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1 Dispersal evolution & Front acceleration

slides Cane Toads.

2 Maladaptation to an environmental change

z = phenotype.

1 A/ Modeling & Results

1 $f(t,x,\theta)$, $t>0$, $x\in\mathbb{R}$, $\theta>0$

$$\frac{\partial f}{\partial t} = \theta \frac{\partial^2 f}{\partial x^2} + f + \frac{\partial^2 f}{\partial \theta^2} - \left(\int f\, d\theta'\right)\cdot f$$

$\rho(t,x) = \int f(t,x,\theta')\,d\theta'$ spatial density.

Goal: long time behaviour $t\to\infty$

Box: the case of a single θ (say $\theta=1$)

$$\begin{cases} \frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} + \rho(1-\rho) \\ \rho(0,x) = \rho_0(x) \end{cases}$$ Fisher's equation.

Th (KPP, Aronson Weinberger, ...)

- $\exists\ \rho(x-ct)$ particular solution for $c\geq 2$ ← minimal speed.
- Suppose ρ_0 is comp supp. then level lines propagate at speed 2 asymptotically as $t\to\infty$.

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~~Heuristics~~ $\cancel{\frac{\partial f}{\partial t} = \Theta \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial \theta^2} - \rho f}$. $\cancel{\to O(t^{3/2})}$

The case of bdd $\theta \in (1, 10)$: similar results / no max people /
by Bouin C ($\exists$ particular solution)
Turanova (speeding)
Bouin - Henderson Ryzhik (refined speeding)

Heuristics for unbounded $\theta \in (1, +\infty)$ (transient regime)

$O(t^{3/2}) \leftarrow$ spatial sorting | B H R PDE / B M R proba

What about the prefactor ? $\longleftrightarrow$ two mechanism of propagation – acceleration –

B/ Methodology : WKB expansion, Large Deviation, ...

$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$.

What about the long time behaviour ?
Trick introduce a time scale
$t = \frac{t'}{\varepsilon}$, $\left(x = \frac{x'}{\varepsilon}\right)$ hyperbolic regime.

$\left[\varepsilon \frac{\partial \rho^\varepsilon}{\partial t'} = \varepsilon^2 \frac{\partial^2 \rho^\varepsilon}{\partial x'^2}\right]$

OK we have explicit formula ... but

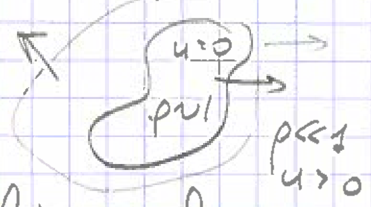
$\rho^\varepsilon(t, x) =$ | Fund sol / Green fn $\frac{1}{\sqrt{4\pi t \varepsilon}} \exp\left(-\frac{|x|^2}{4 t \varepsilon}\right)$

suggests $\left[u^\varepsilon = -\varepsilon \log \rho^\varepsilon\right]$

Back to Fisher's eq. $\frac{\partial u^\varepsilon}{\partial t} + \left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 + 1 - \rho^\varepsilon(t,x) = \varepsilon \frac{\partial^2 u}{\partial x^2}$

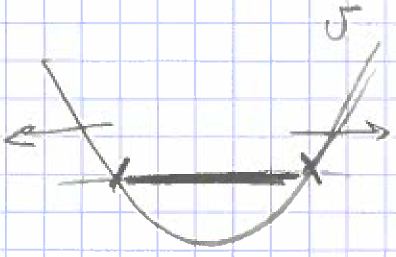
$\underset{\varepsilon \to 0}{\longrightarrow}$ $\rho(t,x) = 0$ if $u > 0$.

| max. people $u \geq 0$ (n-a $\rho \leq 1$).



Freidlin's trick: Compute $v(t,x)$ solution of the "~~...~~ unsaturated pb.

$$\frac{\partial v}{\partial t} + \left|\frac{\partial v}{\partial x}\right|^2 + 1 = 0$$



~~take the~~ $u = \max(0, v)$

fundamental solution $v(t,x) = \frac{|x|^2}{4t} - t \quad \left(e^{-\frac{|x|^2}{4t\varepsilon} + \frac{t}{\varepsilon}} \right)$

c/ Application to the ~~cane toad~~ dispersal evolution

- no max people
- no finite speed of propagation

Alternative: seek $f(t,x,\theta) = \exp(-t\, U(t,x,\theta))$

$$U(t,x,\theta) \underset{t \to +\infty}{\sim} \inf \int_0^t L(\gamma, \dot{\gamma})\, ds + u^0(\gamma(0))$$

Can be made rigorous in weak sense

$$L(\gamma,\dot{\gamma}) = \underbrace{\frac{|\dot{x}|^2}{4\theta(s)} + \frac{|\dot{\theta}|^2}{4} - 1}_{\text{minimize large duration}} + \underbrace{\rho(t, x(t))}_{\text{minimize saturation}}$$

$$\rightsquigarrow \quad \frac{\partial u^\varepsilon}{\partial t} + \left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 = \varepsilon \cancel{\frac{\partial^2 u_\varepsilon}{\partial x^2}}$$

vanishing viscosity limit.

weak solution
lack of unique

First order equation: characteristics curves $x(t)$

$$p^\varepsilon(t) = \frac{\partial u}{\partial x}(t, x(t))$$

$$\dot p(t) = \frac{\partial^2 u}{\partial x^2} \cdot \dot x + \frac{\partial^2 u}{\partial t \partial x} \quad \text{vs.} \quad \frac{\partial^2 u}{\partial t \partial x} + 2 \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} = 0$$

$\longrightarrow$ suggests
$$\begin{cases} \dot x(t) = 2 \dfrac{\partial u}{\partial x} = 2 p(t) \\ \dot p(t) = 0. \end{cases}$$

More generally, $\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}) = 0$ $\quad H(x,p)$ Hamilt. function

$$\begin{cases} \dot x(t) = \dfrac{\partial H}{\partial p} \\ \dot p(t) = -\dfrac{\partial H}{\partial x} \end{cases}$$
Hamilt. system.

Alternative ~~rep~~ formulation (compatible with viscosity solution if H convex // p)

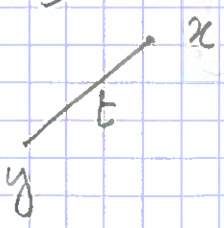
$$u(t,x) = \inf_{\gamma(t) = x} \left\{ \int_0^t L(\gamma(s), \dot\gamma(s))\, ds + u_0(\gamma(0)) \right\}$$

$L(x,v)$ = Convex conj of H: $\frac{\partial L}{\partial v}$ and $\frac{\partial H}{\partial p}$ are reciprocal of each other

$x(t) = \gamma_{opt}$ verifie is sol of Hamilt syste

Heat eq: $u(t,x) = \inf_{y} \left\{ \frac{|x-y|^2}{4t} + u_0(y) \right\}$

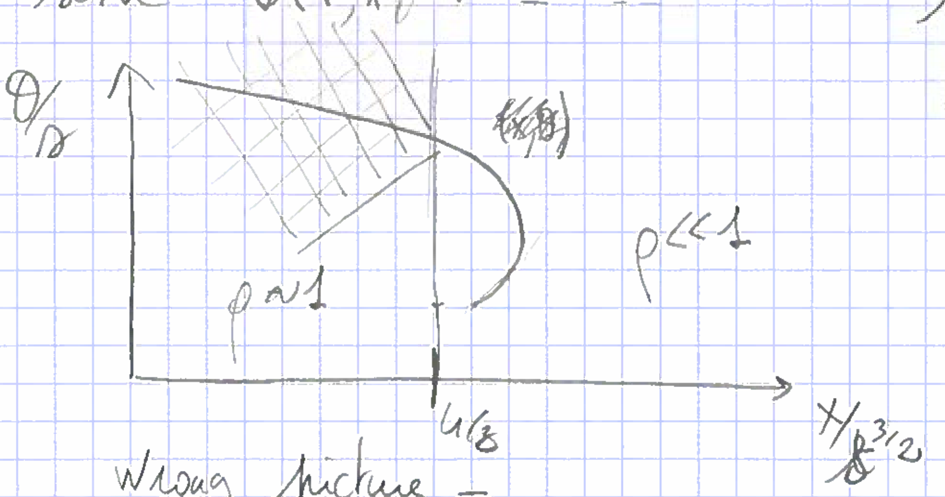
opt. trajectory is a straight line



Main issue. Freidlin's trick does not hold.

~~A~~ solve $\vartheta(t,x,\theta) = \ldots \quad \ldots$ ~~S~~preading

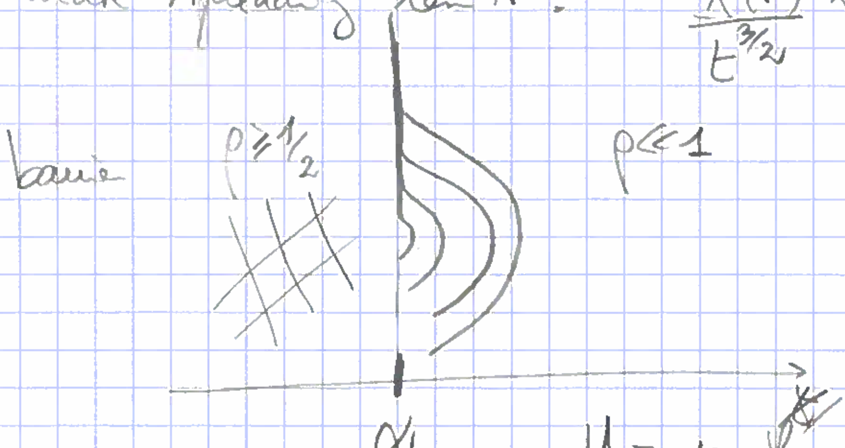
$\frac{\bar{X}(t)}{t^{3/2}} \sim \frac{4}{3}$



Wrong picture.

<u>Th</u> (Henderson, Min Tu)

weak spreading result: $\frac{\bar{X}(t)}{t^{3/2}} \sim (1.315\ldots)$



Assumpt. $\rho(t,x) = \mu \mathbb{1}_{x < \alpha t^{3/2}}$

$\mu \geq \frac{1}{2}$ (level of satur.)

U_α = something tricky.

Pf: For which value of α does the front emerge precisely at $\alpha t^{3/2}$?

Answer $\exists ! \alpha$ root of algebraic eq.

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slides

(2) Maladaptation to a changing environment

$f(t,x)$, $x \in \mathbb{R}^d$ phenotype, trait

$$\frac{\partial f}{\partial t} = B(f) - m(x, \rho(t)) f$$

reproduction | trait dependent / density dependent mortality

Question : stationary state ?
persistence of the population ?

$$0 = B(F) - m(x, \rho) F \qquad \rho = \int F(z)\,dz$$

(number)

2 modes of reproduction : $\cancel{B(F)=}$

(ASEX) • $X \to X + \sigma Y$ → random deviation

(INF) • $(X_1, X_2) \to \frac{X_1+X_2}{2} + \sigma Y$

• $B(F) = K_\sigma * F = \int \frac{1}{\sigma} K\left(\frac{x-x'}{\sigma}\right) F(x')\,dx'$

• $B(F) = \frac{1}{\sqrt{2\pi\sigma^2}} \iint \exp\left(-\frac{1}{2\sigma^2}\left(x - \frac{x_1+x_2}{2}\right)^2\right) \frac{F(x_1)F(x_2)}{\int F}\,dx_1\,dx_2$

(inf operator, Fisher 1918
Bulmer 1980
Mirrahimi Raoul 2013
Barton, Etheridge, Veber 2017)

Main tool: limit $\sigma \to 0$ analogous to semi-classical analysis

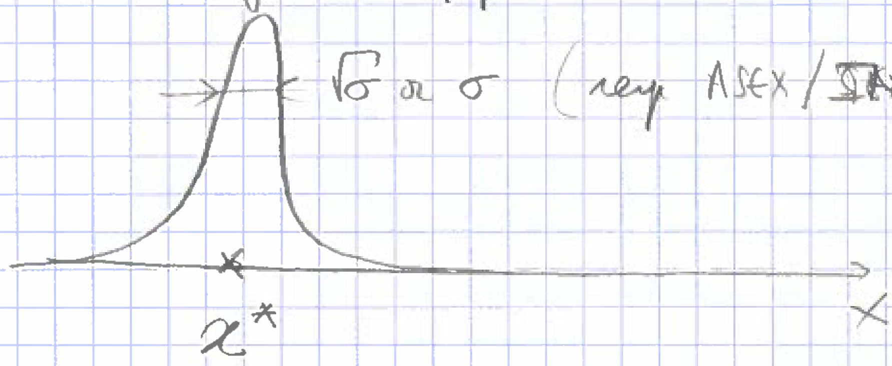
Toy model: (ASEX) $K_\sigma * F = \int K(y)\, F(x-\sigma y)\, dy$

$$= F(x) + \frac{\sigma^2}{2}\left(\int y^2 K\, dy\right) \frac{\partial^2 F}{\partial x^2}$$

→ second derivative operator with ~~small~~ vanishing viscosity (as yesterday).

Heuristics. the population should concentrate around some traits (selection of the fittest)

Variance of the population $\ll 1$



$\sqrt{\sigma}$ or σ (resp ASEX/~~[illegible]~~).

Methodology: ~~to~~ unravel Dirac masses

$$U(t,x) = -\sigma^{\{1,2\}} \log F(t,x)$$

REF Diekmann, Jabin, Mischler, Perthame
Barles - P
B-M-P
Lorenzi —

$$m(x,\rho) = \frac{B_\sigma(F)}{F}$$

ASEX : $\displaystyle\int \frac{1}{\sigma} K\left(\frac{x-x'}{\sigma}\right) e^{-\frac{U(x')}{\sigma}} e^{\frac{U(x)}{\sigma}} \, dx'$

$$= \int K(y) \exp\left(\frac{U(x) - U(x-\sigma y)}{\sigma} \right) dy$$

$$\xrightarrow{\text{Viscosity limit}} \int K(y) \exp(y \cdot \partial_x U(x)) \, dy = \hat{K}(\partial_x U)$$

bi-Laplace Transform

Compatible with max. principle

$m(x,\rho) = \hat{K}(\partial_x U)$ eq. mutation / selection

Application : changing environment at speed σc (given)

$m(\underbrace{x-ct}_{z},\rho) = \boxed{m+\rho}$

$$\partial_t f \leftrightarrow \partial_t f - c\partial_z f$$

Eq. $\quad -c\,\partial_z F + m(z,\rho) F = K * F$

$\quad c\,\partial_z U + m(z,\rho) = \hat{K}(\partial_z U)$ eq. + drift.

z ~~$m(0,\rho)$~~

Simple $\quad \rho = 1 - \underbrace{m(0)}_{\text{strange}} - L(c)$

INF . $\iint \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{\sigma^2}\left(x-\frac{x_1+x_2}{2}\right)^2\right) F(x_1)\frac{F(x_2)}{\int F} dx_1 dx_2$.

Singular integral ?

$*\quad F = \exp\left(-\frac{1}{2\sigma^2}(x-x^*)^2 - U(x)\right)$

$\longrightarrow \quad \frac{B(F)}{F} \longrightarrow \exp\left(U(x) - 2U\left(\frac{x+x^*}{2}\right) + U(x^*)\right)$.

Intuiti

Corrector eq.

$U(x) - 2U\left(\frac{x+x^*}{2}\right) + U(x^*) = -\log(m(x,\rho))$ ⚡

Affin functi- belong to kernel.

Expect . $U(x) = \cancel{ax}\ \cancel{b}$ ~~a+c~~ Quadratic -

$p + q\cdot(x-x^*) + \sum 2^k \log m\left(2^{-k}(x-x^*)+x^*, \rho\right)$

TH (C. Garnier, Pabout)

prove the convergence / identify $q = \frac{\partial^3 m(x^*)}{2\partial_m^2(x^*)}$

Multiple issues : no max peak

lack of unique ~~ne~~ ρ.

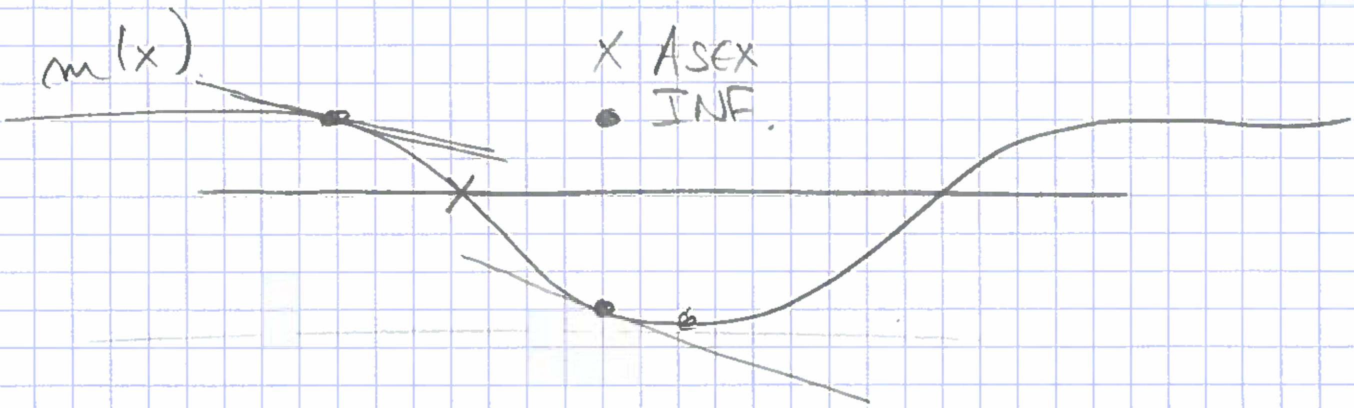
Appl to a Moving optimum : $\underline{\sigma^2 c}$.

$$U(x) - 2U\left(\frac{x+x^*}{2}\right) + U(x^*) = \log\left(m(x^*) + c\,\frac{x-x^*}{2}\right)$$

$$p = 1 - m(x^*) \quad \text{(demographic eq.)}$$

$$\frac{\partial m}{\partial x}(x^*) = -c$$

Illustration : tipping point (Osmond & Klausmeier 2017)



Why? Age selection.