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Spatial segregation limit of a competition-diffusion system with Dirichlet boundary conditions

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Abstract

We consider a competition-diffusion system with inhomogeneous Dirichlet boundary conditions for two competitive species and show that they spatially segregate as the interspecific competition rates become large. The limit problem turns out to be a free boundary problem.

Keywords: Competition-diffusion system; Singular limit problem; Spatial segregation

1. Introduction

The understanding of spatial and/or temporal behaviour of interacting species is a central problem in population ecology. For competitive interactions, coexistence or exclusion of species have been investigated theoretically using different types of mathematical models. In particular, reaction-diffusion (RD) systems have been proposed to study the dynamics of the spatial segregation of competing species. To analyse them, one often has to rely on numerical methods. However, quite recently, methods based

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upon the derivation of spatial segregation limits have been successfully developed. Such methods yield evolution equations describing the time evolution of the boundaries of spatially segregated regions of competing species. In some situations, the limit equations take the form of new types of free boundary problems.

We consider a well-known RD model, namely a system of Gause–Lotka–Volterra type for two competing species. Let $u(x,t)$ and $v(x,t)$ be the population densities of the competing species at the position $x \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^2$, and at the time $t > 0$. The resulting system is given by

$$u_t = d_u \Delta u + (r_u - a_u u - b_u v)u, \quad x \in \Omega, \ t > 0, \tag{1.1}$$

$$v_t = d_v \Delta v + (r_v - b_v u - a_v v)v, \quad x \in \Omega, \ t > 0. \tag{1.2}$$

Here d_u and d_v are the diffusion rates, r_u and r_v are the intrinsic growth rates, a_u and a_v are the intraspecific and b_u and b_v the interspecific competition rates. All of the rates are positive constants. The boundary conditions corresponding to system (1.1) and (1.2) depend on the ecological environment. The most standard ones are the zero-flux boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \ t > 0, \tag{1.3}$$

where n is the outward normal unit vector to $\partial\Omega$. The initial conditions are given by

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega. \tag{1.4}$$

From the mathematical viewpoint, qualitative properties of non-negative solutions (u,v) of (1.1), (1.2) and (1.4) together with the zero-flux boundary conditions (1.3) have been extensively studied. A fundamental result is that the stable attractor of (1.1), (1.2) and (1.3) consists of equilibrium solutions [8,12]. This information indicates that the existence and stability of non-negative equilibrium solutions are important for the study of the asymptotic behaviour of solutions. Let us describe the situation more precisely. Assume that two species are strongly competing, that is,

$$\frac{a_u}{b_v} < \frac{r_u}{r_v} < \frac{b_u}{a_v}. \tag{1.5}$$

Then there are only two stable spatially constant equilibrium solutions, namely $(u,v) = (r_u/a_u, 0)$ and $(u,v) = (0, r_v/a_v)$. Moreover, when the domain Ω is convex, all the spatially non-constant equilibrium solutions are unstable, even if they exist [9]. Thus the only stable equilibria are given by $(r_u/a_v, 0)$ and $(0, r_v/a_v)$, which obviously indicates that two strongly competing species can never coexist in convex habitats. However, if Ω is not convex, the structure of equilibrium solutions depends on the shape of Ω. If, for instance, Ω has a suitable dumb-bell shape, there exist stable non-constant equilibrium solutions, which exhibit spatial segregation of the two competing species [5]. Ecologically speaking, two competing species may possibly coexist, if their habitat has a suitably non-convex shape.

If two competing species prefer rather different environmental conditions, because of different adaptabilities, severe competition occurs essentially in the region of Ω where their main habitats overlap [13]. In this situation, boundary conditions (1.3) should be replaced by the following Dirichlet conditions:

$$u(x,t) = u_{\mathrm{B}}(x,t), \qquad v(x,t) = v_{\mathrm{B}}(x,t), \quad x \in \partial\Omega, \ t > 0. \tag{1.6}$$

The behaviour of the solutions has not been well-understood, even if the domain is simply convex. The difficulty is that the behaviour of the solutions strongly depends not only on the shape of Ω but also on the form of the functions $u_{\mathrm{B}}(x,t)$ and $v_{\mathrm{B}}(x,t)$. In this paper, we study the dynamics of two strongly competing species (cf. (1.5)) whose densities satisfy Eqs. (1.1) and (1.2) together with the Dirichlet boundary conditions (1.6). We search for the spatial segregation limit when the interspecific competition rates b_u and b_v become very large. In this situation, one can expect that the two competing species spatially segregate in the whole domain Ω, because they are very strongly competing. First, we show the results of numerical simulations of the solution of problem (1.1), (1.2), (1.4) and (1.6) in a square domain Ω in $\mathbb{R}^2$. The boundary conditions are imposed as follows. Let Γ_u be a smooth sub-boundary of $\partial\Omega$, with non-empty interior in $\partial\Omega$, and let $\Gamma_v = \partial\Omega \backslash \Gamma_u$. We suppose that u_{B} is strictly positive in the interior of Γ_u, that $u_{\mathrm{B}} = 0$ on Γ_v and that v_{B} is defined in an opposite way. We assume that u and v are initially spatially segregated. In each figure, the parameters are the same except for the values of k with $b_u = b_v = k$. See Section 4 for exact details on the numerical computations. Fig. 1 corresponds to the case that $k = 100$, Fig. 2 corresponds to the case that $k = 1000$, Fig. 3 to the case that $k = 10\,000$ and the initial conditions, which are presented in these figures, are similar for the three cases. We clearly observe spatially segregated regions for u and v if the interspecific competition rates b_u and b_v increase; more precisely the resulting segregating boundaries become sharper. These results suggest that in the limiting situation where the interspecific competition rates b_u and b_v are infinite, one can expect the occurrence of interfaces which are segregating boundaries between u and v.

The aim of this paper is to derive a free boundary problem in the limit that b_u and b_v tend to infinity. The free boundary coincides with the segregating boundary between the two species. To do so, we rewrite (1.1), (1.2) as

$$u_t = d_u \Delta u + f(u) - kuv, \quad x \in \Omega, \ t > 0, \tag{1.7}$$

$$v_t = d_v \Delta v + g(u) - akuv, \quad x \in \Omega, \ t > 0, \tag{1.8}$$

where $f(u) = (r_u - a_u u)u$ and $g(v) = (r_v - a_v v)v$ and where a and k are positive constants derived from b_u and b_v. This paper is organized as follows: in Section 2, we formulate the problem and show some a priori estimates on the solutions. In Section 3, we let k tend to infinity to derive a limiting problem which turns out to be a free boundary problem. For similar studies in the case of Neumann boundary conditions, we refer to [2,7] and to an earlier paper by Evans [6]. Let us also mention an article due to Dancer and Zhang [3] where they study the large time behaviour by means of blow-up-type methods. The new difficulty which we address in this paper is that, in the case of Dirichlet boundary conditions, the boundary terms obtained when performing integrations by parts do not vanish as they do in the zero-flux case; they cannot be estimated independently of k. Therefore, we have to multiply the equalities by suitable

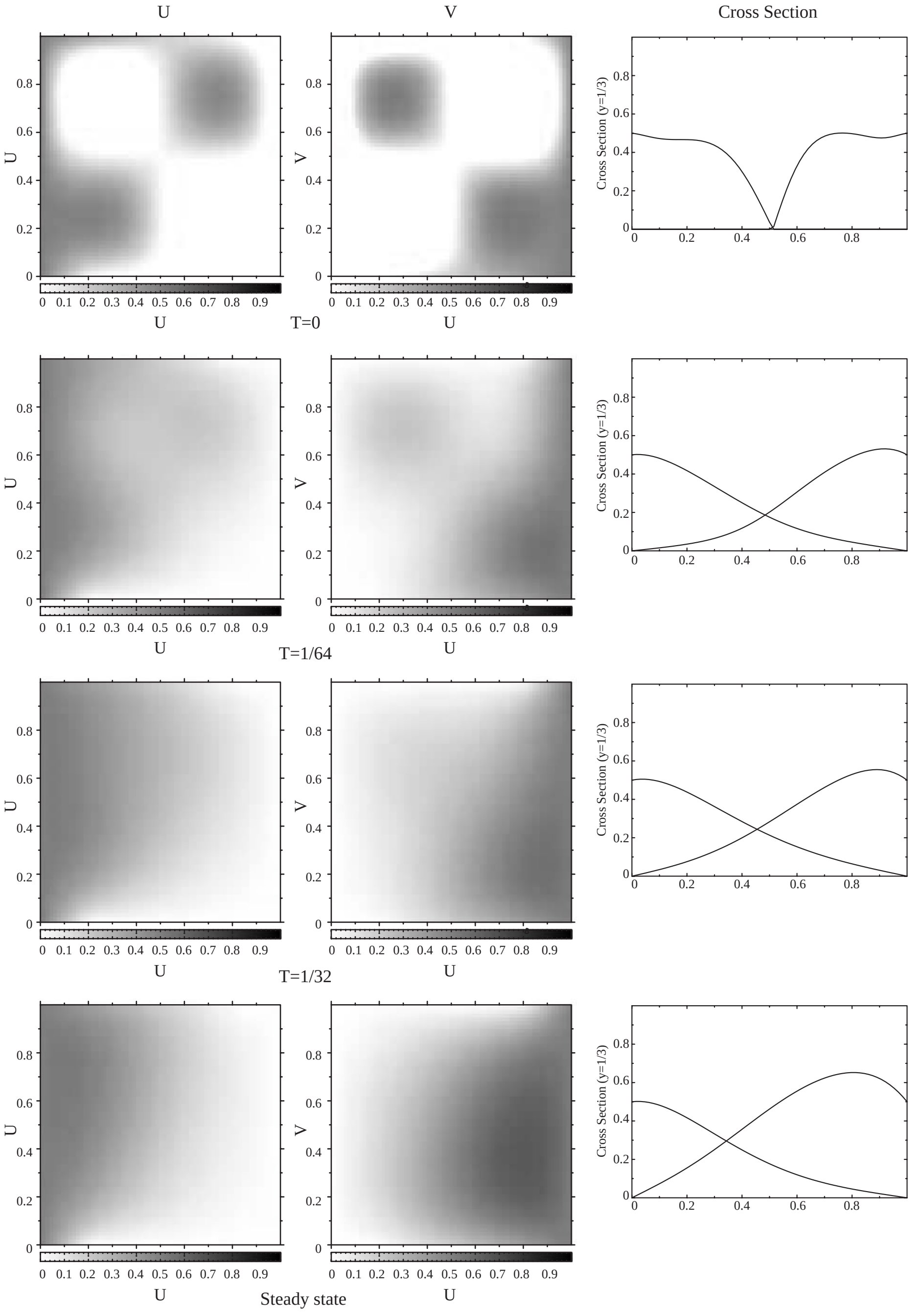

Fig. 1. RDP: $d_u = 1.5$, $d_v = 1.0$, $\lambda = 50$, $k = 10^2$.

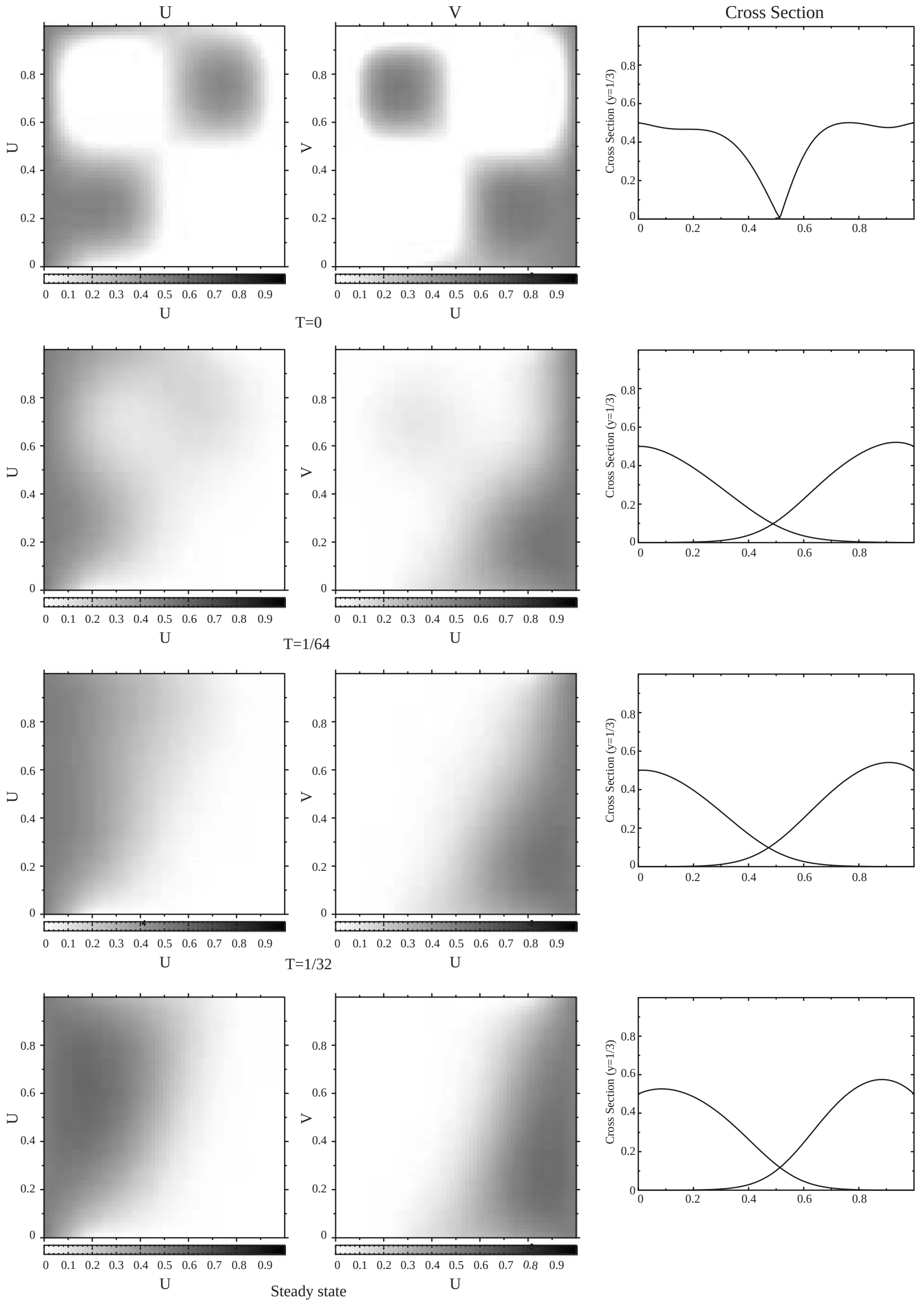

Fig. 2. RDP: $d_u = 1.5$, $d_v = 1.0$, $\lambda = 50$, $k = 10^3$.

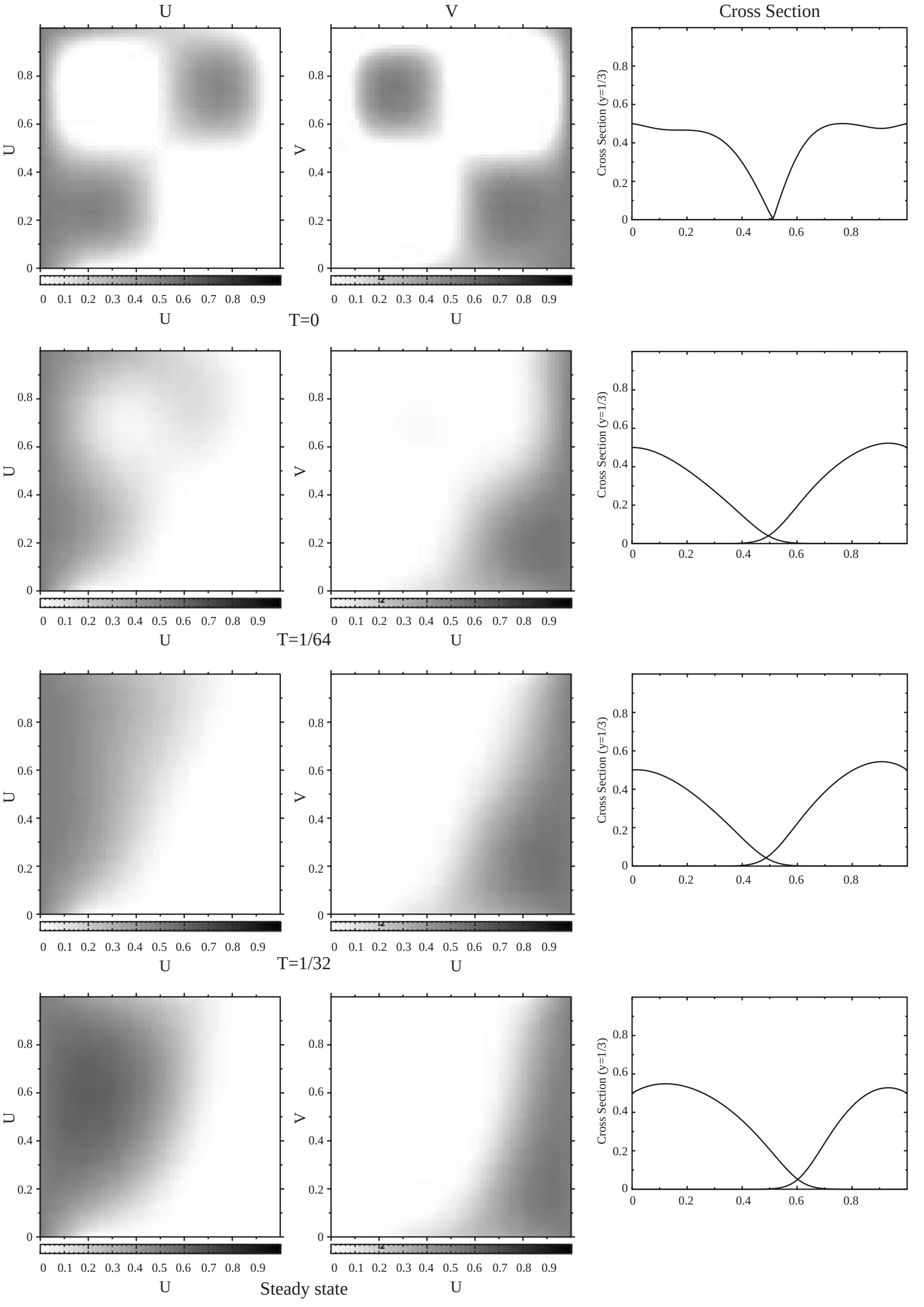

Fig. 3. RDP: $d_u = 1.5$, $d_v = 1.0$, $\lambda = 50$, $k = 10^4$.

test functions which vanish on $\partial\Omega$ before performing the integrations by parts. This yields interior estimates which together with the uniform estimates for the solution pair (u_k, v_k) imply its relative compactness in $L^2(\Omega \times (0,T))$ for every $T > 0$.

2. Formulation of the problem and basic properties

The precise problem which we study is the Dirichlet problem

$$(\mathrm{P}_k) \qquad \begin{cases} u_t = d_1\Delta u + f(u) - kuv & \text{in } Q, \\ v_t = d_2\Delta v + g(v) - \alpha kuv & \text{in } Q, \\ u = m_1^k & \text{on } \partial\Omega \times \mathbb{R}^+, \\ v = m_2^k & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0^k(x), \qquad v(x,0) = v_0^k(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded, open, connected subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $Q := \Omega \times \mathbb{R}^+$. We assume the following:

(i) f and g are continuously differentiable functions on $[0,\infty)$ such that $f(0)=g(0)=0$ and $f(s) < 0$, $g(s) < 0$ for all $s > 1$,
(ii) $m_1^k,\ m_2^k \in C^{2,1}(\bar{\Omega} \times \mathbb{R}^+)$, $0 \leqslant m_1^k,\ m_2^k \leqslant 1$ and $m_1^k \rightharpoonup m_1$, $m_2^k \rightharpoonup m_2$ weakly in $L^2(\partial\Omega \times (0,T))$ for all $T > 0$ as $k \to \infty$.

The initial conditions u_0^k and v_0^k are defined by

$$u_0^k(x) = m_1^k(x,0), \qquad v_0^k(x) = m_2^k(x,0) \quad \text{for } x \in \Omega$$

and $u_0^k \rightharpoonup u_0$, $v_0^k \rightharpoonup v_0$ weakly in $L^2(\Omega)$ as $k \to \infty$.

By a solution of Problem (P_k) we mean a pair (u,v) such that $u, v \in C(\bar{Q}) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ and satisfy pointwise the partial differential equations as well as the boundary and initial conditions in Problem (P_k). We begin with a priori estimates for solutions of Problem (P_k).

Lemma 2.1. *Let (u_k, v_k) be a solution of Problem (P_k). Then*

$$0 \leqslant u_k, \quad v_k \leqslant 1 \quad \text{in } \bar{Q}.$$

Proof. We define

$$\mathscr{L}_1(u) := u_t - d_1\Delta u - f(u) + kuv,$$

$$\mathscr{L}_2(v) := v_t - d_2\Delta v - g(v) + \alpha kuv.$$

Since $\mathscr{L}_i(0) = 0$ and $\mathscr{L}_i(1) \geqslant 0$ for $i = 1, 2$, the assertion follows from the maximum principle. $\square$

Lemma 2.2. *There exists a unique classical solution of Problem* (P_k).

Proof. Define $U := u - m_1^k$, $V := v - m_2^k$. We can apply [11, Proposition 7.3.2, p. 277] to the corresponding problem for U and V with homogeneous boundary conditions to deduce that Problem (P_k) has a unique classical solution. □

Next we define φ as the first eigenfunction of the operator $-\Delta$ in Ω, namely the function φ such that $\|\varphi\|_{H_0^1(\Omega)} = 1$ satisfying

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda > 0$ and $\varphi > 0$ in Ω. In the next three lemmas we obtain a priori bounds for the solution (u_k, v_k) of Problem (P_k) which are uniform with respect to the parameter k in the equations. However these estimates only provide useful information in subdomains ω of Ω such that $\bar{\omega} \Subset \Omega$.

Lemma 2.3. *There exists a positive constant C_1 independent of k such that*

$$\iint_{Q_T} u_k v_k \varphi \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{C_1}{k}. \tag{2.1}$$

Proof. Integrating the equation for u_k over $Q_T := \Omega \times (0, T)$ after multiplication by φ yields

$$\begin{aligned} k &\iint_{Q_T} u_k v_k \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= d_1 \int_0^T \int_{\partial\Omega} \left\{ \frac{\partial u_k}{\partial \nu}\varphi - u_k \frac{\partial \varphi}{\partial \nu} \right\} \mathrm{d}S\,\mathrm{d}t + d_1 \iint_{Q_T} u_k \Delta\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint_{Q_T} f(u_k)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0^k \varphi \,\mathrm{d}x - \int_\Omega u_k(x, T)\varphi \,\mathrm{d}x, \end{aligned}$$

where $\mathrm{d}S$ indicates the $(N-1)$-dimensional area element in $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Since $\varphi = 0$ on $\partial\Omega$ and

$$\sup_{x \in \partial\Omega} \left| \frac{\partial \varphi}{\partial \nu}(x) \right| < \infty,$$

we get (2.1). □

Lemma 2.4. *There exists a positive constant C_2 independent of k such that*

$$\iint_{Q_T} |\nabla u_k|^2 \varphi \,\mathrm{d}x\,\mathrm{d}t, \qquad \iint_{Q_T} |\nabla v_k|^2 \varphi \,\mathrm{d}x\,\mathrm{d}t \leqslant C_2.$$

Proof. We multiply the parabolic equation for u_k by $u_k\varphi$ and integrate by parts. This gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_k^2\varphi\,\mathrm{d}x + d_1\int_\Omega |\nabla u_k|^2\varphi\,\mathrm{d}x + d_1\int_\Omega u_k\nabla u_k\cdot\nabla\varphi\,\mathrm{d}x$$
$$= \int_\Omega f(u_k)u_k\varphi\,\mathrm{d}x - k\int_\Omega u_k^2 v_k\varphi\,\mathrm{d}x.$$

Therefore,

$$\frac{1}{2}\int_\Omega u_k^2(x,T)\varphi\,\mathrm{d}x + d_1\iint_{Q_T}|\nabla u_k|^2\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{1}{2}\int_\Omega (u_0^k)^2\varphi\,\mathrm{d}x - \frac{d_1}{2}\int_0^T\int_{\partial\Omega} u_k^2\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}S\,\mathrm{d}t$$
$$+\frac{d_1}{2}\iint_{Q_T} u_k^2\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} f(u_k)u_k\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

which implies the first result of Lemma 2.4. The second one can be proved in a similar way. □

In order to prove that the sequences $\{u_k\}$ and $\{v_k\}$ are relatively compact in $L^2(Q_T)$ we will apply the following Fréchet–Kolmogorov theorem (e.g. [1, Corollary IV.26, p. 74]).

Proposition 2.5 (Fréchet–Kolmogorov). *Let $\mathscr{F}$ be a bounded subset of $L^p(Q_T)$ with $1 \leqslant p < \infty$. Suppose that*

(i) *for any $\varepsilon > 0$ and any subset $\omega \Subset Q_T$, there exists a positive constant $\delta\,(<\mathrm{dist}\,(\omega,\partial Q_T))$ such that*

$$\|f(x+\xi,t)-f(x,t)\|_{L^p(\omega)} + \|f(x,t+\tau)-f(x,t)\|_{L^p(\omega)} < \varepsilon$$

for all ξ, τ, and $f\in\mathscr{F}$ satisfying $|\xi|+|\tau|<\delta$,

(ii) *for any $\varepsilon > 0$, there exists $\omega \Subset Q_T$ such that*

$$\|f\|_{L^p(Q_T\setminus\omega)} < \varepsilon$$

for all $f\in\mathscr{F}$.

Then $\mathscr{F}$ is precompact in $L^p(Q_T)$.

For that purpose we first present results about differences of space and time translates of u_k and v_k. For $r > 0$ sufficiently small, say $r\in(0,\hat{r})$, we define

$$\Omega_r = \{x\in\Omega \mid B(x,2r)\subset\Omega\}$$

and

$$\Omega_r' = \bigcup_{x \in \Omega_r} B(x,r),$$

where $B(x,r)$ denotes the ball in $\mathbb{R}^N$ with centre x and radius r.

Lemma 2.6. *For each* $r \in (0,\hat{r})$, *there exists a positive constant* C_3 *such that*

$$\int_0^T \int_{\Omega_r} (u_k(x+\xi,t) - u_k(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C_3|\xi|^2,$$

$$\int_0^T \int_{\Omega_r} (v_k(x+\xi,t) - v_k(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C_3|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \leqslant r$.

Proof. It is a direct consequence of Lemma 2.4. Indeed

$$\begin{aligned}
&\int_0^T \int_{\Omega_r} (u_k(x+\xi,t) - u_k(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega_r} \left(\int_0^1 \nabla u_k(x+\theta\xi,t)\cdot\xi \,\mathrm{d}\theta \right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant |\xi|^2 \int_0^1 \int_0^T \int_{\Omega_r} |\nabla u_k(x+\theta\xi,t)|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta \\
&\quad \leqslant |\xi|^2 \int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \frac{|\xi|^2}{\inf_{y\in\Omega_r'} \varphi(y)} \int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|^2 \varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant C_3|\xi|^2.
\end{aligned}$$

A similar proof holds for v_k. □

Lemma 2.7. *For each* $r \in (0,\hat{r})$, *there exists a positive constant* C_4 *such that*

$$\int_0^{T-\tau} \int_{\Omega_\tau} (u_k(x,t+\tau) - u_k(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C_4\tau,$$

$$\int_0^{T-\tau} \int_{\Omega_r} (v_k(x,t+\tau) - v_k(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C_4\tau$$

for all $\tau \in (0,T)$.

Proof. Let $\mu \in C_0^\infty(\Omega_r')$ be such that $0 \leqslant \mu(x) \leqslant 1$ in Ω_r' and $\mu = 1$ on Ω_r. Then

$$\begin{aligned}
&\int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t))^2 \mu(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t)) \left(\int_t^{t+\tau} (u_k)_s(x,s)\,\mathrm{d}s \right) \mu(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t)) \left(\int_0^{\tau} (u_k)_t(x,t+s)\,\mathrm{d}s \right) \mu(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t))\, d_1 \Delta u_k(x,t+s) \mu(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s,$$

$$I_2 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t)) f(u_k(x,t+s)) \mu(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s,$$

$$I_3 := -\int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t))(k u_k v_k)(x,t+s) \mu(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.$$

Since μ vanishes on $\partial\Omega_r'$, we have

$$\begin{aligned}
I_1 &= -d_1 \int_0^\tau \mathrm{d}s \int_0^{T-\tau} \int_{\Omega_r'} \nabla(u_k(x,t+\tau) - u_k(x,t)) \cdot \nabla u_k(x,t+s) \mu(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega_r'} (u_k(x,t+\tau) - u_k(x,t)) \nabla u_k(x,t+s) \cdot \nabla \mu(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\
&\leqslant C_5 \tau \int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + C_6 \tau \int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{C_5 \tau}{\inf_{y \in \Omega_r'} \varphi(y)} \int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|^2 \varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C_7 \tau}{\left(\inf_{y \in \Omega_r'} \varphi(y)\right)^{1/2}} \left(\int_0^T \int_{\Omega_r'} |\nabla u_k(x,t)|^2 \varphi(x)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\leqslant C_8 \tau.
\end{aligned}$$

Similarly, we get

$$I_2 \leqslant C_9 \tau.$$

Finally,

$$I_3 \leqslant \frac{C_{10}\tau}{\inf_{y\in\Omega_r'}\varphi(y)}\int_0^T\int_{\Omega_r'} k u_k v_k \varphi\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{11}\tau.$$

This completes the proof of the first result of Lemma 2.7. The estimate for the function v_k follows in a similar way. □

Furthermore, let ε be arbitrary. Since $\{u_k\}$ and $\{v_k\}$ are bounded by 1, there exist $r_0 > 0$ and $\tau_0 > 0$ such that for $0 \leqslant r \leqslant r_0$ and $0 \leqslant \tau \leqslant \tau_0$,

$$\int_{T-\tau}^T\int_\Omega u_k^2(x,t)\,\mathrm{d}x\,\mathrm{d}t, \qquad \int_0^T\int_{\Omega\setminus\Omega_r} u_k^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon$$

and that similar inequalities hold for v_k.

It follows from this remark, Lemmas 2.6 and 2.7, and Proposition 2.5 that the sequences $\{u_k\}$ and $\{v_k\}$ are relatively compact in $L^2(Q_T)$.

3. The limit problem as $k \to \infty$

We can now state the following convergence result.

Corollary 3.1. *There exists subsequences* $\{u_{k_n}\}$, $\{v_{k_n}\}$, *functions* $u \in L^\infty(Q_T)$ *and* $v \in L^\infty(Q_T)$ *such that*

$$u_{k_n} \to u, \qquad v_{k_n} \to v \quad \textit{strongly in } L^2(Q_T) \textit{ and a.e. in } Q_T,$$

as $k_n \to \infty$.

Lemma 3.2. $uv = 0$ *a.e. in* Q_T.

Proof. It is a consequence of Lemma 2.3 and Corollary 3.1. □

Next we set

$$w_k := u_k - \frac{v_k}{\alpha} \quad \text{and} \quad w := u - \frac{v}{\alpha}. \tag{3.1}$$

We deduce from Corollary 3.1 and Lemma 3.2 that

$$w_{k_n} \to w \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T$$

as $k_n \to \infty$ and furthermore that

$$u = w^+ \quad \text{and} \quad v = \alpha w^-,$$

where $s^+ = \max(s,0)$ and $s^- = \max(-s,0)$. In the sequel we prove that w is the unique weak solution of a limiting free boundary problem. To begin with we show the following integral equality.

Lemma 3.3. *Let $T>0$ be arbitrary. The function pair (u,v) defined in Corollary* 3.1 *is such that*

$$-\iint_{Q_T}\left(u-\frac{v}{\alpha}\right)\psi_t\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega}\left(u_0-\frac{v_0}{\alpha}\right)\psi(x,0)\,\mathrm{d}x$$
$$=-\int_0^T\int_{\partial\Omega}\left(d_1m_1-\frac{d_2m_2}{\alpha}\right)\frac{\partial\psi}{\partial v}\,\mathrm{d}S\,\mathrm{d}t$$
$$+\iint_{Q_T}\left\{\left(d_1u-\frac{d_2v}{\alpha}\right)\Delta\psi+\left(f(u)-\frac{g(v)}{\alpha}\right)\psi\right\}\mathrm{d}x\,\mathrm{d}t \tag{3.2}$$

for all $\psi\in\mathscr{F}_T$ where

$$\mathscr{F}_T:=\{\psi\in C^{2,1}(\overline{Q_T})\,|\,\psi(x,T)=0\ \text{in}\ \Omega\ \text{and}\ \psi=0\ \text{on}\ \partial\Omega\times[0,T]\}.$$

Proof. We take the difference of the partial differential equations for u_k and v_k/α, multiply by ψ and integrate by parts which yields

$$-\iint_{Q_T}\left(u_k-\frac{v_k}{\alpha}\right)\psi_t\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega}\left(u_0^k-\frac{v_0^k}{\alpha}\right)\psi(x,0)\,\mathrm{d}x$$
$$=-\int_0^T\int_{\partial\Omega}\left(d_1m_1-\frac{d_2m_2}{\alpha}\right)\frac{\partial\psi}{\partial v}\,\mathrm{d}S\,\mathrm{d}t$$
$$+\iint_{Q_T}\left\{\left(d_1u_k-\frac{d_2v_k}{\alpha}\right)\Delta\psi+\left(f(u_k)-\frac{g(v_k)}{\alpha}\right)\psi\right\}\mathrm{d}x\,\mathrm{d}t.$$

We then let $k=k_n\to\infty$. Thus we get (3.2). □

We define

$$d(s):=\begin{cases}d_1 & \text{if } s>0,\\ d_2 & \text{if } s<0,\end{cases}$$

$$\mathscr{D}(s):=\begin{cases}d_1s & \text{if } s\geqslant 0,\\ d_2s & \text{if } s<0,\end{cases}$$

$$h(s):=\begin{cases}f(s) & \text{if } s>0,\\ -\dfrac{g(-\alpha s)}{\alpha} & \text{if } s<0.\end{cases}$$

We will show below that w is the unique weak solution of Problem

$$(\mathrm{P})\qquad\begin{cases}w_t=\Delta\mathscr{D}(w)+h(w) & \text{in } Q,\\ \mathscr{D}(w)=d_1m_1-\dfrac{d_2m_2}{\alpha} & \text{on } \partial\Omega\times\mathbb{R}^+,\\ w(x,0)=u_0(x)-\dfrac{v_0(x)}{\alpha} & \text{in } \Omega.\end{cases}$$

We remark that, since the function D is invertible, the boundary condition in Problem (P) is a standard Dirichlet boundary condition.

Definition 3.4. A function w is a weak solution of Problem (P) if it satisfies:

(i) $w \in L^\infty(\Omega \times \mathbb{R}^+)$,

(ii) $\iint_{Q_T} (w\psi_t + \mathscr{D}(w)\Delta\psi + h(w)\psi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\partial\Omega}(d_1m_1 - \dfrac{d_2m_2}{\alpha})\dfrac{\partial\psi}{\partial\nu}\,\mathrm{d}S\,\mathrm{d}t - \int_\Omega w_0\psi(x,0)\,\mathrm{d}x$ for all $T>0$ and $\psi \in \mathscr{F}_T$.

Lemma 3.5. *The function w defined by (3.1) is a weak solution of Problem* (P).

Proof. This follows from (3.2) and the definitions of the various quantities appearing in the integral equation in Definition 3.4. □

In the sequel we prove the uniqueness of the weak solution of Problem (P). We do so by means of several lemmas.

Lemma 3.6. *Let w_1 and w_2 be two solutions of Problem* (P) *with initial functions $w_{0,1}$ and $w_{0,2}$. Then*

$$\iint_{Q_T} |w_1(x,t) - w_2(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant T\int_\Omega |w_{0,1}(x) - w_{0,2}(x)|\,\mathrm{d}x + \iint_{Q_T} (T-t)|h(w_1) - h(w_2)|\,\mathrm{d}x\,\mathrm{d}t.$$

The proof of Lemma 3.6 is based on properties of the solution of the adjoint problem

$$\text{(A)} \quad \begin{cases} \psi_t + \sigma(x,t)\Delta\psi = \eta(x,t), & (x,t) \in Q_T, \\ \psi = 0 & \text{on } \partial\Omega \times (0,T), \\ \psi(x,T) = 0 & \text{for } x \in \Omega. \end{cases}$$

We first show the following result.

Lemma 3.7. *Let $T>0$, $\eta \in C_0^\infty(Q_T)$ be such that $|\eta| \leqslant 1$ and $\sigma \in C^\infty(Q_T)$ be such that there exists a positive constant σ_* with*

$$\sigma(x,t) \geqslant \sigma_* > 0 \quad \text{in } Q_T.$$

Then there exists a unique solution $\psi \in C^{2,1}(\overline{Q_T})$ of Problem (A). *It satisfies*

$$|\psi| \leqslant T - t \quad \text{in } Q_T \tag{3.3}$$

and

$$\iint_{Q_T} (\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{T|\Omega|}{\sigma_*^2}. \tag{3.4}$$

Proof. We first set $\tau = T - t$, $\sigma(x,t) = \bar{\sigma}(x,\tau)$, $\eta(x,t) = \bar{\eta}(x,\tau)$ and $\psi(x,t) = \bar{\psi}(x,\tau)$. Then

$$\bar{\psi}_\tau = -\psi_t, \qquad \Delta\bar{\psi} = \Delta\psi$$

and ψ satisfies the forward in time problem

$$(\mathrm{F}) \qquad \begin{cases} \bar{\psi}_\tau = \bar{\sigma}(x,\tau)\Delta\bar{\psi} - \bar{\eta}(x,\tau), & (x,\tau) \in Q_T, \\ \bar{\psi} = 0 & \text{on } \partial\Omega \times (0,T), \\ \bar{\psi}(x,0) = 0 & \text{for } x \in \Omega. \end{cases}$$

It follows from [10] that Problem (F) has a unique classical solution $\bar{\psi}$, which in turn yields a unique classical solution of the adjoint problem (A).

Since $|\eta| \leqslant 1$, the functions τ and $-\tau$ are upper and lower solutions of Problem (F). Thus

$$-\tau \leqslant \bar{\psi} \leqslant \tau \quad \text{in } Q_T$$

or equivalently

$$-(T-t) \leqslant \psi \leqslant T - t \quad \text{in } Q_T.$$

In order to show (3.4), we multiply the parabolic equation in Problem (A) by $\Delta\psi$ and integrate by parts on Q_T. So,

$$\iint_{Q_T} \{\psi_t \Delta\psi + \sigma(x,t)(\Delta\psi)^2\}\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} \eta(x,t)\Delta\psi\,\mathrm{d}x\,\mathrm{d}t$$

which implies that

$$-\iint_{Q_T} (\nabla\psi)_t \cdot \nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \sigma(x,t)(\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} \eta(x,t)\Delta\psi\,\mathrm{d}x\,\mathrm{d}t,$$

where we have used that $\psi_t = 0$ on $\partial\Omega \times (0,T)$. Thus

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla\psi(x,0)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla\psi(x,T)|^2\,\mathrm{d}x + \iint_{Q_T} \sigma(x,t)(\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{\sigma_*}{2}\iint_{Q_T} (\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\sigma_*}\iint_{Q_T} \eta(x,t)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{1}{2}\iint_{Q_T} \sigma(x,t)(\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\sigma_*}\iint_{Q_T} \eta(x,t)^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Since $\psi(\cdot,T) \equiv 0$, also $\nabla\psi(\cdot,T) \equiv 0$ and we deduce that

$$\iint_{Q_T} \sigma(x,t)(\Delta\psi)^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{T|\Omega|}{\sigma_*}$$

since $|\eta| \leqslant 1$. So

$$\iint_{Q_T} (\Delta\psi)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{\sigma_*} \iint_{Q_T} \sigma(x,t)(\Delta\psi)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{T|\Omega|}{\sigma_*^2}$$

which gives (3.4). □

Proof of Lemma 3.6. Let w_1 and w_2 be two solutions of Problem (P) with initial functions $w_{0,1}$ and $w_{0,2}$. Set $\tilde{w} := w_1 - w_2$, $\tilde{w}_0 := w_{0,1} - w_{0,2}$, $z := h(w_1) - h(w_2)$ and define for all $(x,t) \in Q_T$

$$q(x,t) := \begin{cases} \dfrac{\mathscr{D}(w_1(x,t)) - \mathscr{D}(w_2(x,t))}{w_1(x,t) - w_2(x,t)} & \text{if } w_1(x,t) \neq w_2(x,t), \\ \min\{d_1, d_2\} & \text{otherwise.} \end{cases}$$

Note that

$$\min\{d_1,d_2\} \leqslant q(x,t) \leqslant \max\{d_1,d_2\} \quad \text{in } Q_T.$$

It follows from Definition 3.4 (ii) that

$$\iint_{Q_T} \{\tilde{w}(\psi_t + q\Delta\psi) + z\psi\} \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \tilde{w}_0 \psi(x,0)\,\mathrm{d}x \tag{3.5}$$

for all $\psi \in \mathscr{F}_T$.

Now let $n \in \mathbb{N}$. Using mollifiers one can find a smooth function q_n such that

$$\|q_n - q\|_{L^2(Q_T)} \leqslant \frac{1}{n} \tag{3.6}$$

and

$$\min\{d_1,d_2\} \leqslant q_n(x,t) \leqslant \max\{d_1,d_2\} \quad \text{in } Q_T. \tag{3.7}$$

Fix $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leqslant 1$ and let ψ_n be the solution of Problem (A) with this function η and σ replaced by q_n. Setting $\psi = \psi_n$ in (3.5) gives

$$\int_{Q_T} [\tilde{w}\{(\psi_n)_t + q\Delta\psi_n\} + z\psi_n] \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \tilde{w}_0\psi_n(x,0)\,\mathrm{d}x$$

and hence, since

$$(\psi_n)_t + q_n(x,t)\Delta\psi_n = \eta(x,t),$$

we have

$$\begin{aligned} &\left| \iint_{Q_T} \tilde{w}\{(q - q_n)\Delta\psi_n + \eta\} \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\quad \leqslant \iint_{Q_T} |z\psi_n\tilde{w}| \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\tilde{w}_0\psi_n(x,0)|\,\mathrm{d}x \\ &\quad \leqslant \iint_{Q_T} (T-t)|z| \,\mathrm{d}x\,\mathrm{d}t + T\int_\Omega |\tilde{w}_0|\,\mathrm{d}x \end{aligned} \tag{3.8}$$

by (3.3). Next we show that the first term on the left-hand side of (3.8) vanishes as $n \to \infty$. Indeed,

$$\iint_{Q_T} |\tilde{w}||q(x,t) - q_n(x,t)||\Delta\psi_n| \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant (\|w_1\|_{L^\infty(Q_T)} + \|w_2\|_{L^\infty(Q_T)}) \left(\iint_{Q_T} (q - q_n)^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\times \left(\iint_{Q_T} (\Delta\psi_n)^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant \frac{C_{11} T^{1/2} |\Omega|^{1/2}}{n \min\{d_1, d_2\}},$$

where we have substituted (3.4), (3.6) and (3.7). Letting $n \to \infty$ in (3.8) we obtain

$$\left|\iint_{Q_T} \tilde{w}\eta \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \iint_{Q_T} (T - t)|z| \,\mathrm{d}x\,\mathrm{d}t + T \int_\Omega |\tilde{w}_0| \,\mathrm{d}x \tag{3.9}$$

for each $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leqslant 1$. Take as functions η the elements of a subsequence $\{\eta_m\}$, $(m \in \mathbb{N})$ such that $\{\eta_m\}$ converges to sign $(\tilde{w})$ in $L^1(Q_T)$ as $m \to \infty$. Letting $m \to \infty$ in (3.9) yields

$$\iint_{Q_T} |\tilde{w}| \,\mathrm{d}x\,\mathrm{d}t \leqslant \iint_{Q_T} (T - t)|z| \,\mathrm{d}x\,\mathrm{d}t + T \int_\Omega |\tilde{w}_0| \,\mathrm{d}x \tag{3.10}$$

which completes the proof of Lemma 3.6. □

Corollary 3.8. *There exists at most one weak solution w of Problem* (P). *The function w belongs to $C^{\alpha,\alpha/2}(Q_T)$ for all $\alpha \in (0, 1)$.*

Proof. Suppose that w_1 and w_2 are two weak solutions of Problem (P) with initial data $w_{0,1} = w_{0,2}$ and let $M > 0$ be such that $|w_i| \leqslant M$ $(i = 1, 2)$. Since h is locally Lipschitz continuous on $\mathbb{R}$, there exists a constant L such that

$$|h(w_1) - h(w_2)| \leqslant L|w_1 - w_2| \quad \text{in } Q_T.$$

Applying (3.10) with Q_T replaced by $\Omega \times (t_0, t_0 + \tau)$ gives

$$\int_{t_0}^{t_0+\tau} \int_\Omega |w_1 - w_2| \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant \tau \int_\Omega |w_1(x, t_0) - w_2(x, t_0)| \,\mathrm{d}x + \int_{t_0}^{t_0+\tau} \int_\Omega (t_0 + \tau - t)|h(w_1) - h(w_2)| \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant \tau \int_\Omega |w_1(x, t_0) - w_2(x, t_0)| \,\mathrm{d}x + \tau L \int_{t_0}^{t_0+\tau} \int_\Omega |w_1 - w_2| \,\mathrm{d}x\,\mathrm{d}t$$

from which it follows that

$$\int_{t_0}^{t_0+\tau} \int_{\Omega} |w_1 - w_2| \, dx \, dt \leqslant 2\tau \int_{\Omega} |w_1(x, t_0) - w_2(x, t_0)| \, dx \tag{3.11}$$

for all $\tau \leqslant 1/(2L)$.

Let

$$\tilde{t} := \sup\{t \in [0, T] \mid w_1(x, s) = w_2(x, s) \text{ for } 0 \leqslant s \leqslant t, \; x \in \Omega\}$$

and assume that $\tilde{t} < T$. Let

$$t_0 := \begin{cases} 0 & \text{if } \tilde{t} = 0, \\ \tilde{t} - \varepsilon & \text{if } \tilde{t} > 0 \text{ with } \varepsilon < \min\{\tilde{t}, 1/(2L)\}. \end{cases}$$

Then $w_1(\cdot, t_0) = w_2(\cdot, t_0)$ so that by (3.11),

$$w_1 = w_2 \quad \text{on } \; \Omega \times (t_0, t_0 + \tau)$$

$\tau \in [0, \min\{1/(2L), T - t_0\}]$, which contradicts the definition of $\tilde{t}$. Therefore, Problem (P) has at most one weak solution w. The Hölder continuity of w in Q_T follows from [4, Theorem 1.1, p. 41]. $\square$

Finally, we remark that just as in [2] one can write a strong form of Problem (P), where the equations on the free boundary explicitly appear.

4. Numerical computations of some two-dimensional patterns of the RD system and of the free boundary problem

The equations which we solve are given by

$$u_t = d_u \Delta u + \lambda u(1 - u) - kuv \quad (x, y) \in (0, 1) \times (0, 1), \;\; t > 0, \tag{4.1}$$

$$v_t = d_v \Delta v + \lambda v(1 - v) - kuv \quad (x, y) \in (0, 1) \times (0, 1), \;\; t > 0, \tag{4.2}$$

together with the boundary conditions,

$$(u(x, 0, t), v(x, 0, t)) = \begin{cases} (-2.5x + 0.5, 0), & x \in (0.0, 0.2], \\ (0, -0.625(1 - x) + 0.5), & x \in (0.2, 1.0), \end{cases} \tag{4.3}$$

$$(u(x, 1, t), v(x, 1, t)) = \begin{cases} (-0.625x + 0.5, 0), & x \in (0.0, 0.8], \\ (0, -2.5(1 - x) + 0.5), & x \in (0.8, 1.0) \end{cases} \tag{4.4}$$

and some initial state which is shown in the figures. We also compute the solution of the limiting free boundary problem given by the equation

$$w_t = \Delta \mathscr{D}(w) + \lambda w(1 - |w|), \quad x \in (0, 1) \times (0, 1), \;\; t > 0, \tag{4.5}$$

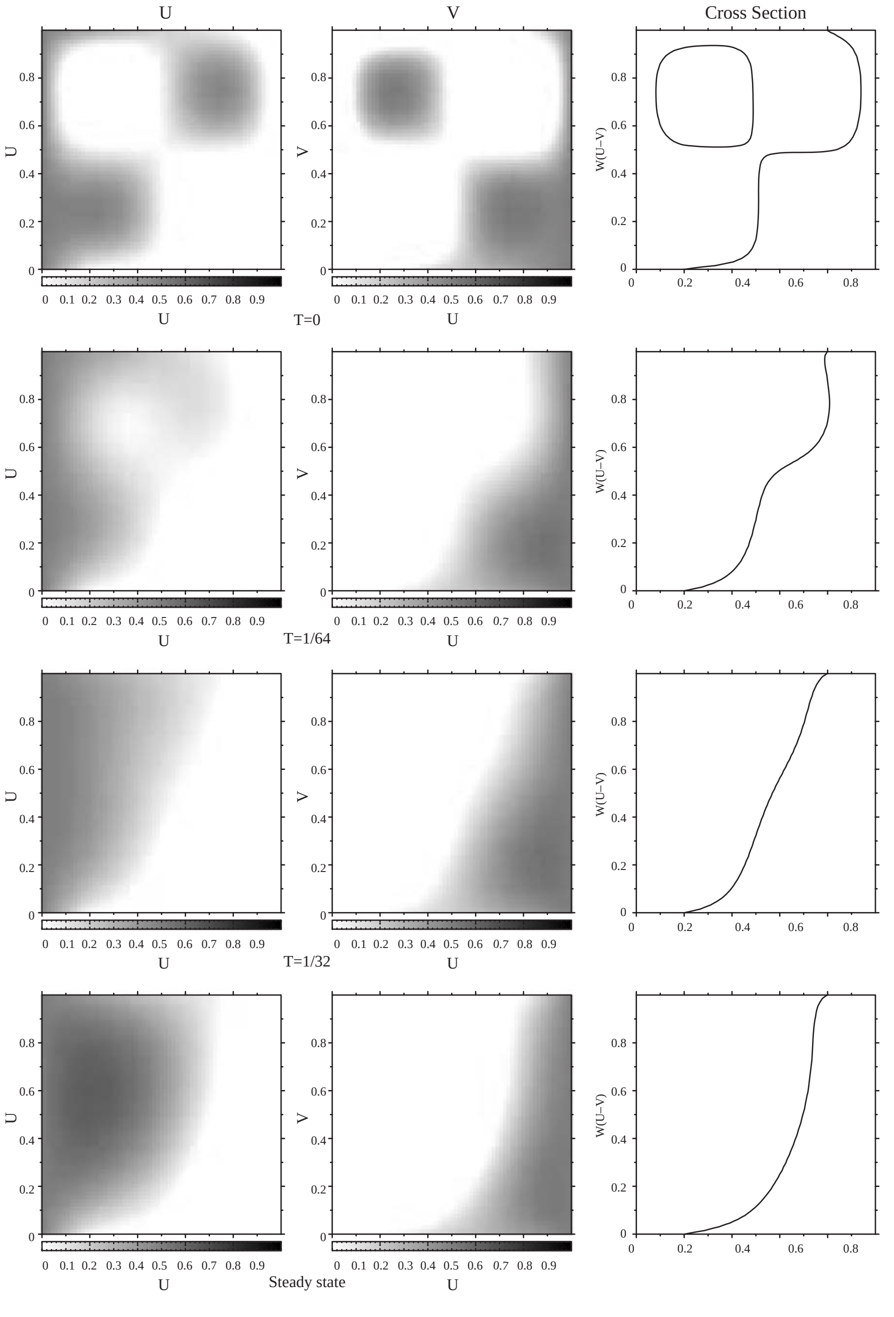

Fig. 4. FBP: $d_u = 1.5$, $d_v = 1.0$, $\lambda = 50$.

together with the boundary conditions,

$$w(x,0,t)=\begin{cases}-2.5x+0.5, & x\in(0.0,0.2],\\ 0.625(1-x)-0.5, & x\in(0.2,1.0),\end{cases} \tag{4.6}$$

$$w(x,1,t)=\begin{cases}-0.625x+0.5, & x\in(0.0,0.8],\\ 2.5(1-x)-0.5, & x\in(0.8,1.0).\end{cases} \tag{4.7}$$

The values of the parameters are

$$d_u=1.5,\qquad d_v=1.0,\qquad \lambda=50,\qquad k=10^2,10^3,10^4. \tag{4.8}$$

The time step is given by $\Delta t=2^{-16}$, and the space steps by $\Delta x=\Delta y=2^{-6}$.

5. Concluding remarks

We have obtained a free boundary problem from the competition-diffusion system by taking a limit as the interspecific competition rates tend to infinity. Fig. 4 shows numerical simulations of the free boundary problem where the initial and boundary conditions are deduced from those of the competition-diffusion system. Comparing with Fig. 3, one can see that it is a good approximation to the reaction-diffusion system.

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