(a) $\beta = 0$



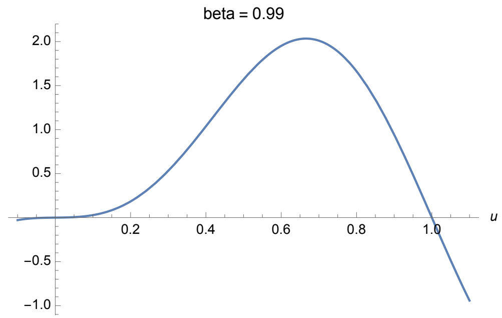
(b) $\beta = 0.25$



(c) $\beta = 0.5$



(d) $\beta = 0.75$



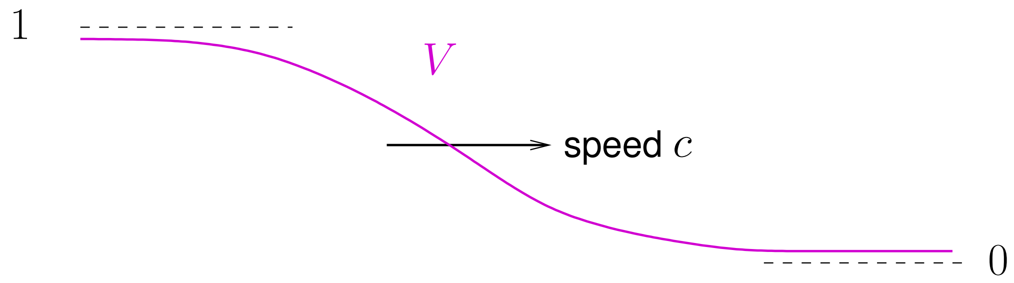
(e) $\beta = 0.99$

- interested in decreasing travelling front solutions

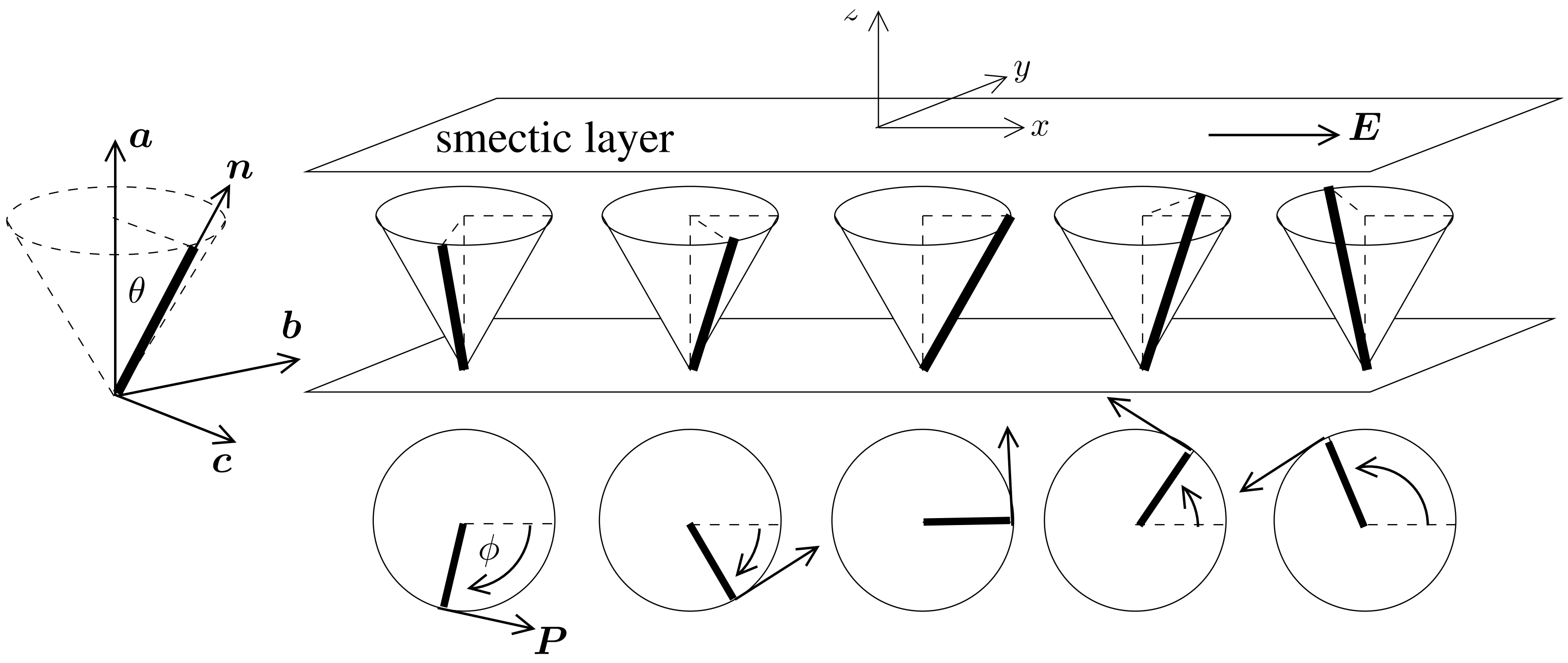
$$v(x,t) = V(x - ct),$$

with

$$\lim_{z \to -\infty} V(z) = 1, \quad \lim_{z \to \infty} V(z) = 0$$



Motivation: smectic C* liquid crystals



- $\boldsymbol{n} =$ director (unit vector giving molecular alignment) $= \boldsymbol{a}\cos\theta + \boldsymbol{c}\sin\theta$
- $\boldsymbol{a} =$ normal to smectic layers
- $\theta =$ (constant) tilt angle, $\phi =$ twist angle (assume depends only on x, t)
- $\boldsymbol{c} = (\cos\phi, \sin\phi, 0)$, $\boldsymbol{b} = (-\sin\phi, \cos\phi, 0)$
- constant electric field $\boldsymbol{E} = E(1, 0, 0)$

Continuum theory [Leslie, Stewart, Nakagawa, 1991]

- free energy density

$$w(\phi) = w_{\text{elastic}}(\phi) + w_{\text{polarisation}}(\phi) + w_{\text{dielectric}}(\phi)$$

- anisotropic elastic energy density

$$w_{\text{elastic}}(\phi) = \frac{1}{2}B_1(\nabla \cdot \boldsymbol{b})^2 + \frac{1}{2}B_2(\nabla \cdot \boldsymbol{c})^2 = \frac{1}{2}B(1 - \xi \cos 2\phi)\phi_x^2,$$

when

$$B_1 = B(1-\xi), \qquad B_2 = B(1+\xi), \qquad \xi \in (-1, 1)$$

- energy density from spontaneous polarisation

$$w_{\text{polarisation}}(\phi) = -P_0\boldsymbol{b} \cdot \boldsymbol{E} = P_0 E \sin\phi$$

- dielectric energy density

$$w_{\text{dielectric}}(\phi) = -\frac{1}{2}\epsilon_0\epsilon_a(\boldsymbol{n} \cdot \boldsymbol{E})^2 = -\frac{1}{2}\epsilon_0\epsilon_a(E\cos\phi\sin\theta)^2$$

- so defining $\beta := -\frac{\epsilon_0\epsilon_a E}{P_0}\sin^2\theta$, we have

$$w(\phi) = \frac{1}{2}B(1 - \xi\cos 2\phi)\phi_x^2 + 2P_0E\left(\frac{1}{2}\sin\phi + \frac{1}{4}\beta\cos^2\phi\right)$$

Dynamics of ϕ and travelling fronts

- L^2-gradient flow

$$\eta\phi_t = -\nabla_{L^2}\left(\int w(\phi)\,dx\right),$$

where η is a rotational viscosity

- setting $v = \frac{1}{2} - \frac{\phi}{\pi}$, and non-dimensionalising x and t gives

$$v_t = \sqrt{1+\xi\cos(2\pi v)}(\sqrt{1+\xi\cos(2\pi v)}v_x)_x + \frac{\sin(\pi v)}{2\pi}[1-\beta\cos(\pi v)]$$

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- travelling front solutions $v(x,t) = V(x-ct)$ with

$$\lim_{z\to-\infty} V(z) = v_1, \quad \lim_{z\to+\infty} V(z) = v_0$$

model switching between two constant states v_1, v_0 of the liquid crystal, and have potential applications to fast electro-optical switches

Isotropic case : $\xi = 0$

(see also Gilding and Kersner, 2004, and van Saarloos et al, 1995)

- with $F(V) = -\frac{dV}{dz}$, phase-plane equation is

$$F\frac{dF}{dV} - cF + f(V) = 0$$

- linear speed becomes

$$c_l(\beta, 0) = \sqrt{2(1-\beta)}$$

Isotropic case : $\xi = 0$

(see also Gilding and Kersner, 2004, and van Saarloos et al, 1995)

- with $F(V) = -\frac{dV}{dz}$, phase-plane equation is

$$F\frac{dF}{dV} - cF + f(V) = 0$$

- linear speed becomes

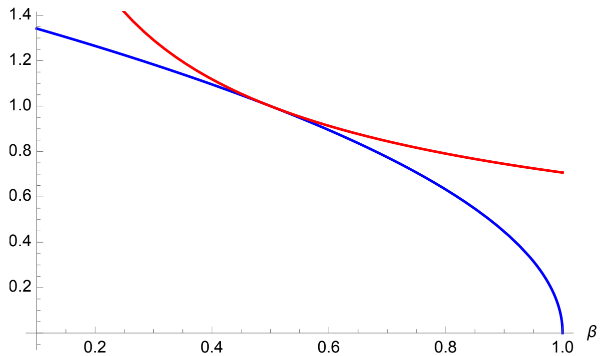
$$c_l(\beta, 0) = \sqrt{2(1-\beta)}$$

- family of explicit solutions [Clarkson and Mansfield, ‘94, Stewart and Momoniat, ’04]

$$F^{\beta}(V) = \frac{1}{\pi}\sqrt{\frac{\beta}{2}}\sin(\pi V), \quad \text{with speed} \quad c_{nl}(\beta, 0) := \frac{1}{\sqrt{2\beta}}$$

- easy to see that

$$c_{nl}(\beta, 0) \geq c_l(\beta, 0) \quad \text{for all} \quad \beta \in [0, 1) \quad \text{and} \quad c_{nl}(\tfrac{1}{2}, 0) = c_l(\tfrac{1}{2}, 0)$$



Theorem ($\xi = 0$) [CGM, van Saarloos et al, Gilding and Kersner]

If $\beta \in [0, 1/2]$,

$$c_*(\beta, 0) = c_l(\beta, 0) = \sqrt{2(1-\beta)}$$

whereas if $\beta \in (1/2, 1)$,

$$c_*(\beta, 0) = c_{nl}(\beta, 0) = \sqrt{\frac{1}{2\beta}} > \ c_l(\beta, 0)$$

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Ideas in proof

- (i) $\beta \in [0, \frac{1}{2}]$: use variational formula for $c_*(\beta, 0)$

$$c_*(\beta, 0) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} \left\{ F'(V) + \frac{f(V)}{F(V)} \right\}$$

  with test functions

$$F_\nu(V) = \nu \sin(\pi V), \quad \nu > 0,$$

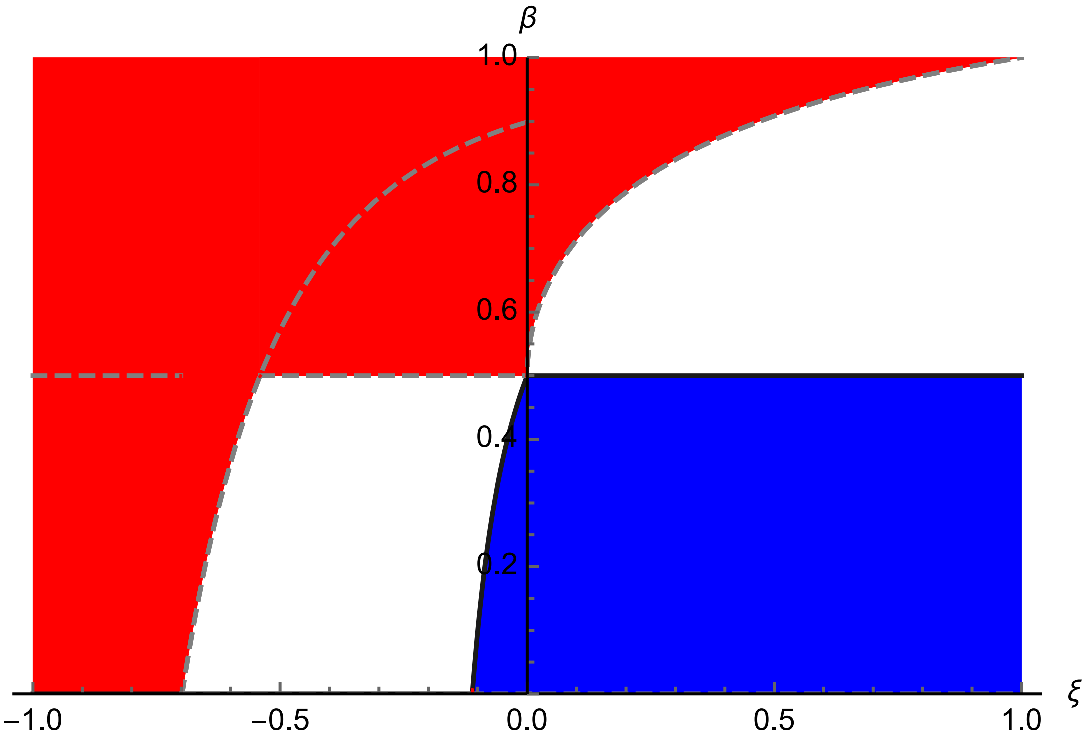
- (ii) $\beta \in (\frac{1}{2}, 1)$: [Lucia, Muratov + Novaga, '04] showed that a front V has minimal speed $c = c_*(\beta, 0)$ if

$$\int_0^\infty e^{cz} \left(V^2(z) + (V')^2(z)\right) \ dz \ < \ \infty$$

  (cf [Rothe, '81]: front has faster of 2 possible rates of decay ⇔ pushed, minimal speed)

Anisotropic case : $\xi \neq 0$

- no explicit travelling wave solutions
- asymmetry between $\xi < 0$ and $\xi > 0$
- summary of results in (ξ, β) plane



blue = linear selection, red = nonlinear selection, white = ???

+ $\exists$ increasing function $\beta(\xi)$, with $\beta(0) = \frac{1}{2}$, separating regions of linear/nonlinear selection

Separating curve between linear/nonlinear selection regions

Proposition If $c_*(\beta^*, \xi^*) = c_l(\beta^*, \xi^*)$, then

$$c_*(\beta^*, \xi) = c_l(\beta^*, \xi) \text{ if } \xi > \xi^* \quad \text{and} \quad c_*(\beta^*, \xi^*) = c_l(\beta, \xi^*) \text{ if } \beta < \beta^*$$

idea of proof....

- define $h_\xi := \sqrt{1 + \xi \cos(2\pi V)}$
- since there exists a decreasing front of speed $c = c_*(\beta^*, \xi^*)$, there exists $\hat{F} \in \Lambda$ such that

$$c_*(\beta^*, \xi^*) = h_{\xi^*}(V) \left\{ \hat{F}'(V) + \frac{f(V)}{\hat{F}(V)} \right\} \quad \text{for all } V \in (0,1).$$

- then

$$\begin{aligned} c_*(\beta^*, \xi) &= \inf_{F \in \Lambda} \sup_{V \in (0,1)} h_\xi(V) \left\{ F'(V) + \frac{f(V)}{F(V)} \right\} \\ &\leq \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)} h_{\xi^*}(V) \left\{ \hat{F}'(V) + \frac{f(V)}{\hat{F}(V)} \right\} \\ &= c_*(\beta^*, \xi^*) \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)} \\ &= \sqrt{2(1-\beta)(1+\xi^*)} \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)}. \end{aligned}$$

idea of proof....ctd

- now

$$\frac{h_\xi(V)}{h_{\xi^*}(V)} = \sqrt{\frac{1+\xi\cos(2\pi V)}{1+\xi^*\cos(2\pi V)}} \le \sqrt{\frac{1+\xi}{1+\xi^*}} \quad \text{for all } V \in (0,1) \text{ if } \ \xi^* < \xi$$

since

$$\frac{1+\xi\cos(2\pi V)}{1+\xi^*\cos(2\pi V)} - \frac{1+\xi}{1+\xi^*} = \frac{(\xi^*-\xi)(1-\cos(2\pi V))}{(1+\xi^*)(1+\xi^*\cos(2\pi V))} \ < \ 0$$

because $\xi^* < \xi$

- so

$$c_*(\beta^*,\xi) \le \sqrt{2(1-\beta)(1+\xi^*)}\sqrt{\frac{1+\xi}{1+\xi^*}} = \sqrt{2(1-\beta)(1+\xi)} = c_l(\beta^*,\xi),$$

$$\Rightarrow \quad c_*(\beta^*,\xi) = c_l(\beta^*,\xi)$$

Fully-funded PhD Project:

Shape optimization problems and reaction-diffusion equations

- joint Université Grenoble-Alpes/Swansea University PhD
- supervised by
  - Emmanuel Russ (Grenoble)
  - Elaine Crooks (Swansea)
  - Norman Dancer (Swansea)
- 18 months in Grenoble, then 18 months in Swansea
- research topic - principal eigenvalues of elliptic operators with drift, applications to spreading speeds
- start date of study: 1st October 2019
- **closing date: 13th May 2019**

https://www.swansea.ac.uk/postgraduate/scholarships/research/mathematics-joint-phd-shape-optimization-2019.php

Thank you for you attention.....