A QUANTITATIVE GENETICS MODEL WITH SEXUAL MODE OF REPRODUCTION IN THE REGIME OF SMALL VARIANCE

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Abstract. We study the asymptotic behavior of stationary solutions to a quantitative genetics model with trait-dependent mortality and sexual reproduction. The infinitesimal model accounts for the mixing of parental phenotypes at birth. Our asymptotic analysis encompasses the case when deviations between the offspring and the mean parental trait are typically small. Under suitable regularity and growth conditions on the mortality rate, we prove existence and local uniqueness of a stationary profile that get concentrated around a local optimum of mortality, with a Gaussian shape having small variance. Our approach is based on perturbative analysis techniques that require to describe accurately the correction to the Gaussian leading order profile. Our result extends previous results obtained with an asexual mode of reproduction, but using an alternative methodology.

1. Introduction

We investigate solutions $(\lambda_\varepsilon, F_\varepsilon)$ of the following stationary problem:

$$(PF_\varepsilon) \qquad \lambda_\varepsilon F_\varepsilon(z) + m(z)F_\varepsilon(z) = \mathcal{B}_\varepsilon(F_\varepsilon)(z)\,, \quad z \in \mathbb{R}^d\,,$$

where z denotes a phenotypic trait variable, $F_\varepsilon(z)$ is the phenotypic distribution of the population, $m(z)$ is the (trait-dependent) mortality rate, and finally $\mathcal{B}_\varepsilon(f)$ is the following non linear, homgeneous operator associated to the infinitesimal model Fisher (1919).

$$\mathcal{B}_\varepsilon(f)(z) := \frac{1}{\varepsilon^d \pi^{\frac{d}{2}}} \iint_{\mathbb{R}^{2d}} \exp\left[-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right] f(z_1)\frac{f(z_2)}{\int_{\mathbb{R}} f(z_2')\,dz_2'}\,dz_1 dz_2.$$

This model is widely used in theoretical evolution see e.g. Bulmer et al. (1980); Ronce and Kirkpatrick (2001); Cotto and Ronce (2014); Turelli (2017), to describe sexual reproduction. The underlying assumption is that, when two individuals with traits z_1, z_2 mate, the trait of the offspring is distributed normally around the mean trait of the parents $(z_1+z_2)/2$ (see Bulmer et al., 1980; Bürger, 2000; Doebeli et al., 2007, for biological details). The positive parameter ε tunes the typical deviation size of the offspring trait from the mean of the parents traits. There exists a broad litterature in evolution theory on closely related model such as Slatkin and Lande (1976); Roughgarden (1972); Slatkin (1970). From a mathematical point of view, this model received recent attention in the field of probability theory ,Barton et al. (2017) and integro-differential equations Mirrahimi and Raoul (2013); Raoul (2017).

The problem (PF_ε) is equivalent to the existence of special solutions of the form $\exp(\lambda_\varepsilon t)F_\varepsilon(z)$, for the following non-linear but one-homogeneous equation which will the subject of future work:

$$(1.1) \qquad \partial_t f(t,z) + m(z)f(t,z) = \mathcal{B}_\varepsilon(f)(t,z)\,, \quad t > 0\,,\ z \in \mathbb{R}^d.$$

This preliminary works on the stationary profile paves the way of systematic analysis of various quantitative genetics models, including multiple effects. The problem (PF_ε) expresses the balance between selection via trait-dependent mortality, and the generation of diversity through reproduction.

Date: November 8, 2018.

2010 *Mathematics Subject Classification.* 35P20;35P30;35Q92;35B40;47G20.

Inspired by similar asymptotics in the case of asexual reproduction, see *e.g.* Perthame (2007) and further discussion below, our goal is to analyze problem (PF_ε) in the limit of vanishing variance $\varepsilon^2 \to 0$. As there is few diversity generated in this asymptotic regime, we expect that the population variance vanishes as well. Actually, there is strong evidence that the leading order profile is a Gaussian distribution with variance ε^2, see below for further discussion. As a matter of fact, any Gaussian distribution with variance ε^2 is invariant by the infinitesimal operator in the absence of selection ($m \equiv 0$, $\lambda_\varepsilon = 1$) Turelli and Barton (1994). This motivates the following decomposition of the solution:

$$F_\varepsilon(z) = \frac{1}{(2\pi)^{\frac{d}{2}}\varepsilon^d} \exp\left(-\frac{(z-z_0)^2}{2\varepsilon^2} - U_\varepsilon(z)\right). \tag{1.2}$$

The latter (1.2) is similar to the Hopf-Cole transform used in adaptative dynamics to study asymptotics problems. In our case U_ε is a corrector term that measures the deviation from the asymptotic Gaussian distribution of variance ε^2. Our analysis reveals that selection determines the center of the distribution z_0, as expected, and also reshapes the distribution F_ε via the corrector U_ε.

The operator $\mathcal{B}_\varepsilon$ is invariant by translation. Up to a translation of m, we may assume that the leading order Gaussian distribution is centered at the origin, *i.e.* $z_0 = 0$. Next, up to a change of $\lambda_\varepsilon \leftarrow \lambda_\varepsilon + m(0)$, we may assume that $m(0) = 0$. Note that we may also assume $U_\varepsilon(0) = 0$ without loss of generality, as the original problem is homogeneous.

Plugging the transformation of (1.2) into (PF_ε) yields the following equivalent problem for U_ε :

$$\lambda_\varepsilon + m(z) = I_\varepsilon(U_\varepsilon)(z) \exp\left(U_\varepsilon(z) - 2U_\varepsilon\left(\frac{z}{2}\right) + U_\varepsilon(0)\right), \quad z \in \mathbb{R}^d. \tag{PU_ε}$$

The residual term from the integral contributions is the following non-local term $I_\varepsilon(U_\varepsilon)$:

(1.3) $I_\varepsilon(U_\varepsilon)(z)$

$$= \frac{\displaystyle\iint_{\mathbb{R}^{2d}} \exp\left[-\frac{1}{2}y_1 \cdot y_2 - \frac{3}{4}\left(|y_1|^2 + |y_2|^2\right) + 2U_\varepsilon\left(\frac{z}{2}\right) - U_\varepsilon\left(\frac{z}{2} + \varepsilon y_1\right) - U_\varepsilon\left(\frac{z}{2} + \varepsilon y_2\right)\right] dy_1 dy_2}{\displaystyle\pi^{d/2} \int_{\mathbb{R}} \exp\left[-\frac{1}{2}|y|^2 + U_\varepsilon(0) - U_\varepsilon(\varepsilon y)\right] dy}.$$

This decomposition appears to be relevant because a formal computation shows that $I_\varepsilon(U_\varepsilon) \to 1$ as $\varepsilon \to 0$. Establishing uniform convergence is actually a cornerstone of our analysis. Thus for small ε, the problem (PU_ε) is presumably close to the following corrector equation, obtained formally at $\varepsilon = 0$:

$$\lambda_0 + m(z) = \exp\left(U_0(z) - 2U_0\left(\frac{z}{2}\right) + U_0(0)\right), \quad z \in \mathbb{R}^d. \tag{PU_0}$$

Interestingly, this finite difference equation admits explicit solutions by means of an infinite series:

$$U_0(z) = \gamma_0 \cdot z + \sum_{k \geqslant 0} 2^k \log\left(\lambda_0 + m(2^{-k}z)\right),$$

However, two difficulty remains: identify *(i)* the linear part $\gamma_0 \in \mathbb{R}^d$ and *(ii)* the unknown $\lambda_0 \in \mathbb{R}$. On the one hand, the linear part γ_0 cannot be recovered from (PU_0) because linear contributions cancel in the right-hand-side of (PU_0). Thus, identifying the coefficient γ_0 will be a milestone of our analysis. On the other hand, two important conditions must be fulfilled to guarantee that the series above converges, namely:

$$\lambda_0 + m(0) = 1, \quad \text{and} \quad \partial_z m(0) = 0.$$

The latter is a constraint on the possible translations that can be operated: the origin must be located at a critical point of m. The former prescribes the value of λ_0 accordingly. These two conditions are necessary conditions for the resolvability of problem (PU_0). Indeed, evaluating

(PU_0) at $z = 0$, we get the first identity. Next, differentiating and evaluating again at $z = 0$, we get the second identity.

In the sequel we make this formal discussion rigourous, following a perturbative approach for ε small enough. Before stating our main result, we need to prescribe the appropriate functional space for the corrector U_ε.

Definition 1.1 (Functional space for U_ε)**.**
For any positive parameter $\alpha \leqslant 2/5$, we define the functional space

$$\mathcal{E}^\alpha = \left\{ u \in \mathcal{C}^3(\mathbb{R}^d) \ : \ u(0) = 0, \ \textit{and} \ \left| \begin{array}{r} |Du(z)| \\ (1+|z|)^\alpha \left|D^2u(z)\right| \\ (1+|z|)^\alpha \left|D^3u(z)\right| \end{array} \right. \in L^\infty(\mathbb{R}^d) \right\},$$

equipped with the norm

$$\|u\|_\alpha = \max\left(\sup_{z\in\mathbb{R}^d} |Du(z)|\,, \ \sup_{z\in\mathbb{R}^d} (1+|z|)^\alpha \left\{ \left|D^2u(z)\right|, \left|D^3u(z)\right| \right\} \right). \tag{1.4}$$

For any bounded set K of $\mathcal{E}^\alpha$, we use the notation $\|K\|_\alpha = \sup_{u\in K} \|u\|_\alpha$. Occasionally we use the notation φ_α for the weight function $\varphi_\alpha(z) = (1+|z|)^\alpha$. Although $2/5$ is not the critical threshold, it happens that the exponent α cannot be taken too large in our approach. We set implicitly $\alpha = 2/5$ in the following results, however we leave it as a parameter to emphasize its role in the analysis, and to pinpoint the apparition of the threshold. Note that $\alpha > 0$ is required in our approach, as one constant collapses in the limit $\alpha \to 0$ (see estimate (5.7) below).

Then, we detail the assumptions on the selection function m.

Definition 1.2 (Assumptions on m)**.**
The function m is a $\mathcal{C}^3(\mathbb{R}^d)$ function, bounded below, that admits a local *non-degenerate minimum at* 0 *such that $m(0) = 0$, and there exists $\mu_0 > 0$ such that $D_z^2 m(0) \geqslant \mu_0\,\mathrm{Id}$ in the sense of symmetric matrices. Furthermore we suppose that $(\forall z)\ 1 + m(z) > 0$ and*

$$(1+|z|)^\alpha \frac{D^k m(z)}{1+m(z)} \in L^\infty(\mathbb{R}^d)\,, \quad \textit{for } \ k = 1,2,3\,. \tag{1.5}$$

The condition (1.5) is clearly verified if m is a polynomial function. It would be tempting to write, in short, that $\log(1+m) \in \mathcal{E}^\alpha$, which is indeed a consequence of (1.5). However, the latter condition also contains the decay of the first order derivative $D\log(1+m)$ with rate $|z|^{-\alpha}$, which is not contained in the definition of $\mathcal{E}^\alpha$ for good reasons.

We also introduce the subset $\mathcal{E}_0^\alpha$:

$$\mathcal{E}_0^\alpha = \left\{ v \in \mathcal{E}^\alpha \ : \ D_z v(0) = 0, \ D_z^2 v(0) \geqslant D_z^2 m(0) \geqslant \mu_0\,\mathrm{Id} \right\}, \tag{1.6}$$

Then, our assumption on m in fact guarantees that

$$\log(1+m) \in \mathcal{E}_0^\alpha. \tag{1.7}$$

The main result of this article is the following theorem :

Theorem 1.3 (Existence and convergence)**.**

(i) There exist K_0 a ball of $\mathcal{E}^\alpha$, and ε_0 a positive constant, such that for any $\varepsilon \leqslant \varepsilon_0$, the problem ($PU_\varepsilon$) admits a unique solution $(\lambda_\varepsilon, U_\varepsilon) \in \mathbb{R} \times K_0$.

(ii) The family $(\lambda_\varepsilon, U_\varepsilon)_\varepsilon$ converges to (λ_0, U_0) as $\varepsilon \to 0$, with

$$\lambda_0 = 1, \tag{1.8}$$

$$U_0(z) = \gamma_0 \cdot z + V_0(z), \tag{1.9}$$

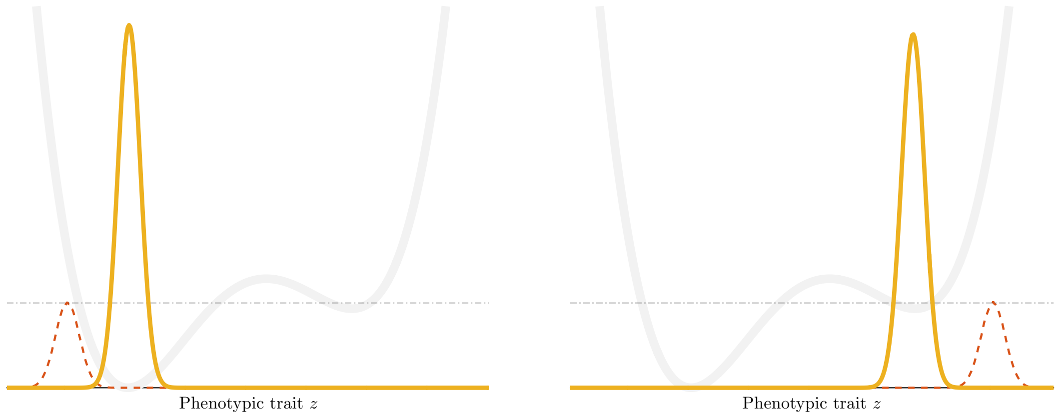

FIGURE 1. Numerical simulations of the stationary problem (PF_ε) with $\varepsilon = 0.1$ in an asymmetric double-well mortality rate (grey line). The numerical equilibrium is in yellow plain line. The only difference between the two simulations is the initial data (red dashed line). The simulations illustrate the lack of uniqueness for problem (PF_ε).

where

$$\gamma_0 = \begin{cases} \dfrac{\partial_z^3 m(0)}{2\partial_z^2 m(0)}, & \text{if} \quad d = 1 \\ \dfrac{1}{2}\left(D^2 m(0)\right)^{-1} D(\Delta m)(0), & \text{if} \quad d > 1 \end{cases} \quad \text{and} \quad V_0 = \sum_{k\geqslant 0} 2^k \log\Big(1 + m(2^{-k}z)\Big). \tag{1.10}$$

Moreover, the convergence $U_\varepsilon \to U_0$ is locally uniform up to the second derivative.

An immediate remark is that the regularity required by (1.5), and particularly the $\mathcal{C}^3$ regularity of m, is consistent with formula (1.9) which involves the pointwise value of third derivatives of m. Alternatively speaking we think that our result is close to optimal in terms of regularity.

It is important to notice that our result holds true for *any* local mimimum z_0 such that

$$m(z_0) < 1 + \inf m. \tag{1.11}$$

One should define the functional spaces $\mathcal{E}^\alpha$ and $\mathcal{E}_0^\alpha$ accordingly (and particularly replace the conditions $u(0) = 0$ and $Du(0) = 0$ by the conditions $u(z_0) = 0$ and $Du(z_0) = 0$), and then adapt (1.8)–(1.9) as follows, for the one-dimensional case:

$$\begin{aligned} &\lambda_0 = 1 - m(z_0), \\ &U_0(z_0 + h) = \gamma_0 \cdot h + \sum_{k\geqslant 0} 2^k \log\Big(1 + m(2^{-k}(z_0 + h)) - m(z_0)\Big), \end{aligned} \tag{1.12}$$

where γ_0 is defined by the same formula as in (1.10) but evaluated at z_0. Immediately, one sees that the compatibility condition (1.11) is necessary to have the positivity of the term inside the log in (1.12). As a consequence, we have:

Corollary 1.4.
If the selection function m has at least two different local non-degenerate minima that verify the compatibility condition (1.11), there exists at least two pairs $(\lambda_\varepsilon, F_\varepsilon)$ solutions of problem (PF_ε) for ε small enough.

We performed numerical simulations to illustrate this phenomenon (see Figure 1). The function m is an asymmetric double well function. We solved the time marching problem (1.1) but on the renormalized density $F_\varepsilon / \int F_\varepsilon$ in order to catch a stationary profile. We clearly observed the co-existence of two equilibria for the same set of parameters, that were obtained for two different initializations of the scheme. However, let us mention that the question of uniqueness in the case of a convex selection function m is an open question, to the extent of your knowledge.

This result is in contrast with analogous eigenvalue problems where $\mathcal{B}_\varepsilon$ is replaced with a linear operator, say $F_\varepsilon + \varepsilon^2 \Delta F_\varepsilon$, as in various quantitative genetics models with asexual mode of reproduction Barles et al. (2009), or in semiclassical analysis of the Schrödinger equation, see *e.g.* Dimassi et al. (1999). In the latter case, λ_ε and F_ε are uniquely determined (up to a multiplicative constant for F_ε) under mild assumptions on the potential m. This is the signature that $\mathcal{B}_\varepsilon$ is genuinely non-linear and non-monotone, so that possible extensions of the Krein-Rutman theorem for one-homogeneous operators, as in Mahadevan (2007), are not applicable.

The existence part (i) has already been investigated in Bourgeron et al. (2017) using the Schauder fixed point theorem and very loose variance estimates. But the approach was not designed to catch the asymptotic regime $\varepsilon \to 0$. The current methodology gives much more precise information on the behavior of the solutions of the problem (PF_ε) in the regime of vanishing variance.

Theorem 1 provides a rigorous background for the connection between problem (PU_ε) and problem (PU_0) in a perturbative setting. It justifies that the problem (PF_ε) is well approximated by the solution (λ_0, U_0) of the problem (PU_0). Quite surprisingly, the value γ_0 of the linear part of the corrector function U_0 is resolved during the analysis although it cannot be obtained readily from problem (PU_0) as mentioned above. It coincides with the heuristics of Bouin et al. (2018) where the same coefficient was obtained by studying the formal expansion up to the next order in ε^2: $U_\varepsilon = U_0 + \varepsilon^2 U_1 + o(\varepsilon^2)$, and by identifying the equation on U_1 in which the value of γ_0 appears as another compatibility condition. Here the value of γ_0 is a by-product of the perturbative analysis.

Our approach is very much inspired, yet different to most of the current literature about asymptotic analysis of asexual models, where the limiting problem is a Hamilton-Jacobi equation, see Diekmann et al. (2005) and subsequent works: Barles and Perthame (2007); Perthame and Barles (2008); Barles et al. (2009); Lorz et al. (2011). To draw a parallel with our problem, let us consider the case where $\mathcal{B}_\varepsilon(f)$ is replaced with the (linear) convolution operator $K_\varepsilon * f$, where the kernel has the scaling property $K_\varepsilon = \frac{1}{\varepsilon} K_1\left(\frac{\cdot}{\varepsilon}\right)$ – again, ε measures the typical size of the deviation between the offspring trait and the sole parental trait. In this case, it is natural to introduce the Hopf-Cole transformation $U_\varepsilon = -\varepsilon \log F_\varepsilon$. Then, the problem is equivalent to the asymptotic analysis of the following equation as $\varepsilon \to 0$:

$$\lambda_\varepsilon + m(z) = \int_{\mathbb{R}^d} K(y) \exp\left(\frac{U_\varepsilon(z) - U_\varepsilon(z - \varepsilon y)}{\varepsilon}\right) dy, \tag{1.13}$$

For this model, it is known that U_ε converges towards the viscosity solution of a Hamilton-Jacobi equation Barles et al. (2009):

$$\lambda_0 + m(z) = \int_{\mathbb{R}^d} K(y) \exp\left(D_z U_0(z) \cdot y\right) dy\,. \tag{1.14}$$

Note that the limiting equation on U_0 (1.14) can be derived formally from (1.13) by a first order Taylor expansion on U_ε.

There are two main discrepancies between the asexual case (1.13)–(1.14) and our problem involving the infintesimal model with small variance. In both cases, ε plays a similar role (measuring typical deviations between offspring and parental traits), but the appropriate normalization differs by a factor ε: it is $-\varepsilon \log F_\varepsilon$ in the asexual case, whereas it is $-\varepsilon^2 \log F_\varepsilon$ in our context, see (1.2). Secondly, the two analogous limiting problems (PU_0) and (1.14) have completely different natures. Moreover, due to the lack of a comparison principle in the original problem (PF_ε), we could not

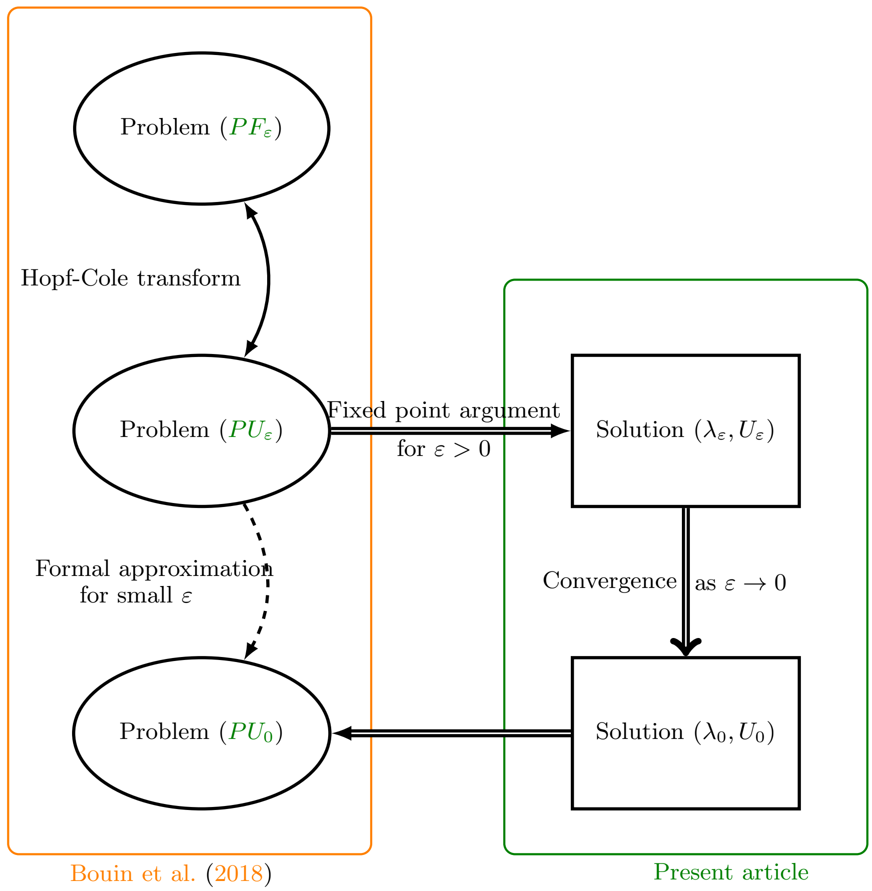

FIGURE 2. Scope of our paper compared to precedent work

envision a similar notion of viscosity solutions for (PU_0). Instead, we use rigid contraction properties and a suitable perturbative analysis to construct strong solution near the limiting problem, as depicted in Figure 2.

Mirrahimi and Raoul (2013) observed that the infinitesimal operator $\mathcal{B}_\varepsilon$ alone enjoys a uniform contraction property with respect to the quadratic Wasserstein distance, with a factor of contraction $1/2$. Recently, this was used by Magal and Raoul (2015) to perform a hydrodynamic limit in a different regime than the one under consideration here. However, the combination of $\mathcal{B}_\varepsilon$ with a zeroth-order heterogeneous mortality seems to destroy this nice structure.

The next section is devoted to the reformulation of problem (PU_ε) into a fixed point problem, introducing a set of notation and the strategy to prove Theorem 1.3.

Up until the last part of the article we implicitly work in dimension $d = 1$, for the readers' convenience. In section 7 we pinpoint the few elements of the proof that are specific to the one-dimensional case and give an extension to the higher-dimensional case in order to complete the proof of Theorem 1.3.

Acknowledgement. The authors are grateful to Laure Saint-Raymond for stimulating discussions at the early stage of this work. Part of this work was completed when VC was on temporary leave to the PIMS (UMI CNRS 3069) at the University of British Columbia. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 639638).

2. Reformulation of the problem as a fixed point

2.1. **Looking for problem** (PU_ε)**.** The equivalence between problem (PF_ε) and problem (PU_ε) through the transform (1.2) is not immediate. It is detailed in Bouin et al. (2018), but we recall here the key steps for the sake of completeness. Plugging (1.2) into problem (PF_ε) yields, with the notation $q(z) = \frac{z^2}{2}$:

$$\lambda_\varepsilon + m(z) = \frac{\iint_{\mathbb{R}^2} \exp\left[-\frac{1}{\varepsilon^2}\left(2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right) - U_\varepsilon(z_1) - U_\varepsilon(z_2) + U_\varepsilon(z)\right] dz_1 dz_2}{\varepsilon\sqrt{\pi}\int_{\mathbb{R}} \exp\left(-\frac{q(z')}{\varepsilon^2} - U_\varepsilon(z')\right) dz'}.$$

When $\varepsilon \to 0$, we expect the numerator integral to concentrate around the minimum of the principal term that is :

$$\operatorname*{argmin}_{(z_1,z_2)} \left[2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right] = \left(\frac{z}{2}, \frac{z}{2}\right).$$

We introduce the notation

$$\overline{z} = \frac{z}{2}.$$

Using the change of variable $(z_1, z_2) = (\overline{z} + \varepsilon y_1, \overline{z} + \varepsilon y_2)$, we obtain the following equation :

$$\lambda_\varepsilon + m(z) = \frac{\iint_{\mathbb{R}^2} \exp\left(-Q(y_1,y_2) - U_\varepsilon(\overline{z}+\varepsilon y_1) - U_\varepsilon(\overline{z}+\varepsilon y_2) + U_\varepsilon(z)\right) dy_1 dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\left(-y^2/2 - U_\varepsilon(\varepsilon y)\right) dy}, \tag{2.1}$$

where

$$\frac{1}{\varepsilon^2}\left[2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right] = \frac{1}{2}y_1 y_2 + \frac{3}{4}(y_1^2 + y_2^2) = Q(y_1, y_2).$$

Definition 2.1.
We denote by Q *the following quadratic form :*

$$Q(y_1, y_2) = \frac{1}{2}y_1 y_2 + \frac{3}{4}(y_1^2 + y_2^2).$$

It is the residual quadratic form after our change of variable. We notice that $\frac{1}{\sqrt{2\pi}}\exp(-Q)$ is the density of a bivariate normal random variable with covariance matrix

$$\Sigma = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}. \tag{2.2}$$

At the denominator of (2.1) naturally arises N the density function of a $\mathcal{N}(0,1)$ random variable.

Finally, (2.1) is equivalent to problem (PU_ε):

$$\lambda_\varepsilon + m(z) = I_\varepsilon(U_\varepsilon)(z) \exp\left(U_\varepsilon(z) - 2U_\varepsilon\left(\overline{z}\right) + U_\varepsilon(0)\right),$$

simply by conjuring $2U_\varepsilon(z/2)$ at the numerator and $U_\varepsilon(0)$ at the denominator, resulting into the defintion of the remainder $I_\varepsilon(U_\varepsilon)$ (1.3) that will be controlled uniformly close to 1 in all our analysis.

In the next section we explain how we reformulate the problem (PU_ε) into a fixed point argument in order to use a Banach-Picard fixed point theorem which prove our results rigorously.

2.2. **Some auxiliary functionals and the fixed point mapping.** This section is devoted to the derivation of an alternative formulation for problem (PU_ε). Let $(\lambda_\varepsilon, U_\varepsilon)$ be a solution of problem (PU_ε) in $\mathbb{R}\times\mathcal{E}^\alpha$.

The first step is to dissociate the study of λ_ε and U_ε. We first evaluate the problem (PU_ε) at $z=0$. It yields the following condition on λ_ε, since $m(0)=0$:

$$\lambda_\varepsilon = I_\varepsilon(U_\varepsilon)(0). \tag{2.3}$$

Considering the terms $\mathcal{I}_\varepsilon$ as a perturbation, we divide problem (PF_ε) by $\mathcal{I}_\varepsilon(U_\varepsilon)(z)$ which is positive, and we take the logarithm on each side. Then we obtain the following equation, considering (2.3) :

$$U_\varepsilon(z) - 2U_\varepsilon(\overline{z}) + U_\varepsilon(0) = \log\left(\frac{I_\varepsilon(U_\varepsilon)(0)+m(z)}{I_\varepsilon(U_\varepsilon)(z)}\right) \tag{2.4}$$

It would be tempting to transform (2.4) into a fixed point problem by inverting the linear operator in the left-hand-side. However, the latter is not invertible as it contains linear functions in its kernel. Therefore we are led to consider linear contributions separately.

Our main strategy is to decompose the unknown U_ε under the form

$$\boxed{U_\varepsilon(z) = \gamma_\varepsilon z + V_\varepsilon(z),} \tag{2.5}$$

with $V_\varepsilon \in \mathcal{E}_0^\alpha$. This is consistent with the analytic shape of our statement in (1.9), where γ_0 and V_0 have quite different features with respect to the function m.

Next, it is natural to differentiate (2.4). One ends up with the following recursive equation for every $z\in\mathbb{R}$

$$\partial_z U_\varepsilon(z) - \partial_z U_\varepsilon(\overline{z}) = \partial_z\left[\log\left(\frac{I_\varepsilon(U_\varepsilon)(0)+m}{I_\varepsilon(U_\varepsilon)(z)}\right)\right](z). \tag{2.6}$$

One simply deduces that, if U_ε exists and is regular, then we must have:

$$\partial_z U_\varepsilon(z) = \partial_z U_\varepsilon(0) + \sum_{k\geqslant 0}\partial_z\left[\log\left(\frac{I_\varepsilon(U_\varepsilon)(0)+m}{I_\varepsilon(U_\varepsilon)(z)}\right)\right](2^{-k}z). \tag{2.7}$$

One can formally integrate back the previous equation to obtain

$$U_\varepsilon(z) = U_\varepsilon(0) + \partial_z U_\varepsilon(0)z + \sum_{k\geqslant 0} 2^k\log\left(\frac{I_\varepsilon(U_\varepsilon)(0)+m}{I_\varepsilon(U_\varepsilon)(z)}\right)(2^{-k}z). \tag{2.8}$$

At this stage we formally identify :

- ▷ $U_\varepsilon(0)=0$, since $U_\varepsilon\in\mathcal{E}^\alpha$. This is not a loss of generality by homogeneity since F_ε is itself defined up to a multiplicative constant in problem (PF_ε).
- ▷ $\gamma_\varepsilon = \partial_z U_\varepsilon(0)$. In fact this is part of the decomposition (2.5) since $V_\varepsilon\in\mathcal{E}_0^\alpha$.

The real number γ_ε is unknown at this stage, but it needs to verify some compatibility condition to make the series converging in (2.6)–(2.8). In particular, if we evaluate (2.6) at $z=0$ we obtain that γ_ε must satisfy

$$0 = \partial_z I_\varepsilon(\gamma_\varepsilon\cdot + V_\varepsilon)(0). \tag{2.9}$$

We will solve (2.9) using an implicit function theorem in order to recover the value γ_ε associated with a given V. Beforehand, we introduce the following notation:

Definition 2.2 (Finite differences operator $\mathcal{D}_\varepsilon$)**.**
We define the finite differences functional $\mathcal{D}_\varepsilon$ as

$$\mathcal{D}_\varepsilon(V)(y_1,y_2,z) = V(\overline{z}) - \frac{1}{2}V(\overline{z}+\varepsilon y_1) - \frac{1}{2}V(\overline{z}+\varepsilon y_2)\,, \quad \overline{z} = \frac{z}{2}\,.$$

We introduce the following auxiliary functional which makes the link between γ_ε and V.

Definition 2.3 (Auxiliary function $\mathcal{J}_\varepsilon$)**.**
We define the functional $\mathcal{J}_\varepsilon : \mathbb{R} \times \mathcal{E}_0^\alpha \to \mathbb{R}$ *as follows*
(2.10)
$$\mathcal{J}_\varepsilon(g, V) = \frac{1}{\varepsilon^2\sqrt{2}\pi} \iint_{\mathbb{R}^2} \exp\left[-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(y_1, y_2, 0)\right] \mathcal{D}_\varepsilon(\partial_z V)(y_1, y_2, 0)\, dy_1 dy_2.$$

The implicit relationship (2.9) is equivalent to $\mathcal{J}_\varepsilon(\gamma_\varepsilon, V_e) = 0$. From this perspective, the following result is an important preliminary step.

Proposition 2.4 (Existence and uniqueness of γ_ε)**.**
For any ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K, such that for all $\varepsilon \leqslant \varepsilon_K$ and for any $V \in K$, there exists a unique solution $\gamma_\varepsilon(V)$ to the equation :

$$\textit{Find } \gamma \in (-R_K, R_K) \textit{ such that: } \quad \mathcal{J}_\varepsilon(\gamma, V) = 0,$$

where the bound $|\gamma_\varepsilon(V)| \leqslant R_K$ is defined as

$$R_K = \max\left(\frac{\|K\|_\alpha \iint_{\mathbb{R}^2} \exp(-Q(y_1, y_2)) \left(y_1^2 + y_2^2\right) dy_1 dy_2 + 8}{2\partial_z^2 m(0)} ; \|K\|_\alpha \right). \tag{2.11}$$

Next we define the main quantity we will work with: the double integral I_ε which is the rescaled infinitesimal operator. For convenience we define it as a mapping on $\mathcal{E}_0^\alpha$. It is compatible with (1.3) because of the decomposition (2.5).

Definition 2.5 (Auxiliary functional $\mathcal{I}_\varepsilon$)**.**
We define the functional $\mathcal{I}_\varepsilon : \mathcal{E}_0^\alpha \to \mathcal{C}^3(\mathbb{R})$ *as follows*

$$\mathcal{I}_\varepsilon(V)(z) = \frac{\iint_{\mathbb{R}^2} \exp\big(- Q(y_1, y_2) - \varepsilon\gamma_\varepsilon(V)(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(y_1, y_2, z)\big) dy_1 dy_2}{\sqrt{\pi} \int_{\mathbb{R}} \exp\left(-y^2/2 - \varepsilon\gamma_\varepsilon(V) y + V(0) - V(\varepsilon y)\right) dy}. \tag{2.12}$$

Finally, in view of (2.8) and (2.5), we see that V_ε must be a solution of this implicit equation :

$$V_\varepsilon(z) = \sum_{k \geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V_\varepsilon)(0) + m}{\mathcal{I}_\varepsilon(V_\varepsilon)(z)} \right)(2^{-k} z), \quad \text{for every } z \in \mathbb{R}. \tag{2.13}$$

This justifies the introduction of our central mapping, upon which our fixed point argument will be based.

Definition 2.6 (Fixed point mapping)**.**
We define the mapping $\mathcal{H}_\varepsilon : \mathcal{E}_0^\alpha \to \mathcal{E}_0^\alpha$ *as follows*

$$\mathcal{H}_\varepsilon(V)(h) = \sum_{k \geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V)(0) + m(2^{-k} h)}{\mathcal{I}_\varepsilon(V)(2^{-k} h)} \right). \tag{2.14}$$

2.3. **Reformulation of the problem.** We are now in position to write our main result for this Section:

Theorem 2.7 (Existence and uniqueness of the fixed point)**.**
There is a ball $K_0 \subset \mathcal{E}_0^\alpha$ and a positive constant ε_0 such that for every $\varepsilon \leqslant \varepsilon_0$, the mapping $\mathcal{H}_\varepsilon$ admits a unique fixed point in K_0.

To conclude, it is sufficient to check that solving problem (PU_ε), on the ball K_0, and seeking a fixed point for $\mathcal{H}_\varepsilon$ in K_0 are equivalent problems for $\varepsilon \leqslant \varepsilon_0$ small enough.

Proposition 2.8 (Reformulation of the problem (PU_ε))**.**
There is a ball K_0' of $\mathcal{E}^\alpha$, and a positive constant ε_0' such that for every $\varepsilon \leqslant \varepsilon_0'$, the following statements are equivalent:

- ▷ *$(\lambda_\varepsilon, U_\varepsilon)$ is a solution of problem (PU_ε) in $\mathbb{R} \times K_0'$.*
- ▷ *$U_\varepsilon = \gamma_\varepsilon(V_\varepsilon) \cdot + V_\varepsilon$, with $V_\varepsilon \in \mathcal{E}_0^\alpha \cap K_0'$, $\mathcal{H}_\varepsilon(V_\varepsilon) = V_\varepsilon$, and $\lambda_\varepsilon = \mathcal{I}_\varepsilon(V_\varepsilon)(0)$.*

Moreover, the statement of Theorem 2.7 holds true in the set $\mathcal{E}_0^\alpha \cap K_0'$.

The main mathematical difficulties are stacked into Theorem 2.7. The rest of the article is organized as follows :

- ▷ In section 3, we justify why the function γ_ε is well defined in Proposition 2.4.
- ▷ Then in section 4, we provide the main properties and the key estimates of the nonlocal operator $\mathcal{I}_\varepsilon$. We point out why this term plays the role of a perturbation between problem (PU_ε) and problem (PU_0). In section 4.2 we prove crucial contraction estimates.
- ▷ Those estimates are the main ingredients of the proof of properties of $\mathcal{H}_\varepsilon$ in section 5: most notably the finiteness of $\mathcal{H}_\varepsilon(V)$, and the fact that $\mathcal{H}_\varepsilon$ is a contraction mapping.
- ▷ This allows us to establish the proof of Theorem 2.7 and Proposition 2.8, and finally to come back to the proof of our main result Theorem 1.3 in the sections 6.1 and 6.2.
- ▷ Section 7 is devoted to those specific arguments that require an extension to the higher dimensional case $d > 1$.

3. Well-posedness of the implicit function γ_ε

3.1. **Heuristics on finding γ_ε.** We consider $V \in \mathcal{E}_0^\alpha$, and we look for solutions γ_ε of $\mathcal{J}_\varepsilon(\gamma_\varepsilon, V) = 0$, or equivalently :
(3.1)
$$0 = \frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big)dy_1dy_2,$$
in accordance with (2.10). We will see here how a Taylor expansion of the right-hand-side around $\varepsilon = 0$ helps to understand why it defines a unique γ_ε in a given interval for small ε. We will show formally why $\mathcal{J}_\varepsilon(\cdot, V)$ can be uniformly approximated by a non-degenerate linear function for small ε.

We expand the right-hand-side with respect to ε:
$$\begin{aligned}
&\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2)\right]\exp\left[-\varepsilon\gamma_\varepsilon(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big)dy_1dy_2 \\
&= -\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2)\right]\left[1 - \varepsilon\gamma_\varepsilon(y_1+y_2) + o\left(\varepsilon\right)\right] \\
&\qquad\qquad \times \left(\frac{\varepsilon}{2}\left(y_1+y_2\right)\partial_z^2 V(0) + \frac{\varepsilon^2}{4}\left(y_1^2+y_2^2\right)\partial_z^3 V(0) + o(\varepsilon^2)\right)dy_1dy_2 \\
&= \frac{1}{\varepsilon^2}\left(\frac{\varepsilon^2}{2}\gamma_\varepsilon\partial_z^2 V(0) - \varepsilon^2\frac{3\partial_z^3 V(0)}{8} + o(\varepsilon^2)\right).
\end{aligned}$$
Then solving
$$0 = -\frac{3\partial_z^3 V(0)}{8} + \frac{1}{2}\gamma_\varepsilon\partial_z^2 V(0) + o(1),$$

we get the expression :

$$\gamma_\varepsilon \underset{\varepsilon\to 0}{\sim} \frac{3}{4}\frac{\partial_z^3 V(0)}{\partial_z^2 V(0)}. \tag{3.2}$$

These heuristics are consistent with the statement in Theorem 1.3, up to the relation between V_0 and m that can be easily read out from (1.10). Note that the denominator involves $\partial_z^2 V(0)$, so that the local convexity of V should be controlled uniformly during our construction. This is the purpose of the restriction in $\mathcal{E}_0^\alpha$ (1.6). In the following, we provide estimates that turn these heuristics into a rigorous proof.

3.2. **Proof of Proposition 2.4.** The aim of this section is to prove the existence and uniqueness of $\gamma_\varepsilon(V)$ stated in Proposition 2.4. We first start with a Lemma providing some useful estimates on the function $\mathcal{J}_\varepsilon$. Combining these estimates with a continuity and monotonicity arguments, we will be able to prove the Proposition 2.4.

Lemma 3.1 (Estimates of $\mathcal{J}_\varepsilon$)**.**
For any ball $K \subset \mathcal{E}_0^\alpha$, there exists $\varepsilon_K > 0$, such that for all $\varepsilon \leqslant \varepsilon_K$ and $V \in K$, the following estimate holds true for all g in the interval $(-R_K, R_K)$:

$$\mathcal{J}_\varepsilon(0,V) = -\frac{1}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1dy_2 + \mathcal{O}(\varepsilon), \tag{3.3}$$

$$\partial_g \mathcal{J}_\varepsilon(g,V) = \frac{\partial_z^2 V(0)}{2} + \mathcal{O}(\varepsilon), \tag{3.4}$$

where, in the former expansion, the variable $\widetilde{y_i}$ is a by-product of Taylor expansions and is such that $|\widetilde{y_i}| \leqslant |y_i| + 1$.

Remark 3.2.
We prove the uniqueness of γ_ε on a uniformly bounded interval. One may think it is a strong restriction not to look at large γ_ε. It is in fact a natural restriction as we have by definition $\gamma_\varepsilon = \partial_z U_\varepsilon(0)$, and $\partial_z U_\varepsilon \in L^\infty$ in our perturbative setting.

We postpone the proof of the technical Lemma 3.1 at the end of this section and we first use it to prove the Proposition 2.4:

Proof of Proposition 2.4. Let K be a ball of $\mathcal{E}_0^\alpha$ and $V \in K$. We deduce from Lemma 3.1 that $|\mathcal{J}_\varepsilon(0,V)| \leqslant G_K + 1$, where

$$G_K = \frac{\|K\|_\alpha}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left(y_1^2 + y_2^2\right) dy_1dy_2\,,$$

for ε small enough. Integrating (3.4) with respect to g, we obtain

$$\mathcal{J}_\varepsilon(g,V) = \mathcal{J}_\varepsilon(0,V) + \frac{\partial_z^2 V(0)}{2} g + \mathcal{O}(\varepsilon)\,,$$

where it is important to notice that the perturbation $\mathcal{O}(\varepsilon)$ is uniform with respect to ε for $g \in (-R_K, R_K)$ and $V \in K$. Since $V \in \mathcal{E}_0^\alpha$, we know that $\partial_z^2 V(0) \geqslant \partial_z^2 m(0) > 0$. Therefore, $\mathcal{J}_\varepsilon$ is uniformly increasing with respect to g on $(-R_K, R_K)$. Moreover, the choice of R_K is such that

$$\mathcal{J}_\varepsilon(R_K,V) \geqslant -1 - G_K + \frac{\partial_z^2 m(0)}{2} R_K + \mathcal{O}(\varepsilon) = 1 + \mathcal{O}(\varepsilon) > 0\,,$$

for ε small enough, and similarly, $\mathcal{J}_\varepsilon(-R_K,V) < 0$. Finally, there exists a unique $\gamma_\varepsilon(V)$ satisfying $\mathcal{J}_\varepsilon(\gamma_\varepsilon(V),V) = 0$ because $\mathcal{J}_\varepsilon$ is continuous with repect to g for $V \in \mathcal{E}_0^\alpha$. □

Proof of Lemma 3.1. Let K be a ball of $\mathcal{E}_0^\alpha$ of radius $\|K\|_\alpha$. In section 3.1, we have used formal Taylor expansions to get a formula for $\gamma_\varepsilon(V)$, morally valid when $\varepsilon = 0$. The idea here is to write exact rests to broaden the formula for small but positive ε.

▷ **Proof of expansion** (3.3)**.** Let us pick $V \in K$ and $\varepsilon > 0$. Recall the expression of $\mathcal{J}_\varepsilon(0, V)$:

$$\mathcal{J}_\varepsilon(0,V) = \frac{1}{\varepsilon^2\sqrt{2\pi}} \iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right] \Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big) dy_1 dy_2.$$

We perform two Taylor expansions, namely:

$$\begin{cases} 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0) = -\dfrac{\varepsilon^2}{2}\left(y_1^2\partial_z^2 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V(\varepsilon\widetilde{y_2})\right) \\ \mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0) = -\dfrac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \dfrac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})), \end{cases} \tag{3.5}$$

where $\widetilde{y_i}$ denote some generic number such that $|\widetilde{y_i}| \leqslant |y_i|$ for $i = 1, 2$. Moreover, we can write

$$\exp(-\varepsilon^2 P) = 1 - \varepsilon^2 P \exp(-\theta\varepsilon^2 P)\,, \quad P = \frac{1}{2}\left(y_1^2\partial_z^2 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V(\varepsilon\widetilde{y_2})\right)\,, \quad |P| \leqslant \frac{1}{2}\left(y_1^2 + y_2^2\right)\|V\|_\alpha\,, \tag{3.6}$$

for some $\theta = \theta(y_1, y_2) \in (0, 1)$. Combining the expansions, we find:

$$\begin{aligned}\mathcal{J}_\varepsilon(0,V) = \frac{1}{\varepsilon^2\sqrt{2\pi}} \iint_{\mathbb{R}^2} &\exp\left[-Q(y_1,y_2)\right]\left(1 - \varepsilon^2 P\exp(-\theta\varepsilon^2 P)\right) \\ &\times \left(-\frac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \frac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2}))\right) dy_1 dy_2\,.\end{aligned}$$

The crucial point is the cancellation of the $O(\varepsilon^{-1})$ contribution due to the symmetry of Q:

$$\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))(y_1+y_2) dy_1 dy_2 = 0\,. \tag{3.7}$$

So, it remains

$$\begin{aligned}\mathcal{J}_\varepsilon(0,V) = &-\frac{1}{4\sqrt{2\pi}} \iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2 \\ &+ \frac{\varepsilon}{2\sqrt{2\pi}} \iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2)) P \exp(-\theta\varepsilon^2 P)(y_1+y_2)\partial_z^2 V(0) dy_1 dy_2 \\ &+ \frac{\varepsilon^2}{4\sqrt{2\pi}} \iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2)) P \exp(-\theta\varepsilon^2 P)\left(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right) dy_1 dy_2\end{aligned}$$

Clearly the last two contributions are uniform $\mathcal{O}(\varepsilon)$ for $V \in K$ and $\varepsilon \leqslant \varepsilon_K$ small enough. Indeed, the term P is at most quadratic with respect to y_i (3.6), so $Q + \theta\varepsilon^2 P$ is uniformly bounded below by a positive quadratic form for ε small enough.

▷ **Proof of expansion** (3.4)**.** The first step is to compute the derivative of J with respect to g:

$$\begin{aligned}\partial_g\mathcal{J}_\varepsilon(g,V) = -\frac{1}{\varepsilon\sqrt{2\pi}} \iint_{\mathbb{R}^2} &\exp\left[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right] \\ &\times (y_1+y_2)\left[\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\right] dy_1 dy_2.\end{aligned}$$

Similar Taylor expansions as above yields:

$$\partial_g \mathcal{J}_\varepsilon(g,V) = -\frac{1}{\varepsilon\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2)\right]\left(1-\varepsilon P'\exp(-\theta\varepsilon P')\right) \times (y_1+y_2)\left(-\frac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \frac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2}))\right) dy_1 dy_2,$$

where $P' = g(y_1+y_2) + y_1\partial_z V(\varepsilon\widetilde{y_1}) + y_2\partial_z V(\varepsilon\widetilde{y_2})$. Interestingly, the leading order term does not cancel anymore, and it remains:

$$\partial_g \mathcal{J}_\varepsilon(g,V) = \frac{1}{2\sqrt{2}\pi}\left(\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2)\right](y_1+y_2)^2\, dy_1 dy_2\right)\partial_z^2 V(0) + \mathcal{O}(\varepsilon)\,.$$

The justification that the remainder is a uniform $\mathcal{O}(\varepsilon)$ is similar as above, except that now P' has a linear part depending on g, but the latter is assumed to be bounded a priory by R_K. □

4. Analysis of the perturbative term $\mathcal{I}_\varepsilon$

4.1. **Lispchitz continuity of some auxiliary functionals.** The function $\mathcal{I}_\varepsilon$ is crucially involved in the definition of the mapping $\mathcal{H}_\varepsilon$. Thus to prove any contraction property on this mapping we will need Lipschitz estimates about $\mathcal{I}_\varepsilon$ and the three first derivatives of its logarithm. But first we show that $\mathcal{I}_\varepsilon$ really plays the role of a perturbative term between problem (PU_ε) and problem (PU_0) that converges to 1 uniformly as $\varepsilon \to 0$.

Proposition 4.1 (Estimation of $\mathcal{I}_\varepsilon$)**.**
For every K ball of $\mathcal{E}_0^\alpha$, for every $\delta > 0$, there exists a constant ε_δ that depends only on K and δ, such that for every $\varepsilon \leqslant \varepsilon_\delta$ and for every $V \in K$:

$$(\forall z \in \mathbb{R}) \quad 1-\delta \leqslant \mathcal{I}_\varepsilon(V)(z) \leqslant 1+\delta\,.$$

Proof. Let V in K. For $\varepsilon \leqslant \varepsilon_K$, one can apply Proposition 2.4 which gives $|\gamma_\varepsilon(V)| \leqslant R_K$. Next it is enough to write that :

$$\underline{C}_\varepsilon \doteqdot \frac{\displaystyle\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) - 2\varepsilon R_K(|y_1|+|y_2|)\right] dy_1 dy_2}{\displaystyle\sqrt{2}\pi\int_{\mathbb{R}} \exp\left(-y^2/2 - 2\varepsilon R_K\,|y|\right) dy} \leqslant \mathcal{I}_\varepsilon(V)(z), \text{ and}$$

$$\mathcal{I}_\varepsilon(V)(z) \leqslant \frac{\displaystyle\iint_{\mathbb{R}^2} \exp\big(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)\big) dy_1 dy_2}{\displaystyle\sqrt{2}\pi\int_{\mathbb{R}} \exp\left(-y^2/2 - 2\varepsilon R_K\,|y|\right) dy} \doteqdot \overline{C}_\varepsilon.$$

We deduce from this lower and upper estimates that the whole $\mathcal{I}_\varepsilon(V)$ converges uniformly to 1 as $\varepsilon \to 0$. □

Next, we show Lipschitz continuity of various quantities of interest.

Proposition 4.2 (Lipschitz continuity of γ_ε)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exist constants $L_K(\gamma)$, and ε_K, depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$

$$|\gamma_\varepsilon(V_1) - \gamma_\varepsilon(V_2)| \leqslant L_K(\gamma)\,\|V_1 - V_2\|_\alpha\,.$$

Proof. Let K be a ball of $\mathcal{E}_0^\alpha$, and let $V_1, V_2 \in K$. Let denote $\Gamma_\varepsilon^I = \gamma_\varepsilon(V_i)$ for $i = 1, 2$. We argue by means of Fréchet derivatives: let $s \in (0,1)$, $\gamma_s = s\gamma_1 + (1-s)\gamma_2$, $V_s = sV_1 + (1-s)V_2$, and consider the following computation:

$$\frac{d}{ds}\mathcal{J}_\varepsilon(\gamma_s, V_s) = \partial_\gamma \mathcal{J}_\varepsilon(\gamma_s, V_s)(\gamma_1 - \gamma_2) + D_V\mathcal{J}_\varepsilon(\gamma_s, V_s) \cdot (V_1 - V_2)\,, \tag{4.1}$$

where the Fréchet derivative of J with respect to V is:

$$\begin{aligned} D_V\mathcal{J}_\varepsilon(\gamma, V)\cdot H = \frac{1}{\varepsilon^2\sqrt{2\pi}} \iint_{\mathbb{R}^2} & \exp\left[-Q(y_1,y_2) - \varepsilon\gamma(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right] \\ & \times \Big(2\mathcal{D}_\varepsilon(H)(y_1,y_2,0)\Big)\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big) dy_1 dy_2 \\ + \frac{1}{\varepsilon^2\sqrt{2\pi}} \iint_{\mathbb{R}^2} & \exp\left[-Q(y_1,y_2) - \varepsilon\gamma(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right] \Big(\mathcal{D}_\varepsilon(\partial_z H)(y_1,y_2,0)\Big) dy_1 dy_2 \end{aligned}$$

We perform similar Taylor expansions as in (3.5),

$$2\mathcal{D}_\varepsilon(W)(y_1,y_2,0) = \begin{cases} -\varepsilon(y_1+y_2) O\left(\|\partial_z W\|_\infty\right) \\ -\varepsilon(y_1+y_2)\partial_z W(0) - (\varepsilon^2/2)\left(y_1^2+y_2^2\right) O\left(\|\partial_z^2 W\|_\infty\right) \end{cases}$$

either for $W = V, H \in \mathcal{E}_0^\alpha$, or $W = \partial_z V, \partial_z H$. We deduce that

$$\begin{aligned} D_V\mathcal{J}_\varepsilon(\gamma, V)\cdot H = \frac{1}{\varepsilon^2\sqrt{2\pi}} \iint_{\mathbb{R}^2} & \exp\left[-Q(y_1,y_2) - \varepsilon\gamma(y_1+y_2) - \varepsilon(y_1+y_2) O\left(\|\partial_z V\|_\infty\right)\right] \\ \times \Bigg[& \left(-\frac{\varepsilon^2}{2}\left(y_1^2+y_2^2\right) O\left(\|\partial_z^2 H\|_\infty\right)\right)\Big(-\varepsilon(y_1+y_2) O\left(\|\partial_z^2 V\|_\infty\right)\Big) \\ & + \left(-\varepsilon(y_1+y_2)\partial_z^2 H(0) - \frac{\varepsilon^2}{2}\left(y_1^2+y_2^2\right) O\left(\|\partial_z^3 H\|_\infty\right)\right)\Bigg] dy_1 dy_2 \end{aligned} \tag{4.2}$$

We proceed as in the previous section for the exponential term: there exists $\theta = \theta(y_1, y_2) \in (0,1)$ such that

$$\exp(-\varepsilon P') = 1 - \varepsilon P' \exp(-\theta\varepsilon P')\,, \quad \text{where} \quad P' = \gamma(y_1+y_2) + (y_1+y_2) O\left(\|\partial_z V\|_\infty\right)$$

Again, the crucial point is the cancellation of the $O(\varepsilon^{-1})$ contribution in (4.2), as in (3.7) What remains is of order one or below, and one can easily show that there exists C_K such that

$$\begin{aligned} |D_V\mathcal{J}_\varepsilon(\gamma, V)\cdot H| &\leqslant C_K\left(\|\partial_z^3 H\|_\infty + |\partial_z^2 H(0)|\left(|\gamma + \|\partial_z V\|_\infty|\right)\right. \\ &\qquad \left. + \varepsilon\|\partial_z^2 H\|_\infty \|\partial_z^2 V\|_\infty + \varepsilon\|\partial_z^3 H\|_\infty\left(|\gamma + \|\partial_z V\|_\infty|\right)\right) \\ &\leqslant C_K\|H\|_\alpha\,, \end{aligned}$$

provided $\varepsilon \leqslant \varepsilon_K$ is small enough.

On the other hand, we have already established that $\partial_\gamma\mathcal{J}_\varepsilon(\gamma, V) = \partial_z^2 V(0)/2 + \mathcal{O}(\varepsilon)$ in Lemma 3.1. Consequently, integrating (4.1) from $s = 0$ to 1, we find:

$$0 = \mathcal{J}_\varepsilon(\gamma_1, V_1) - \mathcal{J}_\varepsilon(\gamma_2, V_2) = \left(\frac{\partial_z^2 V(0)}{2} + \mathcal{O}(\varepsilon)\right)(\gamma_1 - \gamma_2) + \left(\int_0^1 D_V\mathcal{J}_\varepsilon(\gamma_s, V_s)\cdot(V_1 - V_2)\, ds\right)\,.$$

We deduce from the previous estimates and the local convexity condition in (1.6) that

$$|\gamma_1 - \gamma_2| \leqslant C_K\left(\frac{2}{\partial_z^2 m(0)} + C_K\varepsilon\right)\|V_1 - V_2\|_\alpha\,,$$

for some C_K and $\varepsilon \leqslant \varepsilon_K$ small enough. □

In turn, Proposition 4.2 implies the Lipschitz continuity of $\mathcal{I}_\varepsilon$ as a function of V.

Proposition 4.3 (Lipschitz continuity of $\mathcal{I}_\varepsilon$)**.**
For every ball K of $\mathcal{E}_0^\alpha$, there exist constants ε_K, C_K depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$,

$$\sup_{z\in\mathbb{R}} |\mathcal{I}_\varepsilon(V_1)(z) - \mathcal{I}_\varepsilon(V_2)(z)| \leqslant \varepsilon C_K \left\| V_1 - V_2 \right\|_\alpha . \tag{4.3}$$

Proof. The Lipschitz continuity of $\mathcal{I}_\varepsilon$ with respect to V can be proven by composition of Lipschitz functions. With the same notations as in the proof of Proposition 4.2, and with the shortcut notation $\mathcal{I}_\varepsilon = A_\varepsilon/B_\varepsilon$ to separate the numerator from the denominator in (2.12) we have,

$$\frac{d}{ds}A_\varepsilon(V_s)(z) = -\iint_{\mathbb{R}^2} G_\varepsilon^V(y_1,y_2,z)\left(\varepsilon \frac{d}{ds}\gamma_\varepsilon(V_s)(y_1+y_2) + \int_{\overline{z}}^{\overline{z}+\varepsilon y_1} \partial_z(V_1-V_2)(z')dz' - \int_{\overline{z}}^{\overline{z}+\varepsilon y_2} \partial_z(V_1-V_2)(z')dz'\right) dy_1dy_2 ,$$

where we have simply written $V(\overline{z}+\varepsilon y_1) - V(\overline{z}) = \int_{\overline{z}}^{\overline{z}+\varepsilon y_1} \partial_z V(z')\, dz'$, and where G_ε^V denotes the exponential weight:

$$G_\varepsilon^V(y_1,y_2,z) = \frac{1}{\sqrt{2\pi}} \exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right] .$$

We deduce that A_ε is such that:

$$(\forall z) \quad \left|\frac{d}{ds}A_\varepsilon(V_s)(z)\right| \leqslant \varepsilon \iint_{\mathbb{R}^2} G_\varepsilon^V(y_1,y_2,z)\left(L_K(\gamma)\|V_1-V_2\|_\alpha + \|V_1-V_2\|_\alpha\right)(|y_1|+|y_2|)\; dy_1dy_2 .$$

As the weight G_ε^V is uniformly close to a positive quadratic form for small ε, we find that the numerator has a Lipschitz constant of order ε uniformly with respect to z:

$$\sup_{z\in\mathbb{R}} |A_\varepsilon(V_1)(z) - A_\varepsilon(V_2)(z)| \leqslant \varepsilon C_K \left\|V_1 - V_2\right\|_\alpha .$$

The same holds true for the denominator B_ε. In addition, a direct by-product of the proof of Proposition 4.1 is that A_ε and B_ε are uniformly bounded above and below by positive constants for ε small enough. Consequently, the quotient $\mathcal{I}_\varepsilon = A_\varepsilon/B_\varepsilon$ is Lipschitz continuous. □

It is useful to introduce the probability measure dG_ε^V induced by the exponential weight G_ε^V:

$$dG_\varepsilon^V(y_1,y_2,z) = \frac{G_\varepsilon^V(y_1,y_2,z)}{\iint_{\mathbb{R}^2} G_\varepsilon^V(\cdot,\cdot,z)} = \frac{\exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right]}{\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right] dy_1dy_2}.$$

As a consequence of the previous estimates, we obtain the following one:

Lemma 4.4 (Lipschitz continuity of dG_ε^V)**.**
For every ball K of $\mathcal{E}_0^\alpha$, there exist constants ε_K, C_K depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$,

$$\sup_{z\in\mathbb{R}} \left|dG_\varepsilon^{V_1}(y_1,y_2,z) - dG_\varepsilon^{V_2}(y_1,y_2,z)\right| \leqslant \varepsilon C_K \left\|V_1-V_2\right\|_\alpha (1+|y_1|+|y_2|) \exp\Big(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)\Big). \tag{4.4}$$

Furthermore, under the same conditions, we have the following bound, uniform with respect to $z \in \mathbb{R}$:

$$dG_\varepsilon^V(y_1, y_2, z) \leqslant \frac{1}{4} \exp\left[-Q(y_1, y_2) + 2\varepsilon R_K(|y_1| + |y_2|)\right]. \tag{4.5}$$

Proof. We first prove (4.5): the function G_ε^V is such that

$$G_\varepsilon^V(y_1, y_2, z) \gtrless \frac{1}{\sqrt{2\pi}} \exp\left[-Q(y_1, y_2) \mp 2\varepsilon R_K(|y_1| + |y_2|)\right].$$

Therefore, its integral over $(y_1, y_2) \in \mathbb{R}^2$ converges to 1 as $\varepsilon \to 0$, and there exists ε_K depending on K such that $\iint G_\varepsilon^V(y_1, y_2, z)\, dy_1 dy_2 \geqslant 4/\sqrt{2}\pi$ for $\varepsilon \leqslant \varepsilon_K$. This leads to (4.5).

In order to obtain (4.4), we proceed as in the proof of Proposition 4.3, as the denominator of dG_ε^V is the numerator A_ε of $\mathcal{I}_\varepsilon$. For the Lipschitz continuity of the numerator of dG_ε^V, we find:

$$(\forall z) \quad \left|\frac{d}{ds} G_\varepsilon^{V_s}(y_1, y_2, z)\right| \leqslant \varepsilon G_\varepsilon^{V_s}(y_1, y_2, z) \left(L_K(\gamma)\|V_1 - V_2\|_\alpha + \|V_1 - V_2\|_\alpha\right)(|y_1| + |y_2|)\ .$$

We deduce that the quotient $dG_\varepsilon^V = G_\varepsilon^V / A_\varepsilon(V)$ is also Lipschitz continuous:

$$\begin{aligned} \left|dG_\varepsilon^{V_1} - dG_\varepsilon^{V_2}\right| &\leqslant \left|\frac{G_\varepsilon^{V_1} - G_\varepsilon^{V_2}}{A_\varepsilon(V_1)} + \frac{A_\varepsilon(V_2) - A_\varepsilon(V_1)}{A_\varepsilon(V_1)A_\varepsilon(V_2)} G_\varepsilon^{V_2}\right| \\ &\leqslant \varepsilon C_K \|V_1 - V_2\|_\alpha (|y_1| + |y_2|) \exp\Big(-Q(y_1, y_2) + 2\varepsilon R_K(|y_1| + |y_2|)\Big) \\ &\quad + \varepsilon C_K \|V_1 - V_2\|_\alpha \exp\Big(-Q(y_1, y_2) + 2\varepsilon R_K(|y_1| + |y_2|)\Big). \end{aligned}$$

This concludes the proof of (4.4). ☐

To conclude, we have established in this section that $\mathcal{I}_\varepsilon$ is a perturbative term, both in the uniform sense $\mathcal{I}_\varepsilon(V) \to 1$, and in the Lipschitz sense: $\mathrm{Lip}_V \mathcal{I}_\varepsilon = \mathcal{O}(\varepsilon)$. In addition, we have proven a similar Lipschitz smallness property for a probabilty distribution dG_ε^V that will appear frequently in our contraction estimates.

4.2. **Contraction properties (first part).** On the way to estimating the fixed point mapping $\mathcal{H}_\varepsilon$ (2.14), we need good estimates on the logarithmic derivatives of I_ε. For that purpose, we introduce the following quantities for $i = 1, 2, 3$:

$$W_\varepsilon^{(i)}(V)(z) = \frac{\partial_z^i \mathcal{I}_\varepsilon(V)(z)}{\mathcal{I}_\varepsilon(V)(z)}. \tag{4.6}$$

For the sake of conciseness, we omit sometimes the dependency with respect to y_1, y_2 in the notations, as for instance: $dG_\varepsilon^V(y_1, y_2, z) = dG_\varepsilon^V(z)$. The following notation with a duality bracket is useful:

$$\left\langle dG_\varepsilon^V(z), f\right\rangle = \iint_{R^2} dG_\varepsilon^V(y_1, y_2, z) f(y_1, y_2)\, dy_1 dy_2.$$

Indeed, for any $V \in \mathcal{E}_0^\alpha$, we have:

$$W_\varepsilon^{(1)}(V)(z) = \left\langle dG_\varepsilon^V(z), \mathcal{D}_\varepsilon(\partial_z V)(z)\right\rangle. \tag{4.7}$$

Similarly:

$$W_\varepsilon^{(2)}(V)(z) = \left\langle dG_\varepsilon^V(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(z) + (\mathcal{D}_\varepsilon(\partial_z V)(z))^2\right\rangle.$$

And finally :

$$W_\varepsilon^{(3)}(V)(z) = \left\langle dG_\varepsilon^V(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(z) + (\mathcal{D}_\varepsilon(\partial_z V)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(z)\right)\right\rangle. \tag{4.8}$$

In order to obtain estimates on $W^{(i)}$ it seems natural from the previous pattern of differentiation to begin with estimates on the symmetric difference of the derivatives of V.

Lemma 4.5. *For any $V \in \mathcal{E}^\alpha$, and $(y_1, y_2) \in \mathbb{R}^2$, we have:*

$$\sup_z (1+|z|)^\alpha \left|\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,z)\right| \leqslant \varepsilon 2^\alpha \|V\|_\alpha \left[|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right], \tag{4.9}$$

$$\sup_z (1+|z|)^\alpha \left|\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(y_1,y_2,z)\right| \leqslant \varepsilon 2^{\alpha-1} \|V\|_\alpha \left[|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right], \tag{4.10}$$

$$\sup_z (1+|z|)^\alpha \left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1,y_2,z)\right| \leqslant 2^{\alpha-1} \|V\|_\alpha \left(1+\frac{\varepsilon^\alpha}{4}\left[|y_1|^\alpha+|y_2|^\alpha\right]\right). \tag{4.11}$$

It is important to notice that the first two right-hand-sides (resp. first and second derivatives) are of order ε. The third one is larger but controlled by $2^{\alpha-1} < 1$. This is the first occurrence of the contraction property we are seeking. This is the main reason why we make the analysis up to the third derivatives.

Proof. We introduce the additional notation $\varphi_\alpha(z) = (1+|z|)^\alpha$. First, since $\overline{z} = z/2$, we have $\varphi_\alpha(z) \leqslant 2^\alpha \varphi_\alpha(\overline{z})$.

▹ **Proof of (4.9).** By Taylor expansions, we have:

$$\varphi_\alpha(z)\left|\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,z)\right| \leqslant 2^\alpha \varphi_\alpha(\overline{z}) \left|\frac{\varepsilon y_1}{2}\partial_z^2 V(\overline{z}+\varepsilon\widetilde{y_1}) + \frac{\varepsilon y_2}{2}\partial_z^2 V(\overline{z}+\varepsilon\widetilde{y_2})\right|,$$

where $|\widetilde{y_i}| \leqslant |y_i|$. Using the definition of $\|\cdot\|_\alpha$ (1.4), we obtain

$$\varphi_\alpha(\overline{z})\left|\varepsilon y_1 \partial_z^2 V(\overline{z}+\varepsilon\widetilde{y_1})\right| \leqslant \frac{\varepsilon|y_1|\varphi_\alpha(\overline{z})}{\varphi_\alpha(\overline{z}+\varepsilon\widetilde{y_1})}\|V\|_\alpha \leqslant \frac{\varepsilon|y_1|(1+|\varepsilon\widetilde{y_1}|+|\overline{z}+\varepsilon\widetilde{y_1}|)^\alpha}{(1+|\overline{z}+\varepsilon\widetilde{y_1}|)^\alpha}\|V\|_\alpha$$

Since we chose $\alpha < 1$, $|\cdot|^\alpha$ is sub-additive. Thus, we get

$$\begin{aligned}\varphi_\alpha(\overline{z})\left|\varepsilon y_1 \partial_z^2 V(\overline{z}+\varepsilon\widetilde{y_1})\right| &\leqslant \varepsilon|y_1|\left(1+\frac{|\varepsilon\widetilde{y_1}|^\alpha}{(1+|\overline{z}+\varepsilon\widetilde{y_1}|)^\alpha}\right)\|V\|_\alpha \\ &\leqslant \varepsilon|y_1|(1+|\varepsilon y_1|^\alpha)\|V\|_\alpha \leqslant \varepsilon(|y_1|+|y_1|^{1+\alpha}).\end{aligned}$$

By symmetry of the role played by y_1 and y_2, we have proven equation (4.9).

▹ **Proof of (4.10).** The second estimate is a consequence of the first one, applied to the derivative of V. Notice that it is allowed as $\mathcal{E}_0^\alpha$ enables control of derivatives up to the third order.

▹ **Proof of (4.11).** We must be a little more careful in the estimations of the third estimate (4.11), because we cannot go up to the fourth derivative in the Taylor expansions. This is why we do not have an ε bound, but we gain a contraction factor instead. We have

$$\begin{aligned}\varphi_\alpha(z)\left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1,y_2,z)\right| &\leqslant 2^\alpha\varphi_\alpha(\overline{z})\left|\frac{1}{4}\partial_z^3 V(\overline{z}) - \frac{1}{8}\partial_z^3 V(\overline{z}+\varepsilon y_1) - \frac{1}{8}\partial_z^3 V(\overline{z}+\varepsilon y_2)\right| \\ &\leqslant \frac{2^\alpha}{4}\|V\|_\alpha + 2^\alpha\varphi_\alpha(\overline{z})\left|\frac{1}{8}\partial_z^3 V(\overline{z}+\varepsilon y_1) + \frac{1}{8}\partial_z^3 V(\overline{z}+\varepsilon y_2)\right|\end{aligned}$$

We bound separately each term using again the sub-additivity of $|\cdot|^\alpha$. For $\varepsilon \leqslant 1$:

$$\begin{aligned}\varphi_\alpha(\overline{z})\left|\frac{1}{8}\partial_z^3 V(\overline{z}+\varepsilon y_1)\right| &\leqslant \frac{\varphi_\alpha(\overline{z})}{8\varphi_\alpha(\overline{z}+\varepsilon y_1)}\|V\|_\alpha \\ &\leqslant \frac{\|V\|_\alpha}{8}\left(1+\frac{(|\varepsilon y_1|)^\alpha}{(1+|\overline{z}+\varepsilon y_1|)^\alpha}\right) \leqslant (1+|\varepsilon y_1|^\alpha)\frac{\|V\|_\alpha}{8}.\end{aligned}$$

Summing it all up, one ends up with:

$$\varphi_\alpha(z)\left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1,y_2,z)\right| \leqslant 2^{\alpha-1}\left\|V\right\|_\alpha\left(1+\frac{1}{4}\varepsilon^\alpha\left[|y_1|^\alpha+|y_2|^\alpha\right]\right).$$

This is precisely equation (4.11). □

The following proposition is a first step towards contraction properties that will be established in section 5. For convenience, we introduce the following notation:

$$\begin{cases}\triangle W_\varepsilon^{(i)} = W_\varepsilon^{(i)}(V_1) - W_\varepsilon^{(i)}(V_2)\\ \triangle V = V_1 - V_2\end{cases} \tag{4.12}$$

Proposition 4.6 (Lipschitz continuity of W_ε with respect to V)**.**
Let K a ball of $\mathcal{E}_0^\alpha$, and $V_1, V_2 \in K$. There exists constants ε_K, C_K depending only on K such that for all $\varepsilon \leqslant \varepsilon_K$, we have:

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(1)}(z)| \leqslant \varepsilon C_K\|\triangle V\|_\alpha \tag{4.13}$$

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(2)}(z)| \leqslant \varepsilon C_K\|\triangle V\|_\alpha, \tag{4.14}$$

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(3)}(z)| \leqslant \left(2^{\alpha-1}+\varepsilon^\alpha C_K\right)\left\|\triangle V\right\|_\alpha. \tag{4.15}$$

It is also possible to get estimates on $W_\varepsilon^{(i)}(V)$ itself, with the same hypotheses. This is useful to prove the invariance of certain subsets of $\mathcal{E}_0^\alpha$.

Proposition 4.7.
With the same setting as in Proposition 4.6, we also have:

$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(1)}(V)(z)| \leqslant \varepsilon C_K\left\|V\right\|_\alpha,$$
$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(2)}(V)(z)| \leqslant \varepsilon C_K\left\|V\right\|_\alpha,$$
$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(3)}(V)(z)| \leqslant \left(2^{\alpha-1}+\varepsilon^\alpha C_K\right)\left\|V\right\|_\alpha.$$

We do not give the details of the proof of the latter Proposition, since it is a straightforward adaptation of Proposition 4.6. Actually, we cannot readily apply Proposition 4.6 to $(V_1,V_2) = (V,0)$ as $0 \notin \mathcal{E}_0^\alpha$, because of the additional condition on $\partial_z^2 V(0)$ (1.6) which is required to prove boundedness and Lipschitz continuity of γ_ε.

Proof of Proposition 4.6. The proof of theses inequalities is quite tedious because of the numerous non-linear calculations. However, the technique is similar for each inequality, and consists in separating the fully non linear behavior from the quasi-linear parts of the left-hand-sides of equations (4.13) to (4.15).

▷ **Proof of** (4.13)**.** This is the easiest part, because it is quasi-linear with respect to V. Indeed, we have

$$\triangle W_\varepsilon^{(1)}(z) = \left\langle dG_\varepsilon^{V_1}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z)\right\rangle - \left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_2)(z)\right\rangle.$$

We reformulate it in two parts, one involving $V_1 - V_2$, and the other involving $dG_\varepsilon^{V_1} - dG_\varepsilon^{V_2}$:

$$\triangle W_\varepsilon^{(1)}(z) = \left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z \triangle V)(z)\right\rangle + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z)\right\rangle. \tag{4.16}$$

For the first contribution in (4.16), we apply directly Lemma 4.5 to $V_1 - V_2$:

$$\begin{aligned}(1+|z|)^\alpha\left|\left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z\triangle V)(z)\right\rangle\right| &\leqslant \varepsilon 2^\alpha\left\|\triangle V\right\|_\alpha\left\langle dG_\varepsilon^{V_2}(z), \left(|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right)\right\rangle\\ &\leqslant \varepsilon C_K\left\|\triangle V\right\|_\alpha.\end{aligned}$$

For the last inequality we used equation (4.5), which enables to bound uniformly the measure dG_ε^V with respect to z. From Lemmas 4.4 and 4.5, there exists ε_K and C_K such that for $\varepsilon \leqslant \varepsilon_K$, the second contribution in the right-hand-side (4.16) satisfies

$$\begin{aligned}(4.17)\quad & (1+|z|)^\alpha \left|\left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z)\right\rangle\right| \\ & \leqslant \varepsilon^2 C_K \left\|\triangle V\right\|_\alpha \left\|V_1\right\|_\alpha \Big\langle (1+|y_1|+|y_2|)\exp(-Q(y_1,y_2)+2\varepsilon R_K(|y_1|+|y_2|)), \\ & \qquad \left(|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right)\Big\rangle.\end{aligned}$$

The last integral is uniformly bounded for ε small enough, involving moments of a Gaussian distribution. Therefore, the whole quantity is bounded by $\varepsilon^2 C_K \left\|\triangle V\right\|_\alpha$, uniformly with respect to z. This concludes the proof of equation (4.13).

▷ **Proof of (4.14).** To begin with, we have

$$\begin{aligned}\triangle W_\varepsilon^{(2)}(z) = & \left\langle dG_\varepsilon^{V_1}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2\right\rangle \\ & - \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z) + (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^2\right\rangle.\end{aligned}$$

We split the difference into two, as in the previous part,

$$\begin{aligned}\triangle W_\varepsilon^{(2)}(z) = & \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2 - \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z) - (\mathcal{D}_\varepsilon(\partial_z V_2)z))^2\right\rangle \\ & + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2\right\rangle \\ = & A + B\end{aligned}$$

The first contribution can be rearranged as follows, by factorizing the difference of squares:

$$A = \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2\triangle V)(z) + \mathcal{D}_\varepsilon(\partial_z\triangle V)(z)\mathcal{D}_\varepsilon(\partial_z(V_1+V_2))(z)\right\rangle.$$

The term involving V_1+V_2 is bounded uniformly in a crude way: $\|\mathcal{D}_\varepsilon(\partial_z(V_1+V_2))\|_\infty \leqslant 2\|V_1+V_2\|_\alpha$ (in fact it is bounded by a $\mathcal{O}(\varepsilon)$ uniformly with respect to z, but this detail is omitted here). Then, we apply Lemma 4.5 twice with $V_1 - V_2$ to obtain:

$$(1+|z|)^\alpha |A| \leqslant \varepsilon C_K \|\triangle V\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), \left(|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right)\right\rangle$$

To estimate B, the term involving the difference of measures dG_ε^V, we apply (4.4) and Lemma 4.5:

$$(4.18)\quad (1+|z|)^\alpha |B| \leqslant \left\langle \left|dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z)\right|, \varepsilon C\left(\|V_1\|_\alpha^2 + \|V_2\|_\alpha\right)\left(|y_1|+|y_2|+\varepsilon^\alpha|y_1|^{1+\alpha}+\varepsilon^\alpha|y_2|^{1+\alpha}\right)\right\rangle.$$

We find, exactly as above, that the quantity $(1+|z|)^\alpha |B|$ is bounded by $\varepsilon^2 C_K \left\|\triangle V\right\|_\alpha$. Combining both estimates on A, B, we deduce equation (4.14).

▷ **Proof of (4.15).** The full expression for $\triangle W_\varepsilon^{(3)}$ is as follows:

$$\begin{aligned}\triangle W_\varepsilon^{(3)}(z) = & \left\langle dG_\varepsilon^{V_1}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V_1)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z)\right)\right\rangle \\ & - \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_2)(z) + (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V_2)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right)\right\rangle.\end{aligned}$$

We split again in two pieces, one involving $V_1 - V_2$, and the other involving $dG_\varepsilon^{V_1} - dG_\varepsilon^{V_2}$:

$$\triangle W_\varepsilon^{(3)}(V)(z) = \left\langle dG_\varepsilon^{V_2}(z), A_1 + A_2 + A_3 \right\rangle + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), B \right\rangle ,$$

with

$$\begin{aligned}
A_1 &= \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 \triangle V)(z) \\
A_2 &= \left(\mathcal{D}_\varepsilon(\partial_z V_1)(z)\right)^3 - \left(\mathcal{D}_\varepsilon(\partial_z V_2)(z)\right)^3 \\
&= \left(\mathcal{D}_\varepsilon(\partial_z \triangle V)(z)\right) \left[\left(\mathcal{D}_\varepsilon(\partial_z V_1)(z)\right)^2 + \left(\mathcal{D}_\varepsilon(\partial_z V_2)(z)\right)^2 + \left(\mathcal{D}_\varepsilon(\partial_z V_1)(z)\right)\left(\mathcal{D}_\varepsilon(\partial_z V_2)(z)\right) \right] \\
A_3 &= 3\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) \left(\frac{1}{2}\right) \mathcal{D}_\varepsilon(\partial_z V_1)(z) - 3\mathcal{D}_\varepsilon(\partial_z V_2)(z) \left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right) \\
&= 3\mathcal{D}_\varepsilon(\partial_z V_1)(z) \left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 \triangle V)(z)\right) + 3\mathcal{D}_\varepsilon(\partial_z \triangle V)(z) \left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right) \\
B &= \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_1)(z) + \left(\mathcal{D}_\varepsilon(\partial_z V_1)(z)\right)^3 + 3\mathcal{D}_\varepsilon(\partial_z V_1)(z) \left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z)\right).
\end{aligned}$$

We shall estimate all the contributions separately. Firstly, A_1 yields the contraction factor:

$$(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_1| \right\rangle \leqslant 2^{\alpha-1} \left\| \triangle V \right\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), 1 + \frac{\varepsilon^\alpha}{4}\left[|y_1|^\alpha + |y_2|^\alpha \right] \right\rangle \leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \left\| \triangle V \right\|_\alpha .$$

The latter is the main contribution in (4.15). The remaining terms are lower-order contributions with respect to ε. For A_2, we have

$$\begin{aligned}
(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_2| \right\rangle &\leqslant \varepsilon 2^\alpha \left\| \triangle V \right\|_\alpha \Big\langle dG_\varepsilon^{V_2}(z), \left(\|V_1\|_\alpha^2 + \|V_2\|_\alpha^2 + \|V_1\|_\alpha \|V_2\|_\alpha \right) \\
&\qquad \times \left[|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right] \Big\rangle \\
&\leqslant \varepsilon C_K \left\| \triangle V \right\|_\alpha .
\end{aligned}$$

For A_3, we have similarly

$$(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_3| \right\rangle \leqslant \varepsilon C_K \left\| \triangle V \right\|_\alpha .$$

It remains to control the term involving B. We argue as in (4.17) and (4.18):

$$\begin{aligned}
&\left\langle \left| dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z) \right|, (1+|z|)^\alpha |B| \right\rangle \\
&\qquad \leqslant \varepsilon C_K \left\| \triangle V \right\|_\alpha \Big\langle (1 + |y_1| + |y_2|) \exp(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1| + |y_2|)), \\
&2^{\alpha-1} \|V_1\|_\alpha \left(1 + \frac{\varepsilon^\alpha}{4}\left[|y_1|^\alpha + |y_2|^\alpha \right] \right) + C\varepsilon \left(\|V_1\|_\alpha^3 + \|V_1\|_\alpha^2 \right) \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right) \Big\rangle.
\end{aligned}$$

The latter is controlled by $\varepsilon C_K \left\| \triangle V \right\|_\alpha$ for the same reasons as usual.

Combining all the pieces together, we obtain finally (4.15). ∎

5. Analysis of the fixed point mapping $\mathcal{H}_\varepsilon$

In this section we focus on the fixed point mapping $\mathcal{H}_\varepsilon$ (2.14), which is defined through an infinite series. We are first concerned with the convergence of the series for $V \in \mathcal{E}_0^\alpha$.

5.1. **Well-posedness of $\mathcal{H}_\varepsilon$ on balls.** Consider the following decomposition of each term of the series (2.14) in two parts, with the corresponding notations:

$$\Gamma_\varepsilon(z) = \log\left(\frac{\mathcal{I}_\varepsilon(V)(0) + m(z)}{\mathcal{I}_\varepsilon(V)(0)}\right) - \log\left(\frac{\mathcal{I}_\varepsilon(V)(z)}{\mathcal{I}_\varepsilon(V)(0)}\right) = \Gamma_\varepsilon^m(z) - \Gamma_\varepsilon^I(z).$$

They have the following properties :

Lemma 5.1.
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have $\Gamma_\varepsilon^m \in \mathcal{E}_0^\alpha$. Moreover, we have $(1+|z|)^\alpha \partial_z \Gamma_\varepsilon^m \in L^\infty$.

The proof of Lemma 5.1 is a straightforward consequence of Proposition 4.1 and the assumptions on m made in definition 1.2, particularly (1.5).

Lemma 5.2.
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have $\Gamma_\varepsilon^I \in \mathcal{E}^\alpha$, and $\partial_z \Gamma_\varepsilon^I(0) = 0$. Moreover, we have $(1+|z|)^\alpha \partial_z \Gamma_\varepsilon^I \in L^\infty$.

Proof. We begin by verifying the condition $\partial_z \Gamma_\varepsilon^I(0) = 0$. This is in fact equivalent to the choice of $\gamma_\varepsilon(V)$, as can be seen on the following computation:

$$\partial_z \Gamma_\varepsilon^I(0) = \frac{\partial_z \mathcal{I}_\varepsilon(V)(0)}{\mathcal{I}_\varepsilon(V)(0)} = W_\varepsilon^{(1)}(V)(0).$$

Now, comparing (3.1) with (4.7), we see that $\partial_z \Gamma_\varepsilon^I(0) = 0$ is equivalent to $J(\gamma_\varepsilon(V), V) = 0$, provided ε is small enough (for the quantities to be well defined).

Secondly, we need to get uniform bounds on the derivatives of Γ_ε^I to prove that it belongs to $\mathcal{E}^\alpha$. The following formulas relate the successive logarithmic derivatives of $\mathcal{I}_\varepsilon(V)$ to the $W_\varepsilon^{(i)}(V)$ introduced in equation (4.6):

$$\partial_z \Gamma_\varepsilon^I(z) = W_\varepsilon^{(1)}(V)(z) \tag{5.1}$$

$$\partial_z^2 \Gamma_\varepsilon^I(z) = W_\varepsilon^{(2)}(V)(z) - \left[W_\varepsilon^{(1)}(V)(z)\right]^2, \tag{5.2}$$

$$\partial_z^3 \Gamma_\varepsilon^I(z) = W_\varepsilon^{(3)}(V)(z) + 3W_\varepsilon^{(1)}(V)(z)W_\varepsilon^{(2)}(V)(z) + 2\left[W_\varepsilon^{(1)}(V)(z)\right]^3. \tag{5.3}$$

We can use directly the weighted estimates in Proposition 4.7, which include the algebraic decay of the first order derivative. Algebraic combinations are compatible with those estimates because $W_\varepsilon^{(i)}(V) \in L^\infty(\mathbb{R})$. A fortiori those terms are all uniformly bounded and so we obtain that $\Gamma_\varepsilon^I \in \mathcal{E}^\alpha$. □

The main result of this section is the following one:

Proposition 5.3 (Convergence of the series $\mathcal{H}_\varepsilon(V)$)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, the sum $\mathcal{H}_\varepsilon(V)$ is finite.

Before proving this statement, we first establish an auxiliary technical lemma about the following summation operator S:

$$\mathcal{S} : \Lambda \longmapsto \left(h \mapsto \sum_{k \geqslant 0} 2^k \Lambda(2^{-k}h)\right).$$

Lemma 5.4 (Existence of the sum)**.**
Take any function $\Lambda \in \mathcal{E}^\alpha$ such that $\partial_z \Lambda(0) = 0$. Then $\mathcal{S}(\Lambda)(h)$ is well-defined for every $h \in \mathbb{R}$.

Proof. We perform a Taylor expansion: there exists $\widetilde{h}_k$, such that $\Lambda(2^{-k}h) = \frac{1}{2}(2^{-k}h)^2\partial_z^2\Lambda(2^{-k}\widetilde{h_k})$. Therefore, we have immediately

$$\left|\sum_{k\geqslant 0} 2^k\Lambda(2^{-k}h)\right| \leqslant \left(h^2\sum_{k\geqslant 0}2^{-k}\right)\left\|\partial_z^2\Lambda\right\|_\infty < \infty.$$

□

One can now proceed to the proof of the finiteness of the sum of $\mathcal{H}_\varepsilon$ in definition 2.6.

Proof of Proposition 5.3. Let K be the ball of $\mathcal{E}_0^\alpha$ of radius $\|K\|_\alpha$ and take $V \in K$, $z \in \mathbb{R}$. To use the previous lemma, we first notice the identity by definition:

$$\mathcal{H}_\varepsilon(V) = \mathcal{S}(\Gamma_\varepsilon). \tag{5.4}$$

There are two conditions to verify in order to apply Lemma 5.4:

$$\partial_z\Gamma_\varepsilon(0) = 0, \text{ and } \Gamma_\varepsilon \in \mathcal{E}^\alpha.$$

Those properties are verified thanks to Lemmas 5.1 and 5.2. The Proposition 5.3 immediately follows. □

So far, we have not used the algebraic decay condition which is part of the definition of $\mathcal{E}^\alpha$. In the following lemma, we refine the estimate on $\mathcal{S}(\Lambda) \in \mathcal{E}^\alpha$. This foreshadows the same result for the function $\mathcal{H}_\varepsilon(V)$, as stated in the next section.

Lemma 5.5 (Better control of the series)**.**
Assume that $\Lambda \in \mathcal{E}^\alpha$, that $\partial_z\Lambda(0) = 0$, and that $(1+|z|)^\alpha\partial_z\Lambda \in L^\infty$. Then, $\mathcal{S}(\Lambda)$ belongs to $\mathcal{E}^\alpha$, with a uniform estimate:

$$\|\mathcal{S}(\Lambda)\|_\alpha \leqslant C\max\left(\|\Lambda\|_\alpha, \sup_{z\in\mathbb{R}}\,(1+|z|)^\alpha|\partial_z\Lambda(z)|\right) \tag{5.5}$$

There is some subtlety hidden here. In fact, we were not able to propagate the algebraic decay at first order from Λ to $\mathcal{S}(\Lambda)$. What saves the day is that we gain some algebraic decay of the first order derivatives somewhere in our procedure (see *e.g.* Proposition 4.7).

Proof. Recall the notation $\varphi_\alpha(h) = (1+|h|)^\alpha$. We begin with the uniform bound on the first derivative, which is the main reason why we have to impose algebraic decay in our functional spaces.

▷ **Step 1: $\partial_z\mathcal{S}(\Lambda)$ is uniformly bounded.** We split the sum in two parts. Let $h \in \mathbb{R}$, and let $N_h \in \mathbb{N}$ be the lowest integer such that $|h| \leqslant 2^{N_h}$. We consider the two regimes: $k > N_h$ and $k \leqslant N_h$. In the former regime, a simple Taylor expansion yields

$$\left|\sum_{k>N_h}\partial_z\Lambda(2^{-k}h)\right| \leqslant \sum_{k>N_h}2^{-k}|h|\left\|\partial_z^2\Lambda\right\|_\infty \leqslant \left\|\partial_z^2\Lambda\right\|_\infty, \tag{5.6}$$

by definition of N_h. In the regime $k \leqslant N_h$, we use the algebraic decay which is encoded in the space $\mathcal{E}^\alpha$. If $|h| > 1$, we have $N_h \geqslant 1$, and

$$\left|\sum_{k\leqslant N_h}\partial_z\Lambda(2^{-k}h)\right| \leqslant \sum_{k\leqslant N_h}\frac{\|\varphi_\alpha\partial_z\Lambda\|_\infty}{(1+2^{-k}|h|)^\alpha}$$
$$\leqslant \left(\sum_{k\leqslant N_h}\frac{2^{k\alpha}}{|h|^\alpha}\right)\|\varphi_\alpha\partial_z\Lambda\|_\infty = \left(\frac{1}{|h|^\alpha}\frac{2^{(N_h+1)\alpha}-1}{2^\alpha-1}\right)\|\varphi_\alpha\partial_z\Lambda\|_\infty.$$

By definition of N_h, we have $2^{N_h-1} < |h|$, so that the right-hand-side above is bounded by a constant that get arbitrarily large as $\alpha \to 0$ (hence, the restriction on $\alpha > 0$):

$$\left| \sum_{k \leqslant N_h} \partial_z \Lambda(2^{-k} h) \right| \leqslant \left(\frac{4^\alpha}{2^\alpha - 1} \right) \|\varphi_\alpha \partial_z \Lambda\|_\infty. \tag{5.7}$$

The case $|h| \leqslant 1$ is trivial as the sum is reduced to a single term $\partial_z \Lambda(h)$ since $N_h = 0$.

▷ **Step 2:** $\varphi_\alpha \left|\partial_z^2 \mathcal{S}(\Lambda)\right|$ **is uniformly bounded.** This bound and the next one are easier. For any $h \in \mathbb{R}$, we have

$$\varphi_\alpha(h) \left| \sum_{k \geqslant 0} 2^{-k} \partial_z^2 \Lambda(2^{-k} h) \right| \leqslant \left(\sum_{k \geqslant 0} 2^{-k} \frac{\varphi_\alpha(h)}{\varphi_\alpha(2^{-k} h)} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty.$$

Since $1 \geqslant 2^{-k}$, one obtains

$$\varphi_\alpha(h) \left| \sum_{k \geqslant 0} 2^{-k} \partial_z^2 \Lambda(2^{-k} h) \right| \leqslant \left(\sum_{k \geqslant 0} 2^{k(\alpha - 1)} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty = \left(\frac{2}{2 - 2^\alpha} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty.$$

The latter sum is finite since $\alpha < 1$.

▷ **Step 3:** $\varphi_\alpha \left|\partial_z^3 \mathcal{S}(\Lambda)\right|$ **is uniformly bounded.** The proof is similar to the previous argument. □

5.2. **Contraction properties (second part).** In this section we prove that $\mathcal{H}_\varepsilon$ stabilizes some subset of $\mathcal{E}_0^\alpha$. We first show that $\mathcal{H}_\varepsilon$ maps balls into balls with incremental radius that do not depends on the initial ball (Proposition 5.8). This property immediately implies the existence of an invariant subset for $\mathcal{H}_\varepsilon$ (corollary 5.9). Finally, we prove that the mapping $\mathcal{H}_\varepsilon$ is a contraction mapping for ε small enough (Theorem 5.10). To completely justify the definition of $\mathcal{H}_\varepsilon$, it remains to show that $\mathcal{H}_\varepsilon(V) \in \mathcal{E}_0^\alpha$. We begin with the lower bound on the second derivative, which is for free.

Lemma 5.6 (Lower bound on $\partial_z^2 \mathcal{H}_\varepsilon(V)(0)$)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have:

$$\partial_z \mathcal{H}_\varepsilon(V)(0) = 0, \quad \partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \partial_z^2 m(0).$$

Proof. The identity $\partial_z \mathcal{H}_\varepsilon(V)(0) = 0$, and more particularly $\partial_z \Gamma_\varepsilon^I(0) = 0$ is a consequence of the choice of $\gamma_e(V)$ in Proposition 2.4. Indeed, we have, by (5.4),

$$\partial_z \mathcal{H}_\varepsilon(V)(0) = \sum_{k \geqslant 0} \partial_z \Gamma_\varepsilon(V)(0) = 0.$$

For the second estimate, a simple computation yields, using $m(0) = \partial_z m(0) = 0$:

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) = \sum_{k \geqslant 0} 2^{-k} \left[\frac{\partial_z^2 m(0)}{\mathcal{I}_\varepsilon(V)(0)} - W_\varepsilon^{(2)}(V)(0) - W_\varepsilon^{(1)}(V)(0)^2 \right]$$

But since $V \in \mathcal{E}_0^\alpha$, one can use again the uniform estimates of Proposition 4.7 to write that for $\varepsilon \leqslant \varepsilon_K$, that depends only on the ball K :

$$\partial_z^2 \mathcal{H}_\varepsilon(V) = 2 \frac{\partial_z^2 m(0)}{\mathcal{I}_\varepsilon(V)(0)} + \mathcal{O}(\varepsilon), \tag{5.8}$$

where $\mathcal{O}(\varepsilon)$ that depends only on the ball K. Then, we use Proposition 4.1 with $\delta = 1/3$ to deduce that for ε small enough, we have $\mathcal{I}_\varepsilon(V) \leqslant 4/3$. Then (5.8) can be simplified into

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \frac{3\partial_z^2 m(0)}{2} + \mathcal{O}(\varepsilon).$$

Recall that $\partial_z^2 m(0) > 0$ by assumption. Therefore, for ε small enough, we get as claimed

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \partial_z^2 m(0).$$

□

Remark 5.7. *Considering the proof, another way to interpret the result is that automatically for any function $V \in \mathcal{E}^\alpha$ such that $\partial_z V(0) = 0$, the function $\mathcal{H}_\varepsilon$ prescribes a lower bound on $\partial_z^2 V(0)$. Since we are seeking a fixed point $\mathcal{H}_\varepsilon(V) = V$, we may as well put this condition in the subspace $\mathcal{E}_0^\alpha$ without loss of generality.*

Finally, we can establish a first useful estimate on $\|\mathcal{H}_\varepsilon(V)\|_\alpha$, showing more than just its finiteness:

Proposition 5.8 (Contraction in the large)**.**
For every ball $K \in \mathcal{E}_0^\alpha$, there exists an explicit constant $\kappa(\alpha) < 1$, as well as C_m, C_K and ε_K that depend only on K such that, for all $\varepsilon \leqslant \varepsilon_K$, and for every $V \in K$,

$$\|\mathcal{H}_\varepsilon(V)\|_\alpha \leqslant C_m + (\kappa(\alpha) + \varepsilon^\alpha C_K)\,\|V\|_\alpha . \tag{5.9}$$

Proof. Let K be the ball of $\mathcal{E}_0^\alpha$, and take $V \in K$. For clarity we write respectively $\mathcal{I}_\varepsilon(h)$ and $W_\varepsilon^{(i)}(h)$ instead of $\mathcal{I}_\varepsilon(V)(h)$ and $W_\varepsilon^{(i)}(V)(h)$. Combining various estimates derived in Section 5.1, and particularly Lemma 5.5 together with Lemmas 5.1 and 5.2, we find that $\mathcal{H}_\varepsilon(V) = \mathcal{S}(\Gamma_\varepsilon) = \mathcal{S}(\Gamma_\varepsilon^m) - \mathcal{S}(\Gamma_\varepsilon^I)$ belongs to $\mathcal{E}^\alpha$. However, the associated estimate (5.5) is not satisfactory, at least for the $\mathcal{S}(\Gamma_\varepsilon^I)$ and we need to re-examine the dependency of the constants upon ε and α.

The first and second derivatives of Γ_ε^I involve $W_\varepsilon^{(1)}$ and $W_\varepsilon^{(2)}$ which are both of order $\varepsilon C_K \|V\|_\alpha$ thanks to Proposition 4.7. Back to the proof of Lemma 5.5, the quantities $\|\varphi_\alpha \partial_z \Gamma_\varepsilon^I\|_\infty$ and $\|\varphi_\alpha \partial_z^2 \Gamma_\varepsilon^I\|_\infty$ are in fact of order $\varepsilon \|\Lambda\|_\alpha$, and so are $\|\partial_z \mathcal{S}(\Gamma_\varepsilon^I)\|_\infty$ and $\|\varphi_\alpha \partial_z^2 \mathcal{S}(\Gamma_\varepsilon^I)\|_\infty$.

This cannot be extended readily to the third derivative as we lose the order ε at this stage. However, Proposition 4.7 provides an explicit constant that is going to be used. From (5.3), we have:

$$\partial_z^3 \mathcal{S}\left(\Gamma_\varepsilon^I\right)(h) = \sum_{k \geqslant 0} 4^{-k} \left[W_\varepsilon^{(3)}(2^{-k}h) + 3W_\varepsilon^{(2)}(2^{-k}h) W_\varepsilon^{(1)}(2^{-k}h) + 2W_\varepsilon^{(1)}(2^{-k}h)^3 \right].$$

The contributions involving $W_\varepsilon^{(1)}$ and $W_\varepsilon^{(2)}$ are of order ε, and can be handled exactly as above. However, the linear term involving $W_\varepsilon^{(3)}$ requires a careful attention. We obtain from Proposition 4.7 that $\varphi_\alpha W_\varepsilon^{(3)}$ is bounded uniformly by $\left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \|V\|_\alpha$. Therefore,

$$\begin{aligned} \varphi_\alpha(h) \left| \sum_{k \geqslant 0} 4^{-k} W_\varepsilon^{(3)}(2^{-k}h) \right| &\leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \left(\sum_{k \geqslant 0} 4^{-k} \frac{\varphi_\alpha(h)}{\varphi_\alpha(2^{-k}h)} \right) \|V\|_\alpha \\ &\leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \left(\sum_{k \geqslant 0} 2^{k(\alpha-2)} \right) \|V\|_\alpha = \left(\frac{2^{\alpha+1}}{4 - 2^\alpha} + \varepsilon^\alpha C_K \right) \|V\|_\alpha . \end{aligned} \tag{5.10}$$

In view of the latter estimate, we define the explicit constant $\kappa(\alpha)$ as

$$\kappa(\alpha) = \frac{2^{1+\alpha}}{4 - 2^\alpha}. \tag{5.11}$$

A simple calculation shows that $\kappa(\alpha) < 1$ if and only if $\alpha < 2 - \log_2(3) \approx 0.415$. The choice $\alpha < 2/5$ gives some room below this threshold. We conclude that $\left\|\mathcal{S}(\Gamma_\varepsilon^I)\right\|_\alpha \leqslant (\kappa(\alpha) + \varepsilon^\alpha C_K)\left\|V\right\|_\alpha$.

The other contribution to $\mathcal{H}_\varepsilon(V)$, namely $\mathcal{S}(\Gamma_\varepsilon^m)$ can be bounded in an easier way. Indeed, we have

(5.12)
$$\Gamma_\varepsilon^m = \log(1+m) + \log\left(1 + \frac{m}{\mathcal{I}_\varepsilon(0)}\right) - \log(1+m) = \log(1+m) + \log\left(1 + \frac{m}{1+m}\left(\frac{1}{\mathcal{I}_\varepsilon(0)} - 1\right)\right).$$

We define accordingly

$$C_m = \max_{k=1,2,3}\left(\left\|\varphi_\alpha \frac{\partial_z^k m}{1+m}\right\|_\infty\right), \tag{5.13}$$

Moreover, Proposition 4.1 can be easily refined into $|\mathcal{I}_\varepsilon(0) - 1| \leqslant \varepsilon C_m \|V\|_\alpha$, using the definition of R_K in (2.11). Straightforward computations show that the last contribution in (5.12) can be estimated by $\varepsilon C_m \left\|V\right\|_\alpha$.

Combining the estimates obtained for $\mathcal{S}(\Gamma_\varepsilon^m)$ and $\mathcal{S}(\Gamma_\varepsilon^I)$, we come to the conclusion:

$$\left\|\mathcal{H}_\varepsilon(V)\right\|_\alpha \leqslant C_m + (\kappa(\alpha) + \varepsilon^\alpha C_K)\left\|V\right\|_\alpha,$$

□

Proposition 5.8 calls an immediate corollary.

Corollary 5.9 (Invariant subset)**.** *There exist K_0 a ball of $\mathcal{E}_0^\alpha$, and ε_0 a positive constant such that for all $\varepsilon \leqslant \varepsilon_0$ the set K_0 is invariant by $\mathcal{H}_\varepsilon$:*

$$\mathcal{H}_\varepsilon(K_0) \subset K_0.$$

Proof. Let K_0 be the ball of radius $R_0 = 2C_m/(1-\kappa(\alpha))$. We deduce from Proposition 5.8 that, for all $V \in K_0$,

$$\begin{aligned}\left\|\mathcal{H}_\varepsilon(V)\right\|_\alpha \leqslant C_m + (\kappa(\alpha) + \varepsilon^\alpha C_{K_0})R_0 &= C_m\left(1 + \frac{2\kappa(\alpha)}{1-\kappa(\alpha)}\right) + \varepsilon^\alpha C_{K_0} R_0 \\ &= C_m\left(\frac{2}{1-\kappa(\alpha)} - 1\right) + \varepsilon^\alpha C_{K_0} R_0 \\ &= R_0 + C_m\left(-1 + \frac{2\varepsilon^\alpha C_{K_0}}{1-\kappa(\alpha)}\right).\end{aligned}$$

Therefore, the choice $\varepsilon_0 = \left(\dfrac{1-\kappa(\alpha)}{2C_{K_0}}\right)^{\frac{1}{\alpha}}$ guarantees that K_0 is left invariant by $\mathcal{H}_\varepsilon$. □

We are now in position to state the more important result of this section:

Theorem 5.10 (Contraction mapping)**.** *There exists a constant C_{K_0} such that for any $\varepsilon \leqslant \varepsilon_0$, and every function V_1, $V_2 \in K_0$, the following estimate holds true*

$$\left\|\mathcal{H}_\varepsilon(V_1) - \mathcal{H}_\varepsilon(V_2)\right\|_\alpha \leqslant (\kappa(\alpha) + \varepsilon^\alpha C_K)\left\|V_1 - V_2\right\|_\alpha. \tag{5.14}$$

Proof. We denote by $\triangle V$ the difference $V_1 - V_2$, again. The proof is analogous to Proposition 5.8. For clarity we write respectively $\mathcal{I}_\varepsilon^i(h)$ instead of $\mathcal{I}_\varepsilon(V_i)(h)$ and $\triangle W_\varepsilon^{(i)}(h)$ instead of $W_\varepsilon^{(i)}(V_1)(h) - W_\varepsilon^{(i)}(V_2)(h)$. We decompose $\triangle\mathcal{H}_\varepsilon(V)$ as above:

$$\triangle\mathcal{H}_\varepsilon = \triangle\left(\mathcal{S}(\Gamma_\varepsilon^m) - \mathcal{S}(\Gamma_\varepsilon^I)\right) = \triangle\mathcal{H}_\varepsilon^m - \triangle\mathcal{H}_\varepsilon^I. \tag{5.15}$$

We deal with $\triangle\mathcal{H}_\varepsilon^m$ in the following lemma :

Lemma 5.11. *There exists a constant C_0 such that for any $\varepsilon \leqslant \varepsilon_0$, and every function V_1, $V_2 \in K_0$, we have*

$$\left\| \triangle \mathcal{H}^m_\varepsilon \right\|_\alpha \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha .$$

Proof. Recall the following definition:

$$\triangle \Gamma^m_\varepsilon = \log\left(\mathcal{I}^1_\varepsilon(0) + m\right) - \log\left(\mathcal{I}^2_\varepsilon(0) + m\right) - \log\left(\frac{\mathcal{I}^1_\varepsilon(0)}{\mathcal{I}^2_\varepsilon(0)}\right) . \tag{5.16}$$

The first derivative has the following expression,

$$\partial_z \triangle \Gamma^m_\varepsilon = -\frac{\partial_z m}{(\mathcal{I}^1_\varepsilon(0)+m)(\mathcal{I}^2_\varepsilon(0)+m)} \triangle \mathcal{I}_\varepsilon(0) . \tag{5.17}$$

Clearly, $\mathcal{I}^2_\varepsilon(0) + m$ is bounded below, uniformly for ε small enough. Therefore, we can repeat the arguments of Lemma 5.5, with $\Lambda = \log(\mathcal{I}^1_\varepsilon(0) + m)$ in order to get

$$\|\partial_z \mathcal{S}(\triangle \Gamma^m_\varepsilon)\|_\infty \leqslant C_m |\triangle \mathcal{I}_\varepsilon(0)|. \tag{5.18}$$

However, Proposition 4.3 yields that $|\triangle \mathcal{I}_\varepsilon(0)| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha$.

The next order derivatives can be handled similarly. Indeed, the following quantities must be bounded uniformly by $\varepsilon C_0 \left\| \triangle V \right\|_\alpha$:

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \left[\frac{\partial_z^2 m(2^{-k}h)}{\mathcal{I}^1_\varepsilon(0) + m(2^{-k}h)} - \frac{\partial_z^2 m(2^{-k}h)}{\mathcal{I}^2_\varepsilon(0) + m(2^{-k}h)} \right] \right| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha$$

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \left[\frac{\partial_z m(2^{-k}h)^2}{(\mathcal{I}^1_\varepsilon(0) + m(2^{-k}h))^2} - \frac{\partial_z m(2^{-k}h)^2}{(\mathcal{I}^2_\varepsilon(0) + m(2^{-k}h))^2} \right] \right| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha$$

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z^3 m(2^{-k}h)}{\mathcal{I}^1_\varepsilon(0) + m(2^{-k}h)} - \frac{\partial_z^3 m(2^{-k}h)}{\mathcal{I}^2_\varepsilon(0) + m(2^{-k}h)} \right] \right| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha$$

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z m(2^{-k}h)^3}{(\mathcal{I}^1_\varepsilon(0) + m(2^{-k}h))^3} - \frac{\partial_z m(2^{-k}h)^3}{(\mathcal{I}^2_\varepsilon(0) + m(2^{-k}h))^3} \right] \right| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha ,$$

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z^2 m(2^{-k}h)\partial_z m(2^{-k}h)}{(\mathcal{I}^1_\varepsilon(0) + m(2^{-k}h))^2} - \frac{\partial_z^2 m(2^{-k}h)\partial_z m(2^{-k}h)}{(\mathcal{I}^2_\varepsilon(0) + m(2^{-k}h))^2} \right] \right| \leqslant \varepsilon C_0 \left\| \triangle V \right\|_\alpha .$$

The first and the third items are handled similarly as for the first derivative. The three other items are handled analogously. For the sake of concision, we focus on the second line: We have,

$$\begin{aligned} &\frac{\partial_z m(z)^2}{(\mathcal{I}^1_\varepsilon(0) + m(z))^2} - \frac{\partial_z m(z)^2}{(\mathcal{I}^2_\varepsilon(0) + m(z))^2} \\ &\qquad = \left[\frac{\partial_z m(z)}{\mathcal{I}^1_\varepsilon(0) + m(z)} + \frac{\partial_z m(z)}{\mathcal{I}^2_\varepsilon(0) + m(z)} \right] \left[\frac{-\partial_z m(z) \triangle \mathcal{I}_\varepsilon(0)}{(\mathcal{I}^1_\varepsilon(0) + m(z))(\mathcal{I}^2_\varepsilon(0) + m(z))} \right] \end{aligned}$$

The first factor is uniformly bounded by assumption (1.5), for ε small enough. The second factor is the same as above, so we can conclude directly. ☐

It remains to handle $\triangle\mathcal{H}^I_\varepsilon$. We have the following formulas for the two first derivatives (5.1)–(5.3):

$$\partial_z\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}\triangle W^{(1)}_\varepsilon(2^{-k}h),$$
$$\partial_z^2\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}2^{-k}\left[\triangle W^{(2)}_\varepsilon(2^{-k}h) - \triangle\left(W^{(1)}_\varepsilon(2^{-k}h)^2\right)\right]$$

Finally the formula for the third derivative is:

$$\partial_z^3\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}4^{-k}\left[\triangle W^{(3)}_\varepsilon(2^{-k}h) + 3\triangle\left(W^{(2)}_1 W^{(1)}_1(2^{-k}h)\right) + 2\triangle\left(W^{(1)}_\varepsilon(2^{-k}h)^3\right)\right]. \tag{5.19}$$

The combination of Proposition 4.6 and Lemma 5.5 yields

$$\|\partial_z\triangle\mathcal{H}^I_\varepsilon\|_\infty \leqslant \varepsilon C_0\,\|\triangle V\|_\alpha. \tag{5.20}$$

In the same way, we get the bound for the second derivative, using the factorization

$$\triangle\left(W^{(1)}_\varepsilon(z)^2\right) = \left(W^{(1)}_\varepsilon(V_1)(z) + W^{(1)}_\varepsilon(V_2)(z)\right)\triangle\left(W^{(1)}_\varepsilon(z)\right), \tag{5.21}$$

together with the uniform bound in Proposition 4.7.

As in the proof of Proposition 5.8, the third order derivative must be handled with care, as it does not yield a $\mathcal{O}(\varepsilon)$ bound.

Exactly as above, the contribution involving $\triangle W^{(3)}_\varepsilon$ in (5.19) is the one that yields the contraction factor, the remaining part being of order $\mathcal{O}(\varepsilon)\,\|\triangle V\|_\alpha$. Actually, we have precisely:

$$\varphi_\alpha(h)\left|\sum_{k\geqslant 0}4^{-k}\triangle W^{(3)}_\varepsilon(2^{-k}h)\right| \leqslant (\kappa(\alpha) + \varepsilon^\alpha C_0)\,\|\triangle V\|_\alpha$$

as in (5.10). This concludes the proof of the main contraction estimate. □

6. Existence of a (locally) unique U_ε, and convergence as $\varepsilon \to 0$.

6.1. **Solving problem (PU_ε) – Theorem 1.3(i).** First of all, Theorem 5.10 immediately implies Theorem 2.7, that is the existence of a unique fixed point $\mathcal{H}_\varepsilon(V_\varepsilon) = V_\varepsilon$ in the invariant subset K_0, for $\varepsilon \leqslant \varepsilon_0$. Note that ε_0 could possibly be reduced to meet the requirement of the last estimate in (5.14).

However, due to the peculiar role played by the linear part $\gamma_\varepsilon(V_\varepsilon)$, it is convenient to enlarge slightly the set K_0. More precisely, after corollary 5.9 we define K_0' the ball of radius

$$R_0' = R_0 + \sup_{V\in K_0}|\gamma_\varepsilon(V)|. \tag{6.1}$$

It is clear that, up to reducing further ε_0 to ε_0' in order to control the new constant $C_{K_0'}$, the set K_0' is also invariant for $\varepsilon \leqslant \varepsilon_0'$. The same contraction estimate as in Theorem 5.10 holds, obviously. Furthermore, the fixed point on K_0' coincides with the fixed point on the smaller ball K_0, by uniqueness.

Next, we show that finding this fixed point is equivalent to solving problem (PU_ε), as claimed in Proposition 2.8. We prove in fact the two sides of the equivalence.

▷ The easy part consists in saying that, being given V_ε the unique fixed point in K_0, the function $U_\varepsilon = \gamma_\varepsilon(V_\varepsilon)\cdot + V_\varepsilon$ belongs to K_0' by definition of K_0' (6.1), and it solves problem (PU_ε) by construction.

▷ On the other side, suppose that $(\lambda_\varepsilon, U_\varepsilon) \in \mathbb{R} \times K_0'$ is a solution of the problem (PU_ε). As in section 2, evaluating (PU_ε) at $z = 0$ yields the following necessary condition on λ_ε, since $m(0) = 0$:

$$\lambda_\varepsilon = I_\varepsilon(U_\varepsilon)(0).$$[1]

Then, we focus on U_ε. We decompose it as $U_\varepsilon = \gamma_U \cdot + V_U$, with $\gamma_U = \partial_z U_\varepsilon(0)$, and $\partial_z V_U(0) = 0$. Our purpose is threefold: (i) first, we show that $\gamma_U = \gamma_\varepsilon(V_U)$, then (ii) we prove that $V_U \in \mathcal{E}_0^\alpha$, and finally (iii), we prove that $\mathcal{H}_\varepsilon(V_U) = V_U$.

We can reformulate problem (PU_ε) as follows:

$$I_\varepsilon(\gamma_U \cdot + V_U)(0) + m(z) = I_\varepsilon(\gamma_U \cdot + V_U) \exp\left(V_U(z) - 2V_U\left(\overline{z}\right) + V_U(0)\right). \tag{6.2}$$

Since we assume $U_\varepsilon \in \mathcal{E}^\alpha$, we can differentiate the previous equation, and evaluate it at $z = 0$ to get :

$$\partial_z I_\varepsilon(\gamma_U \cdot + V_U)(0) = 0.$$

As in Section 2, a direct computation shows that γ_U and V_U are linked by the following relation:

$$0 = \mathcal{J}_\varepsilon(\gamma_U, V_U), \tag{6.3}$$

In order to invert this relationship, and deduce that $\gamma_U = \gamma_\varepsilon(V_U)$, it is important to prove that $V_\varepsilon \in \mathcal{E}_0^\alpha$, which amounts to showing that $\partial_z^2 V_\varepsilon(0) \geqslant \partial_z^2 m(0)$, the other conditions being clearly verified.

Differentiating the problem (PU_ε) twice, and evaluating at $z = 0$, we get:

$$\begin{aligned}\partial_z^2 m(0) &= \partial_z^2 I_\varepsilon(U_\varepsilon)(0) + I_\varepsilon(U_\varepsilon)(0)\frac{\partial_z^2 U_\varepsilon(0)}{2}.\\ &= I_\varepsilon(U_\varepsilon)(0)\left(\frac{\partial_z^2 I_\varepsilon(U_\varepsilon)(0)}{I_\varepsilon(U_\varepsilon)(0)} + \frac{\partial_z^2 U_\varepsilon(0)}{2}\right).\end{aligned}$$

Then, using straightforward adaptations of Propositions 4.1 and 4.7, where V should be replaced with $V_U \in \mathcal{E}^\alpha$ and $\gamma_\varepsilon(V)$ should be replaced by γ_U, we find that

$$\partial_z^2 m(0) \leqslant \frac{3}{2}\left(\varepsilon C_{K_0'} + \frac{\partial_z^2 U_\varepsilon(0)}{2}\right).$$

for ε sufficiently small. We deduce that the missing condition is in fact a consequence of the formulation (PU_ε):

$$\partial_z^2 U_\varepsilon(0) \geqslant \partial_z^2 m(0).$$

By definition, $\partial_z^2 U_\varepsilon(0) = \partial_z^2 V_\varepsilon(0)$, so we have established that $V_\varepsilon \in \mathcal{E}_0^\alpha$.

Hence, we can legitimately invert (6.3), so as to find $\gamma_U = \gamma_\varepsilon(V_U)$, where the function γ_ε is defined in Proposition 2.4. Since $U_\varepsilon \in K_0'$ by assumption, we have in particular $\|V_U\|_\alpha \leqslant R_0'$. Of course, V_U is the candidate of being the unique fixed point of $\mathcal{H}_\varepsilon$ in K_0' (but also in K_0). The proof of this claim follows the lines of section 2.2, checking that all manipulations are justified.

First, we divide (6.2) by $I_U = I_\varepsilon(\gamma_U \cdot + V_U) = I_\varepsilon(\gamma_\varepsilon(V_U) \cdot + V_U) = \mathcal{I}_\varepsilon(V_U)$. According to Proposition 4.1, this quantity is uniformly close to 1 for ε small, so it does not vanish. Taking the logarithm on both sides, we get for all $z \in \mathbb{R}$:

$$V_U(z) - 2V_U(\overline{z}) + V_U(0) = \log\left(\frac{I_U(0) + m(z)}{I_U(z)}\right).$$

[1] We use the notation $I_\varepsilon(U_\varepsilon)$ introduced in equation (1.3), that should not be confused with $\mathcal{I}_\varepsilon(V_\varepsilon)$. It is the purpose of the present argument to show that the two quantities do coincide.

We differentiate the last equation to end up with the following recursive equation for every $z \in \mathbb{R}$

$$\partial_z V_U(z) - \partial_z V_U(\overline{z}) = \partial_z \log\left(\frac{I_U(0)+m}{I_U(z)}\right)(z).$$

One simply deduces, that for all $z \in \mathbb{R}$, we necessarily have:

$$\partial_z V_U(z) = \partial_z V_U(0) + \sum_{k\geqslant 0} \log\left(\frac{I_U(0)+m(2^{-k}z)}{I_U(2^{-k}z)}\right).$$

Note that the $\mathcal{C}^1$ continuity at $z=0$ is used here. Moreover, $\partial_z V_U(0) = 0$ by definition of V_U. The analysis performed in Proposition 5.3 guarantees that this sum is indeed finite. Finally, integrating back the previous identity yields

$$V_U(z) = \sum_{k\geqslant 0} 2^k \log\left(\frac{I_U(0)+m(2^{-k}z)}{I_U(2^{-k}z)}\right) = \sum_{k\geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V_U)(0)+m(2^{-k}z)}{\mathcal{I}_\varepsilon(V_U)(2^{-k}z)}\right).$$

The last expression is nothing but $\mathcal{H}_\varepsilon(V_U)$, by definition (2.14). Therefore, $V_U = \mathcal{H}_\varepsilon(V_U)$ is the unique fixed point of $\mathcal{H}_\varepsilon$ in K_0'.

6.2. **Convergence of $(\lambda_\varepsilon, U_\varepsilon)$ towards (λ_0, U_0) – Theorem 1.3(ii).** As previously, we decompose $U_\varepsilon = \gamma_\varepsilon \cdot + V_\varepsilon$, where γ_ε stands for $\gamma_\varepsilon(V_\varepsilon)$. Firstly, we have $\lambda_\varepsilon = \mathcal{I}_\varepsilon(V_\varepsilon)(0) \to 1$, using Proposition 4.1. Secondly, using an argument of diagonal extraction, there exists a subsequence ε_n, and a limit function V_0 such that

$$\lim_{\varepsilon\to 0} \partial_z V_\varepsilon = \partial_z V_0\,, \quad \text{in } L^\infty_{\text{loc}}, \tag{6.4}$$

$$\lim_{\varepsilon\to 0} \partial_z^2 V_\varepsilon = \partial_z^2 V_0\,, \quad \text{in } L^\infty_{\text{loc}}. \tag{6.5}$$

We have used the Arzela-Ascoli theorem and the uniform $\mathcal{C}^3$ bound in order to get the convergence up to the second derivative. However, there is no reason why the convergence should hold for the third derivative, due to the lack of compactness.

Looking at (PU_ε), we see that $\mathcal{I}_\varepsilon(U_\varepsilon)$ converges uniformly to 1, and, for every given $z \in \mathbb{R}$, (6.4) implies that

$$U_\varepsilon(z) - 2U_\varepsilon(\overline{z}) + U_\varepsilon(0) = V_\varepsilon(z) - 2V_\varepsilon(\overline{z}) + V_\varepsilon(0) \underset{\varepsilon\to 0}{\longrightarrow} V_0(z) - 2V_0(\overline{z}) + V_0(0). \tag{6.6}$$

Passing to the pointwise limit in problem (PU_ε), we get that V_0 solves the following problem:

$$1 + m(z) = \exp\left(V_0(z) - 2V_0(\overline{z}) + V_0(0)\right).$$

Then, we have necessarily:

$$V_0(z) = \sum_{k\geqslant 0} 2^k \log\left(1 + m(2^{-k}z)\right). \tag{6.7}$$

This completes the proof of Theorem 1.3(ii), up to the identification of the limit of γ_ε, if it exists. In our approach, this goes through the characterization of the functional $\mathcal{J}_\varepsilon$ (2.10). This was indeed the purpose of Lemma 3.1. Here comes an important difficulty, as compactness estimates are not sufficient to pass to the limit in $\mathcal{J}_\varepsilon(0, V_\varepsilon)$ as $\varepsilon \to 0$ (3.3), as it would formally involve the pointwise value $\partial_z^3 V_0(0)$ which is beyond what our compactness estimates can provide. Note that passing to the limit in $\partial_g \mathcal{J}_\varepsilon(g, V_\varepsilon)$ as $\varepsilon \to 0$ is not an issue, as it can be encompassed by (6.5), see (3.4).

It remains to prove that the following limit holds true

$$\lim_{\varepsilon\to 0} \frac{1}{4\sqrt{2}\pi} \iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2)) \left[y_1^2 \partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2 \partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2 = \frac{1}{2}\partial_z^3 m(0)\,. \tag{6.8}$$

Indeed, this would directly imply that

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(g,V_\varepsilon) = -\frac{1}{2}\partial_z^3 m(0) + g\partial_z^2 m(0)\,, \tag{6.9}$$

as $\partial_z^2 V_0(0) = 2\partial_z^2 m(0)$ as a consequence of (6.7). We could deduce immediately that the root $\gamma_\varepsilon(V_\varepsilon)$ converges to the expected value (1.10).

In the absence of compactness, we call the contraction argument, in order to prove the following key result:

Lemma 6.1. *For every $\delta > 0$, there exists $R_1(\delta) > 0$, such that, for every $R \geqslant R_1(\delta)$, there exists $\varepsilon_1(\delta, R)$ such that for all $\varepsilon \leqslant \varepsilon_1(\delta, R)$, we have:*

$$\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \leqslant \delta\,. \tag{6.10}$$

Proof. To begin with, we differentiate the problem (PU_ε) three times:

$$\partial_z^3 V_\varepsilon(z) - \frac{1}{4}\partial_z^3 V_\varepsilon(\overline{z}) = \partial_z^3 \log\left(\lambda_\varepsilon + m(z)\right) - \partial_z^3 \log\left(\mathcal{I}_\varepsilon(U_\varepsilon)(z)\right).$$

We expand the right hand side as usual:

$$\begin{aligned}\partial_z^3 V_\varepsilon(z) - \frac{1}{4}\partial_z^3 V_\varepsilon(\overline{z}) = \frac{\partial_z^3 m(z)}{\lambda_\varepsilon + m(z)} + \frac{3\partial_z^2 m(z)\partial_z m(z)}{(\lambda_\varepsilon + m(z))^2} + \frac{2\partial_z m(z)^3}{(\lambda_\varepsilon + m(z))^3}&\\ - W_\varepsilon^{(3)}(z) - 3W_\varepsilon^{(2)}(z)W_\varepsilon^{(1)}(z) - 2W_\varepsilon^{(1)}(z)^3.&\end{aligned}$$

We subtract $\partial_z^3 m(z)$ on each side, and we reorganize the terms in order to conjure the difference $\partial_z^3 V_\varepsilon(z) - (4/3)\partial_z^3 m(z)$ we are interested in:

$$\begin{aligned}\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z) - \frac{1}{4}\left(\partial_z^3 V_\varepsilon(\overline{z}) - \frac{4}{3}\partial_z^3 m(\overline{z})\right) + \frac{1}{3}\left(\partial_z^3 m(z) - \partial_z^3 m(\overline{z})\right) =&\\ \partial_z^3 m(z)\left(\frac{1}{\lambda_\varepsilon + m(z)} - 1\right) + \frac{3\partial_z^2 m(z)\partial_z m(z)}{(\lambda_\varepsilon + m(z))^2} + \frac{2\partial_z m(z)^3}{(\lambda_\varepsilon + m(z))^3}&\\ - W_\varepsilon^{(3)}(z) - 3W_\varepsilon^{(2)}(z)W_\varepsilon^{(1)}(z) - 2W_\varepsilon^{(1)}(z)^3.&\end{aligned} \tag{6.11}$$

We estimate below each term of (6.11). First, the terms involving m and its derivatives on the right hand side of (6.11) converge to zero, uniformly for $|z| \leqslant \varepsilon R$, as $\varepsilon \to 0$, simply because $m(0) = \partial_z m(0) = 0$, and $\lambda_\varepsilon \to 1$. Actually, the same holds true for the difference of $\partial_z^3 m(z) - \partial_z^3 m(\overline{z})$ by continuity of $\partial_z^3 m$ at the origin.

Second, from Proposition 4.7, we know that

$$\max\left(\left\|W_\varepsilon^{(1)}\right\|_\infty, \left\|W_\varepsilon^{(2)}\right\|_\infty\right) = \mathcal{O}(\varepsilon). \tag{6.12}$$

The remaining term, $W_\varepsilon^{(3)}(z)$ is more delicate to handle. In fact, it will result in a contraction estimate, exactly as in section 5. We recall the expression of $W_\varepsilon^{(3)}$ (4.8):

$$W_\varepsilon^{(3)}(z) = \left\langle dG_\varepsilon^{V_\varepsilon}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)(z) + \left(\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)(z)\right)^3 + \frac{3}{2}\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)(z)\mathcal{D}_\varepsilon(\partial_z^2 V_\varepsilon)(z)\right\rangle.$$

As in the proof of equations (4.13) to (4.15), we get that the last two contributions involving the non-linear and lower order terms $\left(\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)\right)^3$ and $\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)\mathcal{D}_\varepsilon(\partial_z^2 V_\varepsilon)$ are $\mathcal{O}(\varepsilon)$. It remains the term $\left\langle dG_\varepsilon^{V_\varepsilon}, (1/4)\mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)\right\rangle$, which is a double integral in variables (y_1, y_2) that we split in two regions of integration: $\Omega = \{|y_1| \leqslant R/2,\ \text{and}\ |y_2| \leqslant R/2\}$ and $\Omega^{\mathsf{c}} = \{|y_1| > R/2,\ \text{or}\ |y_2| > R/2\}$.

Let $\delta > 0$. We can choose $R_1(\delta)$ large enough so that, for all $R \geqslant R_1(\delta)$, we have

$$\frac{1}{4}\left|\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_{\Omega^c}(y_1,y_2), \mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)(z)\right\rangle\right| \leqslant \frac{1}{2}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_{\Omega^c}(y_1,y_2), 1\right\rangle \|K_0\|_\alpha \leqslant \frac{\delta}{10}. \tag{6.13}$$

In the region where y_1 and y_2 are both below $R/2$, we introduce the difference with $\partial_z^3 m$, as in (6.10):

$$\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\partial_z^3 V_\varepsilon(\overline{z}) - \frac{1}{8}\partial_z^3 V_\varepsilon(\overline{z}+\varepsilon y_1) - \frac{1}{8}\partial_z^3 V_\varepsilon(\overline{z}+\varepsilon y_2)\right\rangle = A + B,$$

where

$$\begin{aligned} A = \Bigg\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\left(\partial_z^3 V_\varepsilon(\overline{z}) - \frac{4}{3}\partial_z^3 m(\overline{z})\right) \\ -\frac{1}{8}\left(\partial_z^3 V_\varepsilon(\overline{z}+\varepsilon y_1) - \frac{4}{3}\partial_z^3 m(\overline{z}+\varepsilon y_1)\right) - \frac{1}{8}\left(\partial_z^3 V_\varepsilon(\overline{z}+\varepsilon y_2) - \frac{4}{3}\partial_z^3 m(\overline{z}+\varepsilon y_2)\right)\Bigg\rangle \end{aligned}$$

and

$$B = \frac{1}{6}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \left(\partial_z^3 m(\overline{z}) - \partial_z^3 m(\overline{z}+\varepsilon y_1)\right) + \left(\partial_z^3 m(\overline{z}) - \partial_z^3 m(\overline{z}+\varepsilon y_2)\right)\right\rangle.$$

By construction, we have $|\overline{z}+\varepsilon y_i| \leqslant \varepsilon R/2 + \varepsilon R/2 \leqslant \varepsilon R$. Therefore, we have

$$\begin{aligned} |A| &\leqslant \left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| + \frac{2}{8}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|\right\rangle \\ &\leqslant \frac{1}{2}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|. \end{aligned}$$

As for B we find:

$$|B| \leqslant \frac{1}{3}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \operatorname*{osc}_{|z|\leqslant\varepsilon R}(\partial_z^3 m)\right\rangle \leqslant \frac{1}{3}\operatorname*{osc}_{|z|\leqslant\varepsilon R}\left(\partial_z^3 m\right) \xrightarrow[\varepsilon\to 0]{} 0.$$

Going back to (6.11), we have shown that for $R \geqslant R_1$, there exists $\varepsilon_1 > 0$ small enough such that for all $\varepsilon \leqslant \varepsilon_1$ we have:

$$\begin{aligned} \sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| &\leqslant \frac{\delta}{4} + \frac{1}{4}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(\overline{z}) - \frac{4}{3}\partial_z^3 m(\overline{z})\right| + \frac{1}{2}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \\ &\leqslant \frac{\delta}{4} + \frac{3}{4}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|. \end{aligned}$$

As a consequence, we find that

$$\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \leqslant \delta.$$

This completes the proof of Lemma 6.1. □

Back to (6.8), we recall that $|\widetilde{y_i}| \leqslant |y_i| + 1$, as a by-product of Taylor expansions. Let $\delta > 0$, and take R sufficiently large such that

$$\frac{1}{4\sqrt{2\pi}}\iint_{\Omega^c}\exp(-Q(y_1,y_2))\left(\|K_0\|_\alpha + \frac{1}{2}\partial_z^3 m(0)\right)\left[y_1^2+y_2^2\right]dy_1dy_2 \leqslant \frac{\delta}{10}, \tag{6.14}$$

where $\Omega = \{|y_1| \leqslant R-1,\ \text{and}\ |y_2| \leqslant R-1\}$. The other part of the double integral is:

$$\frac{1}{4\sqrt{2\pi}}\iint_\Omega \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right]dy_1dy_2. \tag{6.15}$$

Using Lemma 6.1 and the continuity of $\partial_z^3 m$ at $z=0$, we can find $\varepsilon_1>0$ such that for all $\varepsilon\leqslant\varepsilon_1$,

$$\left|\frac{1}{4\sqrt{2}\pi}\iint_\Omega \exp(-Q(y_1,y_2))\left[y_1^2\left(\partial_z^3V(\varepsilon\widetilde{y_1})-\frac{4}{3}\partial_z^3m(0)\right)+y_2^2\left(\partial_z^3V(\varepsilon\widetilde{y_2})-\frac{4}{3}\partial_z^3m(0)\right)\right]dy_1dy_2\right|$$
$$\leqslant\left(\frac{1}{4\sqrt{2}\pi}\iint_\Omega \exp(-Q(y_1,y_2))\left[y_1^2+y_2^2\right]dy_1dy_2\right)\frac{\delta}{10}.$$

Putting all the pieces together, and using that $\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))\left[y_1^2+y_2^2\right]dy_1dy_2=\frac{3}{2}$ (2.2), we deduce that:

$$\left|\frac{1}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3V(\varepsilon\widetilde{y_1})+y_2^2\partial_z^3V(\varepsilon\widetilde{y_2})\right]dy_1dy_2-\frac{1}{2}\partial_z^3m(0)\right|\leqslant\delta\,. \tag{6.16}$$

Hence, the limit announced in (6.8) holds true. This completes the proof of the asymptotic behavior $(\lambda_\varepsilon,U_\varepsilon)\to(\lambda_0,U_0)$ as described in Theorem 1.3(ii).

7. Extension to higher dimensions

Our methodology can be extended to higher dimension, without too much effort. This section is devoted to the generalization of the elements of proof that were specific to the one-dimensional case.

All the estimates on the operator $\mathcal{H}_\varepsilon$ and its constitutive pieces are still operational in higher dimension. The only part of our proof that requires some specific attention is the construction of the linear part $\gamma_\varepsilon(V_\varepsilon)$ which was performed in Section 3. Indeed, we used a monotonicity argument to show that $\gamma_\varepsilon(V_\varepsilon)$ can be defined in a unique way.

We proceed as in Section 3. First we show formally how to obtain the expression of the vector γ_0 (1.10) via suitable Taylor expansions. Then, we justify these Taylor expansions, and we exhibit a monotonic function that enables to conclude, exactly as in dimension 1.

7.1. The formal expression of the linear part γ_0.

Following the very same heuristics as in section 3.1, but being careful during the Taylor expansions, we formally end up with the following matrix valued identity:

$$D^2V(0)\left(\frac{1}{(\sqrt{2}\pi)^d}\iint_{\mathbb{R}^{2d}}e^{-Q(y_1,y_2)}(y_1\otimes y_1+y_1\otimes y_2)dy_1dy_2\right)\gamma_0$$
$$=\frac{1}{2}D^3V(0)\left(\frac{1}{(\sqrt{2}\pi)^d}\iint_{\mathbb{R}^{2d}}e^{-Q(y_1,y_2)}y_1\otimes y_1dy_1dy_2\right). \tag{7.1}$$

The quadratic form Q yields the multivariate centered gaussian distribution associated with the following covariance matrix $\Sigma\in\mathcal{M}_{2d}(\mathbb{R})$:

$$\Sigma=\frac{1}{4}\begin{pmatrix}3\,\mathrm{Id} & -\mathrm{Id}\\ -\mathrm{Id} & 3\,\mathrm{Id}\end{pmatrix}.$$

The Kronecker product $y_1\otimes y_1$ yields a matrix of moments, and so the relation (7.1) can be simplified, similarly to the one dimensional case, so as to obtain:

$$\left(D^2V(0)\left(\frac{3}{4}-\frac{1}{4}\right)\mathrm{Id}\right)\gamma_0=\frac{1}{2}D^3V(0)\frac{3}{4}\,\mathrm{Id},$$
$$\frac{1}{2}D^2V(0)\gamma_0=\frac{3}{8}D^3V(0)\,\mathrm{Id}\,.$$

The righ hand side is a tensor applied to a matrix yields a vector that can be simplified even further using tensorial properties: $D^3V(0)\,\mathrm{Id}=D(\Delta V)(0)$. Then, provided that $D^2V(0)$ is non degenerate, we obtain the limited expected value of γ_0 in dimension higher than 1, that is a generalization of (3.2):

$$\gamma_0(V) = \frac{3}{4}\left(D^2V(0)\right)^{-1} D(\Delta V)(0).$$

In the case where V_0 is given by (1.10) through the fixed point procedure, we obtain

$$\gamma_0(V_0) = \frac{1}{2}\left(D^2m(0)\right)^{-1} D(\Delta m)(0). \tag{7.2}$$

7.2. **Extension of the proof of Proposition 2.4 (section 3.2).** We now fix $V \in K$, where K is a ball of $\mathcal{E}_0^\alpha$. The purpose is to prove that there is a unique solution in $\mathbb{R}^d$ of the following problem:

$$\mathcal{J}_\varepsilon(\gamma, V) = 0. \tag{7.3}$$

We insist upon the fact that the variable g belongs to $\mathbb{R}^d$ and the function $\mathcal{J}_\varepsilon(\cdot, V)$ is now defined as a vector field on $\mathbb{R}^d$, $\mathcal{J}_\varepsilon : \mathbb{R}^d \times \mathcal{E}^\alpha \to \mathbb{R}^d$.

As in section 3.2, we can obtain the following estimate

$$\mathcal{J}_\varepsilon(g, V) = \mathcal{J}_\varepsilon(0, V) + \frac{1}{2}D^2V(0)g + \mathcal{O}(\varepsilon)\,, \tag{7.4}$$

by means of refined Taylor expansions, where $\mathcal{J}_\varepsilon(0, V)$ is bounded *a priori*, independently upon $\varepsilon > 0$ for $V \in K$. To prove the existence of a root γ_ε, we used the mean value theorem in the proof of Proposition 2.4. The analogous statement in higher dimension is the Brouwer fixed point theorem. Indeed, (7.3) can be recast as follows:

$$g = \left(\mathrm{Id} + \frac{1}{2}D^2V(0)\right)^{-1}\left(g - \mathcal{J}_\varepsilon(0, V) + \mathcal{O}(\varepsilon)\right) = \mathcal{T}(g).$$

Thus, we are led to finding a fixed point of a continuous function. As in the one-dimensional case, thanks to the lower bounded $D^2V(0) \geqslant \mu_0 \,\mathrm{Id}$ encoded in the definition of $\mathcal{E}_0^\alpha$ (1.6), we can show easily that there exists R_K such that the ball of radius R_K in $\mathbb{R}^d$ is left invariant by $\mathcal{T}$. Brouwer's fixed point theorem guarantees that there exists a fixed point γ_ε to $\mathcal{T}$, which is also a root of (7.3).

For the uniqueness part, we can use strict monotonicity, similarly as in the one dimensional case. This is possible, thanks to (3.4) :

$$D_g\mathcal{J}_\varepsilon(g, V) = \frac{1}{2}D^2V(0) + \mathcal{O}(\varepsilon). \tag{7.5}$$

We deduce from this strong estimate that the vector field $\mathcal{J}_\varepsilon(\cdot, V)$ is locally uniformly monotonic, in the sense that there exists μ_K such that the following inequality holds true for all ε sufficient small, and every $g_1, g_2 \in B(0, R_K)$:

$$\left(\mathcal{J}_\varepsilon(g_1, V) - \mathcal{J}_\varepsilon(g_2, V)\right) \cdot (g_1 - g_2) \geqslant \frac{1}{2}\mu_K \left\|g_1 - g_2\right\|^2. \tag{7.6}$$

This monotonicity condition is clearly satisfied, as it is equivalent to the following first order condition,

$$\frac{1}{2}\left(D_g\mathcal{J}_\varepsilon(g, V) + D_g\mathcal{J}_\varepsilon(g, V)^\top\right) \geqslant \mu_K Id, \tag{7.7}$$

It is immediate that any strictly monotonic vector field admits at most one root. This completes the proof of uniqueness of $\gamma_\varepsilon(V)$.

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