MINI-PROJECT: SPATIAL SEGREGATION IN COMPETITION-DIFFUSION SYSTEMS

This project centres around a study of the research article

- Crooks, E.C.M.; Dancer, E.N.; Hilhorst, D.; Mimura, M.; Ninomiya, H., *Spatial segregation limit of a competition-diffusion system with Dirichlet boundary conditions.* Nonlinear Anal. Real World Appl. 5 (2004), no. 4, 645-665.

which is concerned with spatial segregation in the strong-competition limit ($k \to \infty$) of solutions of the competition-diffusion system

$$(\mathrm{P_k}) \qquad \begin{cases} u_t = d_1 \triangle u + f(u) - kuv & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = d_2 \triangle v + g(v) - \alpha kuv & \text{in } \Omega \times \mathbb{R}^+, \\ u = m_1^k & \text{on } \partial\Omega \times \mathbb{R}^+, \\ v = m_2^k & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0^k(x), \ v(x,0) = v_0^k(x) & x \in \Omega. \end{cases}$$

Here u and v correspond to the population densities of two competing species, and m_1^k, m_2^k are given functions of space and time that describe the behaviour of u and v on the boundary of the spatial domain Ω. Prototype examples for the nonlinear self-interaction functions $f(u)$ and $g(v)$ are the logistic functions

$$f(u) = u(r_u - a_u u), \quad g(v) = v(r_v - a_v v), \qquad (\dagger)$$

where r_u, r_v, a_u, a_v are positive constants, in which case the system satisfied by u, v is sometimes known as a Lotka-Volterra competition-diffusion system.

Following the introduction in Section 1, which covers both ecological background and previous work, some important estimates are presented in Section 2, before these estimates and some compactness properties of solutions are used to discuss the limit as $k \to \infty$ in Section 3. Section 4 is concerned with numerical simulation of solutions.

Here are some suggestions of things to think about/discuss, organised by the sections of the paper. You might like to choose which aspects to focus on according to your own backgrounds/interests. The parts marked * are perhaps the most straightforward.

- **Section 1: Introduction**

  (i)* What is the ecological motivation for the boundary condition $u = m_1^k, v = m_2^k$ on $\partial\Omega$? What alternative boundary conditions could one consider, and what do they correspond to ecologically?

  (ii)* In the case when $f(u)$ and $g(v)$ are given by ($\dagger$), find all pairs of constant equilibrium solutions (u_0, v_0) of the system $u_t = d_1 \triangle u + f(u) - kuv$, $v_t = d_2 \triangle v + g(v) - \alpha kuv$ (forget about the boundary condition for this), and find their stability as solutions of the ODE system $u_t = f(u) - kuv$, $v_t = g(v) - \alpha kuv$. Your answers will depend on the parameters, of course. What happens when k is large?

- **Section 2: Formulation of the problem and basic properties**

  This section focuses mainly on proof of *a priori* estimates of solutions of ($\mathrm{P_k}$) independent of the parameter k, which are then used in Section 3 to pass to the strong-competition limit as $k \to \infty$. The most important proofs to try to work through and understand are Lemma 2.1, Lemma 2.3, Lemma 2.4, Lemma 2.6 and Lemma 2.7. Here are a few extra points to think about:

  (i)* Which result is the key to proving spatial segregation between u and v? This underpins the whole paper, and application of this approach to similar problems will always need an estimate of this type.

  (ii) In Lemma 2.3 and Lemma 2.4, the factor ϕ is present in the integral because of the boundary condition imposed in (P_k). Can you prove analogues of these two results with ϕ replaced by the constant 1 if the boundary conditions are replaced by $\partial u/\partial \nu = 0 = \partial v/\partial \nu$ on $\partial\Omega \times \mathbb{R}^+$?

  (iii)* The proof of Lemma 2.1 uses the maximum principle (or more precisely, comparison principle) for *scalar* parabolic equations, which informally says: if $\underline{u}$ and $\overline{u}$ are sub- and super-solutions respectively of $u_t = d\Delta u + f(x,u)$, that is

  $$\underline{u}_t \le d\Delta\underline{u} + f(x,\underline{u}) \quad \text{and} \quad \overline{u}_t \ge d\Delta\overline{u} + f(x,\overline{u}) \quad \text{for all} \ \ (x,t) \in \Omega \times \mathbb{R}^+,$$

  and

  $$\underline{u}(x,0) \le \overline{u}(x,0) \ \text{ for all } \ x \in \Omega, \qquad \underline{u}(x,t) \le \overline{u}(x,t) \ \text{ for all } \ (x,t) \in \partial\Omega \times \mathbb{R}^+,$$

  then

  $$\underline{u}(x,t) \le \overline{u}(x,t) \qquad \text{for all} \ \ (x,t) \in \Omega \times \mathbb{R}^+.$$

  The proof given in the paper is quite brief - can you write out a fuller version, with more details? You might like first to explain why $0 \le u_k$ and $0 \le v_k$ in $\bar{Q} := \overline{\Omega} \times \mathbb{R}^+$, before showing that $u_k \le 1$ and $v_k \le 1$ in $\bar{Q}$.

  (iv) Lemmas 2.6 and 2.7 establish estimates on the differences between space and time translates of solutions respectively. In one of these results, the bounds obtained do not depend on the diffusion coefficients d_1, d_2, whereas in the other, they do. Which is which? Here, dependence on d_1, d_2 is not important, but there might be other modelling situations where we would like to have estimates independent ofd_1 or d_2. Can you think of any?

- **Section 3: The limit problem as $k \to \infty$**

  It is important to work through and understand the proof of Lemma 3.3, since this is the key to understanding where the limit problem (P) comes from.

  (i)* Corollary 3.1 is a straightforward consequence of results in Section 2. Can you write a short proof, referencing the results that are needed?

  (ii) Lemma 3.6, Lemma 3.7 and Corollary 3.8 are concerned with showing uniqueness of the weak solution of the initial-boundary problem (P). Why is uniqueness important here?

  (iii) Study of Lemma 3.6 and Lemma 3.7 can be left until later if you do not have time now, but it is good to try to understand how the proof of Corollary 3.8 uses Lemma 3.6, particularly if you have not seen arguments like this before.

(iv)* On page 656, it is stated that if $w := u - \frac{v}{\alpha}$ then $u = w^+$ and $v = \alpha w^-$. Can you write down a proof of this? Which property of the limit functions u, v is crucial for this proof?

(v) Suppose, for concreteness, that $m_1 > 0$ and $m_2 = 0$ on $\partial\Omega \times [0, T]$. Suppose also that there exists a smooth family of closed hypersurfaces $\Gamma = \cup_{t\in[0,T]}\Gamma(t)$, with $\Gamma(t) \subset\subset \Omega$, and subdomains $\Omega_u(t)$, $\Omega_v(t)$ such that for each $t \in [0, T]$,

$$\overline{\Omega} = \overline{\Omega_u(t)} \cup \overline{\Omega_v(t)}, \qquad \Gamma(t) = \overline{\Omega_u(t)} \cap \overline{\Omega_v(t)},$$

and

$$w(\cdot, t) > 0 \quad \text{on} \quad \Omega_u(t), \qquad w(\cdot, t) < 0 \quad \text{on} \quad \Omega_v(t).$$

Then $\Gamma(t)$ is a so-called *free boundary*, depending on time, between the region where $w > 0$ and where $w < 0$. Let

$$\bar{u} := w+, \qquad \bar{v} := \alpha w^-,$$

where

$$w^+ := \max\{w, 0\}, \quad w^- := \max\{-w, 0\}.$$

Assuming that $(\bar{u}, \bar{v})$ is sufficiently smooth up to $\Gamma(t)$, can you derive a re-formulation of the limit problem (P) in terms of $\bar{u}, \bar{v}$ and the free boundary Γ?

You should obtain a PDE satisfied by $\bar{u}$ in $\cup_{t\in[0,T]}\Omega_u(t)$, another PDE satisfied by $\bar{v}$ in $\cup_{t\in[0,T]}\Omega_v(t)$, initial conditions for $\bar{u}, \bar{v}$, boundary conditions for $\bar{u}, \bar{v}$ on $\partial\Omega$, and conditions on $\bar{u}, \bar{v}$ and the relationship between $\partial\bar{u}/\partial\nu$ and $\partial\bar{v}/\partial\nu$ on the free boundary Γ, where in each case, ν denotes the *outward* normal to $\Omega_u(t), \Omega_v(t)$ respectively.

Start by writing the equation in Definition 3.4(ii) in terms of $\bar{u}, \bar{v}, \Omega_u(t)$ and $\Omega_v(t)$. You might like to consult Section 4 of the article

– Hilhorst, Danielle; Martin, Sebastien; Mimura, Masayasu, *Singular limit of a competition-diffusion system with large interspecific interaction* J. Math. Anal. Appl. 390 (2012), 488-513.

where some similar arguments are used.

- **Section 4: Numerical computations (see also Figures 1-3)**

(i) Can you reproduce the simulations in Figures 1-4?

(ii) What happens if you make different choices of the boundary functions m_1^k, m_2^k or the initial conditions u_0^k, v_0^k? (note that in the simulations in the paper, the boundary and initial conditions actually do not depend on the parameter k)

See also the further paper

- Squassina, Marco; Zuccher, Simone, *Numerical computations for the spatial segregation limit of some 2D competition-diffusion systems.* Adv. Math. Sci. Appl. 18 (2008), no. 1, 83-104.

for some more discussion on numerical computation for systems of this type (http://www.dmf.unicatt.it/ squassin/papers/lavori/squzuc-numerical.pdf)

- **Some extensions/more things to think about**

  (i) In this article, α is a (strictly positive) constant. But in principle, α (and k) may depend on x. What would such heterogeneity this correspond to ecologically? Can you try to extend the results of this article to the case when α is a smooth (say C^2) function of x with

  $$\alpha(x) \geq \delta > 0, \quad x \in \Omega,$$

  for some constant $\delta > 0$ What about if $\alpha(x) \geq 0$ for all $x \in \Omega$ but there are some x (or regions of x) where $\alpha(x) = 0$?

  (ii) Not much is known, even now (15 years after this article was published), about the relationship between the initial and boundary conditions and the precise patterns that are observed in solutions of the limit problem when time t are both large, even in one space dimension. Can you make some conjectures about this?

  (iii) If $d_1 = d_2$, more results can be shown. Why is this? Can you prove any additional properties?

A few pieces of suggested additional reading

- Namba, Toshiyuki; Mimura, Masayasu *Spatial distribution of competing populations.* J. Theoret. Biol. 87 (1980), no. 4, 795-814.

- Dancer, E. N.; Zhang, Zhitao, *Dynamics of Lotka-Volterra competition systems with large interaction.* J. Differential Equations 182 (2002), no. 2, 470-489.

- Dancer, E. N.; Hilhorst, D.; Mimura, M.; Peletier, L. A., *Spatial segregation limit of a competition-diffusion system.* European J. Appl. Math. 10 (1999), 97-115.

- Crooks, E.C.M.; Dancer, E.N. ; Hilhorst, D., *On long-time dynamics for competition-diffusion systems with inhomogeneous Dirichlet boundary conditions.* Topol. Methods Nonlinear Anal. 30 (2007), no. 1, 1-36.