# MINI-PROJECT: TRAVELLING FRONTS AND LINEAR DETERMINACY

This mini-project is concerned with travelling-front solutions of reaction-diffusion-(convection) equations, which are ubiquitous throughout mathematical biology and represent the transition from one steady state to another at a constant rate as time evolves, and in particular, the question of whether the minimal speed of a family of travelling waves, which can be characterised as a spreading speed, is or is not determined by the linearisation of the underlying reaction-diffusion equation about the zero (or extinction) steady state. The article

- Al-Kiffai, A., Crooks, E.C.M., *Lack of symmetry in linear determinacy due to convective effects in reaction-diffusion-convection problems.* Tamkang J. Math. 47 (2016), no. 1, 51-70.

discusses some similar questions and can be read/studied alongside the exercises given below.

**Section 1: Introduction** Reaction-diffusion equations of the form

$$u_t = du_{xx} + f(u), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$f(0) = f(1) = 0, \qquad f'(0) > 0, \quad f'(1) < 0, \qquad f(u) > 0 \text{ for all } u \in (0,1), \tag{2}$$

arise widely in the modelling of biological (and other) phenomena, the prototype being the famous Fisher-Kolmogorov-Petrovskii-Piskounov equation (Fisher-KPP), discussed in 1937 both by Fisher and by Kolomorov where

$$f(u) = ru(1-u).$$

Such equations are said to be 'monostable', since there is one stable steady state at $u = 1$ and one unstable steady state at $u = 0$. It is well-known that there exist decreasing travelling-front solutions $u(x,t) = w(x-ct)$, where the profile $w$ satisfies

$$w'(\xi) < 0 \text{ for all } \xi \in \mathbb{R}, \quad w(\xi) \to 1, 0 \text{ as } \xi \to -\infty, \infty, \tag{3}$$

if and only if the speed $c \geq c^*$, where[1]

$$c^* = \min_{\rho\in\Gamma} \max_{\xi\in\mathbb{R}} \left\{ \frac{d\rho''(\xi) + f(\rho(\xi))}{-\rho'(\xi)} \right\}, \tag{4}$$

and

$$\Gamma = \{\rho \in C^1(\mathbb{R};\mathbb{R}) : \rho'(\xi) < 0 \text{ for all } \xi \in \mathbb{R},\ \rho \to 1, 0 \text{ as } \xi \to -\infty, \infty\}.$$

Importantly for applications, in a famous 1978 paper, Aronson and Weinberger showed that the minimal travelling-front speed $c^*$ can be characterised as a *spreading speed*, in the sense that for an initial condition $u(x,0) = u_0(x) \in [0,1]$ with

$$u_0(x) = 0 \text{ for } x >> 0 \quad \text{and} \quad u_0(x) \in (\alpha, 1) \text{ for } x << 0,$$

the solutions $u$ of (1) satisfy $\lim_{t\to\infty} \sup_{x\geq ct} u(x,t) = 0$ if $c > c^*$, whereas if $c < c^*$, then $\lim_{t\to\infty} \inf_{x\leq ct} u(x,t) = 1$.

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[1] Note that *if* there is a travelling-front profile $w$ with speed $c$, then $w \in \Gamma$ and $\frac{dw''(\xi)+f(w(\xi))}{-w'(\xi)} = c$ for all $\xi \in \mathbb{R}$, so it is easy to see (exercise - check this) that if a front exists its speed must satisfy $c \geq c^*$ where $c^*$ is defined in (4), that is, $c^* \leq c$ is *necessary* for a travelling front satisfying (1) and (3) to exist. It turns out that $c^* \leq c$ is also *sufficient* for a travelling front satisfying (1) and (3) to exist; see, for example, the reference Volpert, Volpert and Volpert '94, Gilding and Kersner '04, or Crooks '03 for a proof of this in more general settings.

The *linear value* $\bar{c}$ is defined to be the minimal value of $c \in \mathbb{R}$ for which the linear equation

$$dw'' + cw' + f'(0)w = 0 \tag{5}$$

has a solution $w(\xi) = \exp(-\mu\xi)$ with real $\mu > 0$, and hence the minimal value for which the linearisation of (1) admits a decreasing travelling-front solution that decays to 0 at $+\infty$. It can be shown that

$$\bar{c} = 2\sqrt{df'(0)},$$

and that

$$c^* \geq \bar{c},$$

since the existence of $-\mu < 0$ is necessary for a front-solution $w(x - ct)$ of (1), (3) to exist.

> **Exercise 1:** Verify that $\bar{c} = 2\sqrt{df'(0)}$ and draw a sketch of the phase plane for $(w, w')$ satisfying (5) in a neighbourhood of $(0, 0)$ in each of the cases $c > c^*, c = c^*$ and $c < c^*$. Use the sketch to explain why $c^* \geq \bar{c}$.

In general, we may have

$$c^* > \bar{c} \quad \text{('pushed' case)} \quad \text{or} \quad c^* = \bar{c} \quad \text{('pulled' case)},$$

and say that the minimal-front speed (or spreading speed) is *linearly determinate* if

$$c^* = \bar{c}.$$

Clearly, in the linearly determinate case, we have an explicit value for the minimal travelling-front speed (and hence the spreading speed), which is important for applied problems such as determining the speed of biological invasions that are modelled by equations of this type. In addition, the type of stability results that can be shown for travelling fronts, in the sense of for what initial conditions, and in what sense (e.g. in which norm, or...) do solutions of the initial-value problem for (1) converge to the travelling front as $t \to \infty$, differ depending on whether or not the front has minimal speed, that is, whether or not $c = c^*$, and on whether or not the minimal-speed front is linearly determinate, that is, $c^* = \bar{c}$ or $c^* > \bar{c}$.

So it is important to know whether the spreading speed is linearly determinate or not for a given equation. In practice, there are two tools that are particularly useful for this:

- Firstly, the 'min-max' formula (4), can be used to establish upper bounds for $c^*$ be substituting specific choices of test function $\rho \in \Gamma$.

- Secondly, if $c > c^*$, since the front behaves like $w(\xi) \approx e^{-\mu\xi}$ where $e^{-\mu\xi}$ solves the linearisation (5), there are two possible rates of decay of the front to 0, namely

  $$-\mu_1 = \frac{-c + \sqrt{c^2 - 4df'(0)}}{2d}, \qquad -\mu_2 = \frac{-c - \sqrt{c^2 - 4df'(0)}}{2d}.$$

  Clearly, $-\mu_2$ is more negative than $-\mu_1$, and it is known (see the references Rothe '81, Lucia, Muratov, Novaga '04) that the *only* situation in which a front of speed $c$ decays like $-\mu_2$ is when

  $$c = c^* > \bar{c}.$$

  This implies that, for a given equation, if you have an explicit formula for a travelling front and thus can check its rate of decay to 0, you know that it must be a minimal-speed front that is *not* linearly determined if it decays like $-\mu_2$.

In order to take advantage of these tools, it is useful to introduce an alternative characterisation of the minimal-front speed $c^*$, namely

$$c^* = \inf_{\psi\in\Lambda} \sup_{0<w<1} \left\{ d\psi^{'}(w) + \frac{f(w)}{\psi(w)} \right\} \tag{6}$$

where the set $\Lambda$ is defined by

$$\Lambda := \Big\{\psi : [0,1] \to [0,\infty) : \psi \text{ is continuously differentiable, } \psi(0) = 0,\ \psi^{'}(0) > 0, \\ \text{and } \psi(w) > 0 \text{ for } w \in (0,1)\Big\}. \tag{7}$$

This formula was first derived in the famous reference of Hadeler and Rothe '75, and is related to the important fact that, for a strictly monotone front profile $w$, we can think of $w'$ as being a function of $w$ (exercise - why is this not possible if the front is not monotone?), and so write

$$\psi(w) = -w'. \tag{8}$$

**Exercise 2:** For a strictly monotone continuously differentiable function $w$ and $\psi$ as defined in (8), show that

$$\frac{dw''(\xi) + f(w(\xi))}{-w'(\xi)} = d\psi'(w) + \frac{f(w)}{\psi(w)},$$

and that

$$dw''(\xi) + cw'(\xi) + f(w(\xi)) = 0 \quad \Leftrightarrow \quad d\psi'(w)\psi(w) - c\psi(w) + f(w) = 0.$$

The next exercise illustrates the fact that it is possible to have either $c^* = \bar{c}$ or $c^* > \bar{c}$, using the two tools given above. This example is based on one given in Hadeler-Rothe '75. It takes advantage of a family of explicit travelling-front solutions, known as 'Huxley pulses'.

**Exercise 3:** Consider the reaction-diffusion equation

$$u_t = u_{xx} + u(1-u)(1+au),$$

where $a$ is a positive constant. Then $f(u) := u(1-u)(1+au)$ satisfies the conditions (2), so it follows from the discussion above that for each $a > 0$, there exists $c^*(a)$ such that a travelling-front solution satisfying (3) exists if and only if $c \geq c^*(a)$, and that $c^*(a) \geq \bar{c}(a)$, where $\bar{c}(a)$ is the $a$-dependent linear value.

(i) Show that for each $a > 0$, the linear value $\bar{c}(a) = 2$.

(ii) Use the formula (6) to show that $c^*(a)$ is a non-decreasing function of $a$.

(iii) Show that for each $a > 0$, the function $w_a(x - c_a t)$ with

$$w_a(\xi) := \frac{1}{1 + \exp\left(\sqrt{\frac{a}{2}}\,\xi\right)}, \qquad c_a = \frac{a+2}{\sqrt{2a}},$$

is a travelling-front solution of (1), (3).

(iv) By analysing the limit as $\xi \to \infty$ of $w_a'(\xi)/w_a(\xi)$, show that $w_a$ decays at the faster of the two possible rates of decay whenever $a > 2$.

(v) Deduce that

$$c^*(a) = \begin{cases} 2 & \text{if } 0 \leq a \leq 2, \\ \frac{a+2}{\sqrt{2a}} & \text{if } a \geq 2 \end{cases}$$

and hence that

$$c^*(a) = \bar{c}(a)$$

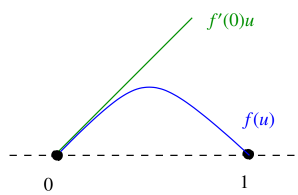
if and only if $a \in [0, 2]$.

The following is a famous sufficient condition for linear determinacy (see Hadeler-Rothe '75): if

$$f(w) \leq f'(0)w \quad \text{for all} \quad w \in (0,1) \tag{9}$$

then

$$c^* = \bar{c}.$$




*i.e.* no Allee effect

This condition is clearly satisfied for the prototype $f(u) = ru(1-u)$, implying that for the Fisher-KPP equation,

$$c^* = \bar{c} = 2\sqrt{dr},$$

as is well known. It is easy to check that (9) is also satisfied whenever $f$ is concave.

**Exercise 4:** Use the formula (6) together with the family of test functions

$$\psi_\kappa(w) = \kappa w$$

to prove that the condition (9) is sufficient to ensure that $c^* = \bar{c}$.

**Exercise 5:** For the example in Exercise 3, show that the condition (9) is satisfied if and only if $a \in [0, 1]$, and deduce that (9) is not a necessary condition for linear determinacy.

The following exercise explores the role of the diffusion coefficient $d$ in linear determinacy for the reaction-diffusion equation (1).

**Exercise 6:** Show that if $w$ is a travelling-front profile of speed $c$ when the diffusion coefficient $d = 1$, then for any diffusion coefficient $d > 0$, the re-scaled function

$$y(\xi) := w\left(\frac{\xi}{\sqrt{d}}\right)$$

is a travelling-front profile of speed $\sqrt{d}c$, and that the linear values satisfy

$$\bar{c}(d) = \sqrt{d}\bar{c}(1),$$

where $\bar{c}(d)$ denotes the linear value when the diffusion coefficient is $d$. Deduce that the equation is either linearly determinate for all $d > 0$, or for no $d > 0$, that is, linearly determinacy is not affected by the diffusion coefficient $d$.

**Section 2: Problems with convection** The equation (1) can be modified by the addition of a nonlinear convection term, *e.g.*,

$$u_t + h'(u)u_x = du_{xx} + f(u). \tag{10}$$

Such additional terms can arise due to, for instance, chemotaxis, as discussed in the article of Al Kiffai-Crooks '16. Can you think of any other modelling situations where nonlinear convection might arise?

An important consequence of the inclusion of convective effects is that the symmetry between decreasing and increasing fronts that exists in the convection-less case no longer holds.

**Exercise 7:** (i) Show that if $u(x, t)$ is a solution of the convection-less equation (1), then the function

$$\hat{u}(x, t) := u(-x, t)$$

is again a solution of (1), and deduce that if $\hat{w}(\xi) = w(-\xi)$, then $w(x - ct)$ is a decreasing front with speed $c$ if and only if $\hat{w}(x - (-c)t)$ is an increasing front with speed $-c$.

[Note that here we use the word 'speed', in keeping with common usage, when we should really say 'velocity' since we allow $c \in \mathbb{R}$ and understand $c < 0$ as corresponding to propagation to the left.]

(ii) Now consider the equation (10) that includes a convective term $h'(u)u_x$. What equation is now satisfied by the function $\hat{u}(x, t) := u(-x, t)$ if $u$ satisfies (10)?

Analogously to the convection-less case, it can be shown that there exists a decreasing travelling-front solution of (10)of speed $c$ if and only if $c \geq c^*$, where

$$c^* = \inf_{\rho\in\Lambda} \sup_{w\in(0,1)} \left\{ \psi'(w) + h'(w) + \frac{f(w)}{\psi(w)} \right\}, \tag{11}$$

and $\Lambda$ is as above, and again,

$$c^* \geq \bar{c},$$

where here (exercise - check this)

$$\bar{c} = h'(0) + 2\sqrt{df'(0)}.$$

**Exercise 8:** (i) Use this result for the existence of decreasing fronts to formulate a corresponding result for the existence of increasing fronts.'

(ii) What is the analogue of $\bar{c}$ for increasing fronts?

(iii) Using the respective linear values, or otherwise, show that if $w(x - c_1 t)$ and $w(x - c_2 t)$ are increasing and decreasing travelling-front solutions of (10) respectively, where $c_1, c_2 \in \mathbb{R}$ are the speeds (strictly speaking, velocities) of the fronts, then $c_1 < c_2$.

The next exercise extends (9) to establish a sufficient condition for linear determinacy for (10).

**Exercise 10:** Use (11) to show that if

$$h'(w) + \sqrt{d}\frac{f(w)}{\sqrt{f'(0)}w} \leq h'(0) + \sqrt{d}\sqrt{f'(0)} \quad \text{for all } w \in (0,1), \tag{12}$$

then

$$c^* = \bar{c}.$$

The paper of Al Kiffai-Crooks '16 contains some alternative sufficient conditions for linear determinacy, as well as examples illustrating, for instance,

- how convection can render an equation linearly determinate when it would not be linearly determinate in the absence of convection
- the fact that an equation might be linearly determinate for decreasing fronts but not increasing fronts, or vice versa.

**Exercise 11:** Think about:

- when convection is present, do you think that whether or not an equation is linearly determinate (for increasing or decreasing fronts, or both) is independent of the diffusion coefficient $d$, as in the convection-less case, or not?//
- can you use the 'min-max' formula (11) with different choices of test functions to derive some alternative speed estimates?

It is also possible to derive sufficient conditions for an equation not to be linearly determinate. The most famous of these conditions, which is discussed in Berestycki-Nirenberg '92, is that $c^* > \bar{c}$ if

$$\sqrt{2\int_0^1 f(u)\,du} \;>\; 2\sqrt{f'(0)}.$$

Similar ideas can be used to to derive sufficient conditions for the equation (10) with convection.

**Section 3: Linear determinacy for systems**

Section 3 of the Al Kiffai-Crooks '16 article discusses linear-determinacy for reaction-diffusion-(convection) systems. The key ingredient that allows arguments that work in the scalar case to be extended in a relatively straightforward way to systems is to have a *comparison principle* - see Theorem 3.1 of Al Kiffai-Crooks '16. This enables sub- and super-solution arguments to be used to estimate solutions of the system. Such a principle can be proved under the condition that the term $f$ in the system is co-operative (sometimes also called quasi-monotone).

A function $f : \mathbb{R}^N \to \mathbb{R}^N$ is said to be *co-operative* if

$$\frac{\partial f_i}{\partial u_j}(u) \geq 0, \quad \text{if } i \neq j.$$

Comparison principles are typically proved by applying a maximum principle. - see, for instance, the book of Volpert, Volpert and Volpert '94. To see why co-operativity is important, let $v : [0,L]\times[0,T] \to \mathbb{R}^N$ be such that

$$v(x,t) \geq 0 \quad \text{for all} \;\; (x,t) \in [0,L]\times[0,t_0],$$

$$v_i(x_0,t_0) = 0 \quad \text{for some} \;\; x_0 \in (0,L),\; t_0 > 0,$$

and

$$L[v] := v_t - A(x,t)v_{xx} - B(x,t)v_x - C(x,t)v > 0 \quad \text{for all } (x,t) \in (0,L)\times(0,t_0), \tag{13}$$

where $A$ is a positive definite diagonal matrix, $B$ is a diagonal matrix, the $N \times N$ matrix satisfies $C_{ij} \geq 0$ if $i \neq j$, and inequalities are understood componentwise. Then the $i$-th component of (13) gives

$$(v_i)_t - A_{ii}(v_i)_{xx} - B_{ii}(v_i)_x - \sum_{j=1}^{N} C_{ij}v_j \;>\; 0, \tag{14}$$

but at $(x_0,t_0)$, we know that

$$(v_i)_t(x_0,t_0) \leq 0, \quad (v_i)_x(x_0,t_0) = 0, \quad (v_i)_{xx}(x_0,t_0) \geq 0, \tag{15}$$

and

$$\sum_{j=1}^{N} C_{ij}v_j = \sum_{j=1, j\neq i}^{N} C_{ij}v_j \;\geq\; 0 \qquad \text{since } C_{ij} \geq 0,\;\; i \neq j,$$

which contradicts (15).

**Exercise 12:** Consider the system

$$\begin{aligned} u_t &= u_{xx} - v \\ v_t &= v_{xx}, \end{aligned}$$

for $(x,t) \in (0,1) \times (0,5)$.

(i) Why is this not a co-operative system?

(ii) Construct pairs of functions $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, such that for all $(x,t) \in (0,1) \times (0,5)$,

$$\underline{u}_t \leq \underline{u}_{xx} - \underline{v}, \quad \underline{v}_t \leq \underline{v}_{xx},$$
$$\overline{u}_t \geq \overline{u}_{xx} - \overline{v}, \quad \overline{v}_t \geq \overline{v}_{xx},$$

and for each $(x,t) \in ([0,1] \times \{0\}) \cup (\{0,1\} \times [0,5])$,

$$\underline{u} \leq \overline{u}, \qquad \underline{v} \leq \overline{v},$$

but there exists $(x_0, t_0) \in (0,1) \times (0,5)$ such that

$$(\underline{u}, \underline{v})(x_0, t_0) \nleq (\overline{u}, \overline{v})(x_0, t_0).$$

[Hint: play around with constants, linear functions, and the function $(x,t) \mapsto t(x - x^2)$.]

This shows that a comparison principle does not hold for this system. Similar ideas can be used to show that a comparison principle does not necessarily hold if the dependence on $(u_x, v_x)$ is not diagonal, *e.g.*, if the $u$ equation depends on $v_x$.

In the convection-less case, the sufficient condition (9) for linear determinacy can be extended to co-operatives systems - see the references Lui '89 and Weinberger, Lewis, and Li '02. Sufficient conditions for linear determinacy for co-operative systems with convection are discussed in Al Kiffai-Crooks '16.

**Some references**: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