Anisotropic Diffusion in Oriented Environments can lead to Singularity Formation

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Abstract

We consider a non-isotropic diffusion equation of the form $u_t = \nabla\nabla(D(x)u)$ in two dimensions, which arises in various applications including the modelling of wolf movement along seismic lines and the invasive spread of certain brain tumours along white matter neural fiber tracts. We consider a degenerate case, where the diffusion tensor $D(x)$ has a zero-eigenvalue for certain values of x.
Based on a regularisation procedure and various pointwise and integral a-priori estimates, we establish the global existence of very weak solutions to the degenerate limit problem. Moreover, we show that in the large time limit these solutions approach profiles which exhibit a Dirac-type mass concentration phenomenon on the boundary of the region in which diffusion is degenerate, which is quite surprising for a linear diffusion equation. The results are illustrated by numerical examples.

Key words: anisotropic diffusion, degenerate diffusion, large time behaviour, singularity formation, pattern formation
AMS Classification: 35B36, 35B44, 35K67, 92B05

1 Introduction

In this paper we consider a linear parabolic equation of the form

$$u_t = \nabla\nabla(D(x)u) \tag{1.1}$$

on a bounded domain in $\mathbb{R}^n$ with homogeneous Neumann boundary conditions. The diffusion coefficient $D(x) = (D^{ij}(x))_{i,j}$ is an n-dimensional tensor which describes anisotropic diffusion in different directions of the environment. We use the notation

$$\nabla\nabla(Du) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(D^{ij}(x)u(x)).$$

Here we assume that $D(x)$ is positive semi-definite, and we show an example where model (1.1) has solutions that converge to Dirac-type singularities as $t \to \infty$.

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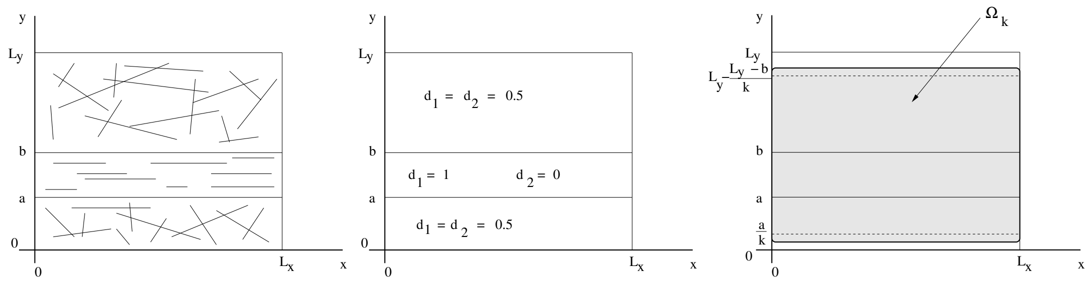


Figure 1: Left: schematic of high anisotropy on the path between $y = a$ and $y = b$, and isotropic diffusion off the path. Middle: diffusion coefficients for the degenerate problem (1.4) are smooth approximations of the values that are indicated in this figure. Right: Construction of the smooth domain Ω_k which is used for the smooth approximation in problem (3.2)

This problem arises in the modelling of individual movement in strongly anisotropic environments. Two examples, both under current investigation, are the patterns of wolf movement [18, 13] and the invasive spread of gliomas (a certain class of brain tumour) [9, 2, 4]. In the former, wolves in Northern Canada have been observed to exploit the linear roads and seismic lines cut into the forest by oil exploration companies to increase their hunting range. This presents a significant thread to caribou populations and strategies to reduce the impact of these lines are required. In the latter, glioma cells are believed to follow the aligned fiber tracts in white matter, facilitating the invasive spread into healthy tissue. Predicting the pattern of glioma growth promises the design of more efficient treatment strategies [14, 5].

A common feature to these and other problems is the directional guidance provided by roads, seismic lines or white matter tracts. Mathematically these linear features present a highly anisotropic environment, where individuals preferentially move along these features [13]. Here we focus on an idealised stretch of road or white matter tract: oriented horizontally and embedded in an otherwise homogeneous tissue, as illustrated in Figure 1 (left). We specifically consider the singular case of individuals that never escape a linear feature once entered. Although degenerate, we argue that this model offers a good explanation of overshooting, which we observe in numerical simulations. In Figure 2 we show a numerical simulation where in the aligned region, the diffusion in y-direction is close to zero. We see clearly the formation of highly concentrated aggregates along the lines $y = a$ and $y = b$. We will prove in Theorem 1.2 that for $\varepsilon \to 0$ these solutions approach δ-singularities. The details of the simulations are given in Section 6.

Models of the form (1.1) have been derived from detailed transport equations for the movement of wolf, or cells, respectively (see Section 1.1 below and [11, 13]). Usually, anisotropic diffusion is associated with a term in divergence form[1].

$$u_t = \nabla(D(x)\nabla u). \tag{1.2}$$

[1]Note that (1.1) is also in divergence form, however, the term "divergence form" is usually associated with (1.2)

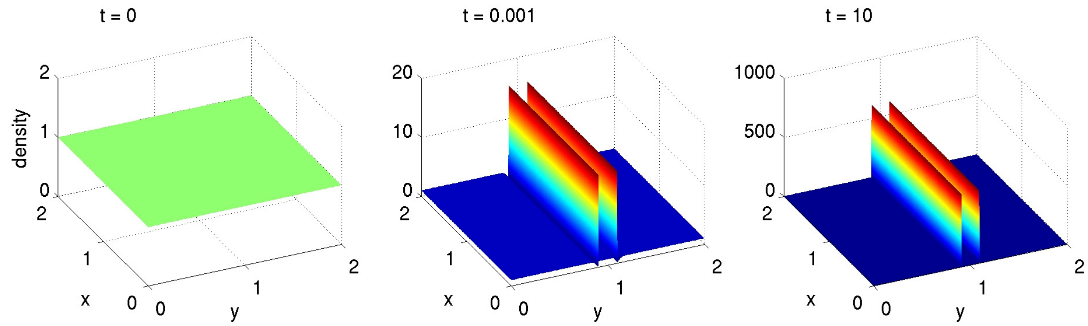


Figure 2: Simulation of the anisotropic diffusion model, system (1.4) together with $u_0(x,y) = 1$ and smooth diffusion coefficients given in (6.1). Time evolution showing solutions using $\epsilon = 0.001$, $a = 0.9$ and $b = 1.1$. The population accumulates into two extremely concentrated ridges at the interface between the isotropic and aligned regions, with negligible subsequent movement within simulation timescales. Simulations performed as described in Section 6.

This equation (1.2) obeys the maximum principle, and steady states under homogeneous Neumann boundary conditions are constant solutions. The maximum principle does not apply to (1.1) and non-constant steady states are typical for (1.1) (see [13]). This can easily be understood in the one-dimensional situation: Considering $u_t = (d(x)u)_{xx}, d(x) > 0$ and defining $v(x,t) := d(x)u(x,t)$, we see that v solves $v_t = d(x)v_{xx}$, which is a standard diffusion equation. Hence v satisfies a maximum principle and, under homogeneous boundary conditions, v has constant steady states, e.g. $\bar{v}(x) \equiv c$. The corresponding steady state for u is then a non-homogeneous solution given by $\bar{u}(x) = \frac{c}{d(x)}$.

In this paper, we go one step further and show that in a certain degenerate limit case, equations of the form (1.1) when posed in bounded two-dimensional domains under no-flux boundary conditions can give rise to solutions that exhibit δ-singularities in the long time limit as $t \to \infty$. The model below characterises a typical piece of a road (or white matter track), and the singular behaviour describes individuals which get trapped at the side of the road.

1.1 Model derivations

The classical anisotropic diffusion model (1.2) follows directly from the assumption that the particle flux is a linear transformation of the particle gradient (like a Fourier law, or a Fick law) of the form $J = -D\nabla u$. The flux of the fully anisotropic model (1.1) is $J = -\nabla\cdot(Du)$, i.e. the divergence of a matrix quantity, and a direct physical interpretation is uncertain. There are, however, biological models which naturally lead to forms such as (1.1) and we briefly discuss three such models:

(i) Transport equations: We have extensively worked on the modelling of cell (or organism) movement with transport equations (see [12, 21, 10, 11, 23, 13]): this work is the main motivation for the present study of (1.1). The transport equation is a mesoscopic model for movement, treating cell density as a continuum and based on individual movement parameters such as speeds, direction of movement, turning rates and turning angles. Developing a full theory of transport equations for cell movement (see, for example, [12, 21, 10, 11, 23, 13]) is outside the scope of the current paper and we simply summarize the major steps.

Our principal application has been the movement of individuals in a heterogeneous environment that contains directional information, e.g. collagen fibers in tissue, neural fiber tracks in the brain, roads or seismic lines in forests or the slope of the terrain. The directional information can be encoded in a directional distribution $q(t,x,\theta)$, $\theta \in \mathbb{S}^{n-1}$, $\int q(t,x,\theta)d\theta = 1$, $q \geq 0$, and individuals moving in such an environment are modelled through their density $p(t,x,v)$ that satisfies the transport equation

$$p_t + v \cdot \nabla p = \mu \left(\tilde{q} \int p dv - p \right),$$

where $v \in V$ and V is a bounded set of possible cell velocities. The parameter μ describes a turning rate and $\tilde{q}$ denotes the distribution q lifted to the space V ($\tilde{q}(t,x,v) = \beta q(t,x,v/||v||)$, where β is chosen such that $\int_V \tilde{q}(t,x,v)dv = 1$). As shown in detail in [11, 13], a parabolic scaling of the form $\tau = \varepsilon^2 t, \xi = \varepsilon x$ leads (in the limit $\varepsilon \to 0$) to the fully anisotropic diffusion model (1.1) and a convergence result for the isotropic case is given in [12]. For example, if q is symmetric ($q(t,x,-v) = q(t,x,v)$), then

$$D = \frac{1}{\mu} \int_V v v^T \tilde{q}(t,x,v) dv,$$

i.e. the diffusion tensor is the variance-covariance matrix of the underlying fiber network distribution (see [11, 13]). The formulation of a diffusion tensor from the underlying network structure allows one to directly connect the impact of environmental structure on the movement paths of a typical individual to diffusion-type models: for example, in [13] we employed this formulation to connect a network of seismic lines in boreal forest to the spread of wolves, while in [24] we connected brain-imaging data to diffusion models for anisotropic invasion of gliomas (a class of brain tumour). For further details on the employment of anisotropic diffusion models in brain tumor spread, see the references in [14, 15, 19, 24].

(ii) Random walks: A random walk on a one-dimensional equidistant grid can be described through a master equation for the density $u(x,t)$ of stochastic independent random walkers as follows

$$\frac{d}{dt} u_i = T_{i-1}^+ u_{i-1} + T_{i+1}^- u_{i+1} - (T_i^+ + T_i^-) u_i,$$

where $u_i = u(x_i,t)$ and $T_i^{\pm}\Delta t$ are the transitional probabilities for a jump to the right (+) or left (-) per unit of time Δt. The choice of

$$T_i^{\pm} = (\Delta x)^{-2} T(x_i) \tag{1.3}$$

leads in the limit of $\Delta x \to 0$ to the fully nonhomogeneous model (in 1-D) [22, 20]:

$$u_t = (Tu)_{xx}.$$

Other choices of $T_i^{\pm}$ lead to other diffusion models. For example $T_i^{\pm} = (\Delta x)^{-2} T(x_{i\pm 1/2})$ leads to the physical form of $u_t = (Tu_x)_x$. The choice of (1.3) is a natural choice, for example to model a "myopic walker" that jumps according to information at its current location, and hence the model of the form (1.1) is a natural candidate for nonhomogeneous and nonisotropic diffusion.

(iii) Ideal free distribution: The ideal free distribution refers to a spatial distribution of species in heterogeneous landscapes, where each individual has the same fitness [6, 7]. Given a spatially nonhomogeneous landscape, as described by a non-constant carrying capacity $\mu(x)$, the concept of an ideal free distribution requires the existence of a nonhomogeneous steady state proportional to $\mu(x)$. Cosner, Cantrel, Lewis and others [6, 7, 17] have studied which forms of standard reaction-diffusion model support the ideal free distribution, showing that a choice of $D(x) = \mu(x)^{-1}$ and

$$u_t = (D(x)u)_{xx} + \alpha u(\mu(x) - u)$$

has this capacity. Again, the diffusion part has the form (1.1).

The above example provides a number of motivating reasons for a deeper understanding of models of the form (1.1).

In the following section we describe the degenerate limit problem studied in this paper, along with a regularisation through which we shall obtain solutions for the degenerate problem when the regularisation parameter approaches zero. We next list our main results, the first of which ensures stabilisation of the above solutions to certain limit profiles in the large time limit (Theorem 1.1); secondly, Theorem 1.2 then shows that when reduced to a spatially one-dimensional framework on integration with respect to one of the two space variables, this convergence involves δ singularities in the long time asymptotics. The proofs of the main results are based on a priori estimates which will be provided in Section 3. These will be used to show global existence in the degenerate limit problem (Section 4), and to characterise the large time behaviour of solutions to both the regularised and the degenerate problem (Section 5). In Section 6 we show some typical numerical simulations that support our results.

1.2 The Model

We consider the initial-boundary value problem

$$\begin{cases} u_t = \Big(d_1(y)u\Big)_{xx} + \Big(d_2(y)u\Big)_{yy} & (x,y) \in \Omega,\ t > 0, \\ \Big((d_1(y)u)_x, (d_2(y)u)_y\Big) \cdot \nu = 0 & (x,y) \in \partial\Omega,\ t > 0, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{1.4}$$

in the two-dimensional rectangle $\Omega := (0, L_x) \times (0, L_y)$ with $L_x > 0$ and $L_y > 0$, where ν denotes the outward normal vector field on $\partial\Omega$.

The coefficient functions d_1 and d_2 are supposed to be smooth approximations of the prototypical choices

$$d_{1,prot}(y) = \begin{cases} 1 & \text{if } y \in [a,b], \\ \frac{1}{2} & \text{if } y \in [0,L_y] \setminus [a,b], \end{cases} \qquad \text{and} \quad d_{2,prot}(y) = \begin{cases} 0 & \text{if } y \in [a,b], \\ \frac{1}{2} & \text{if } y \in [0,L_y] \setminus [a,b], \end{cases} \tag{1.5}$$

where $0 < a < b < L_y$, as shown in Figure 1 (middle). More precisely, we shall assume that there exist $a, b \in (0, L_y)$ such that $a < b$ and

$$\begin{cases} d_1 \in C^0([0,L_y]) \text{ is positive in } [0,L_y], & \text{and that} \\ d_2 \in C^2([0,L_y]) \text{ is positive in } [0,L_y] \setminus [a,b] & \text{and} \\ d_2 \equiv 0 \text{ in } [a,b]. \end{cases} \tag{1.6}$$

As for the initial data, we shall assume that

$$u_0 \in C^0(\bar{\Omega}) \quad \text{ is nonnegative.} \tag{1.7}$$

Observe that according to (1.6) the diffusion in (1.4) is degenerate throughout the subdomain $\Omega_{ab} := (0, L_x) \times (a, b)$ of Ω. Not only for technical reasons, but also in order to compare the respective solution properties, we shall study (1.4) along with certain regularised problems with non-degenerate diffusion. For this purpose, we suppose that we are given two families $(d_{1\varepsilon})_{\varepsilon \in (0,1)}$ and $(d_{2\varepsilon})_{\varepsilon \in (0,1)}$ of functions $d_{1\varepsilon}, d_{2\varepsilon} \in C^\infty([0, L_y])$ such that

$$\begin{cases} \varepsilon \le d_{2\varepsilon} \le d_2 + 1 & \text{in } [0,L_y] \quad \text{for all } \varepsilon \in (0,1), \\ d_{2\varepsilon} \equiv \varepsilon & \text{in } [a,b] \quad \text{for all } \varepsilon \in (0,1) \quad \text{and} \\ (d_{1\varepsilon}, d_{2\varepsilon}) \to (d_1, d_2) & \text{in } C^0([0,L_y]) \times C^2([0,L_y]) \quad \text{as } \varepsilon \searrow 0. \end{cases} \tag{1.8}$$

For instance, if both d_1 and d_2 are smooth in $[0, L_y]$, this is consistent with the choices $d_{1\varepsilon} \equiv d_1$ and $d_{2\varepsilon} \equiv d_2 + \varepsilon$. For $\varepsilon \in (0, 1)$ we then consider the problem

$$\begin{cases} u_{\varepsilon t} = \Big(d_{1\varepsilon}(y) u_\varepsilon\Big)_{xx} + \Big(d_{2\varepsilon}(y) u_\varepsilon\Big)_{yy} & (x,y) \in \Omega, \ t > 0, \\ \Big((d_{1\varepsilon}(y)u_\varepsilon)_x, (d_{2\varepsilon}(y)u_\varepsilon)_y)\Big) \cdot \nu = 0 & (x,y) \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x,y,0) = u_0(x,y) & (x,y) \in \Omega, \end{cases} \tag{1.9}$$

with u_0 as before. For later reference we call (1.9) the u_ε-problem. Since (1.9) is a linear uniformly parabolic problem in a bounded domain with Lipschitz boundary, various standard approaches may be applied to see that (1.9) indeed is solvable in a natural weak sense. In order to be able to deal with smooth functions, we prefer a method based on smooth approximations of Ω (cf. (3.2) and Lemma 4.1 below).

We shall then see that indeed some globally defined generalised solution u of (1.4) can be obtained as the limit of the above solutions u_ε along an appropriate sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ in the sense specified in (4.5) below (cf. Definition 2.1).

1.3 Main results

The main results of this paper characterise the large time behaviour of this solution. First, we show that outside the closure of the alignment domain $\Omega_{ab} = (0, L_x) \times (a, b)$ the solution converges to zero and inside Ω_{ab} it converges to a steady state which is independent of x.

Theorem 1.1 *Let u denote the global very weak solution determined by (4.5) below, and let $\Omega_{ab} = (0, L_x) \times (a, b)$.*

i) We have

$$u(\cdot, t) \to 0 \qquad \text{in } L^2_{loc}(\bar{\Omega} \setminus \bar{\Omega}_{ab}) \quad \text{as } t \to \infty. \tag{1.10}$$

ii) The solution satisfies

$$u(x, y, t) \rightharpoonup \overline{u}_0(y) \qquad \text{in } L^2_{loc}([0, L_x] \times (a, b)) \quad \text{as } t \to \infty, \tag{1.11}$$

where $\overline{u}_0 \in C^0([a, b])$ is the function defined by

$$\overline{u}_0(y) := \frac{1}{L_x} \int_0^{L_x} u_0(x, y) dx, \qquad y \in [a, b]. \tag{1.12}$$

Since u enjoys a natural mass conservation property (see Corollary 4.3), (1.10) and (1.11) entail that a mass concentration must occur at the horizontal boundaries $(0, L_x) \times \{a\}$ and $(0, L_x) \times \{b\}$ of Ω_{ab}. Since one can show that $u(\cdot, t)$ is bounded in $L^\infty(\Omega)$ for each finite t (Proposition 4.2), this happens only in the limit $t \to \infty$. Indeed, we have the following.

Theorem 1.2 *Let u denote the global very weak solution of (1.4) given by (4.5), and set*

$$U(y, t) := \int_0^{L_x} u(x, y, t) dx, \qquad y \in [0, L_y], \ t > 0, \tag{1.13}$$

and

$$U_0(y) := \int_0^{L_x} u_0(x, y) dx, \qquad y \in [0, L_y]. \tag{1.14}$$

Then in the sense of Borel measures over $[0, L_y]$ we have

$$U(y, t) \overset{\star}{\rightharpoonup} \chi_{(a,b)}(y) \cdot U_0(y) + m_1 \cdot \delta(y - a) + m_2 \cdot \delta(y - b) \qquad \text{as } t \to \infty, \tag{1.15}$$

where $\chi_{(a,b)}$ is the characteristic function of (a, b), δ denotes the one-dimensional Dirac measure and

$$m_1 := \int_0^a \int_0^{L_x} u_0(x, y) dx dy \qquad \text{and} \qquad m_2 := \int_b^{L_y} \int_0^{L_x} u_0(x, y) dx dy.$$

The above type of behaviour is in sharp contrast to the asymptotics in each of the regularised problems (1.9), since for $\varepsilon > 0$ the solution converges to an x-independent steady state:

Proposition 1.3 *For all $\varepsilon \in (0,1)$, the weak solution u_ε of (1.9) constructed in Lemma 4.1 satisfies*

$$u_\varepsilon(\cdot,t) \to u_{\varepsilon\infty} \quad \text{in } L^2(\Omega) \qquad \text{as } t \to \infty, \tag{1.16}$$

where

$$u_{\varepsilon\infty}(x,y) := \frac{A_\varepsilon}{d_{2\varepsilon}(y)}, \quad (x,y) \in \Omega, \qquad \text{with} \qquad A_\varepsilon := \frac{\int_\Omega u_0}{L_x \cdot \int_0^{L_y} \frac{1}{d_{2\varepsilon}(y)} dy}. \tag{1.17}$$

Taking the limit of $\varepsilon \searrow 0$ in (1.17) we directly see that the asymptotic profiles of the solutions of the u_ε-problem (1.9) approach a step-type distribution

$$u_{\varepsilon\infty}(x,y) \to \begin{cases} \frac{\int_\Omega u_0}{L_x \cdot (b-a)} & \text{if } y \in (a,b), \\ 0 & \text{if } y \in (0,L_y) \setminus [a,b] \end{cases}$$

a.e. in Ω, which is different from (1.15) and no extreme mass concentration phenomenon occurs.

Before going into details, let us finally mention that in view of our choice (1.6) of the diffusion tensor, investigating the dynamics in (1.4) and in (1.9) partially reduces to studying the corresponding one-dimensional initial-boundary value problem

$$\begin{cases} U_t = (d_2(y)U)_{yy}, & y \in (0,L_y), \ t > 0, \\ U_y = 0, & y \in \{0, L_y\}, \ t > 0, \\ U(y,0) = U_0(y), & y \in (0,L_y). \end{cases} \tag{1.18}$$

formally satisfied by the function U defined in (1.13), with U_0 as in (1.14). Accordingly, our analysis on (1.4) will in many places reflect the distinctiveness of the direction of the spatial variable y. In particular, it will turn out that U in fact is a very weak solution of (1.13) in an appropriate sense (see Proposition 4.4), and that this solution in fact is unique within a certain function class (cf. Proposition 4.5).
Of course, one-dimensional parabolic problems with prescribed spatially fixed degeneracies have been studied quite thoroughly in the literature, yielding expected ([3]) and unexpected results ([8]). An important peculiarity of the problem considered here, however, is that the diffusivity is supposed to vanish in a spatial region which has *positive measure*, and which does *not* touch the boundary parts where $y \in \{0, L_y\}$. Correspondingly, phenomena like the somewhat counterintuitive observations made in Theorem 1.2 and Proposition 1.3 apparently have not been detected before in any related context.

2 Very weak solutions

To be able to include the mass concentration phenomenon in our solution theory, and to be able to pass to the appropriate limits for $\varepsilon \searrow 0$, we define very weak solutions for the degenerate problem (1.4)

Definition 2.1 *Let $T \in (0,\infty]$. By a* very weak solution *of (1.4) in $\Omega \times (0,T)$ we mean a function $u \in L^1_{loc}(\bar\Omega \times [0,T))$ which satisfies*

$$-\int_0^T \int_\Omega u\varphi_t = \int_\Omega u_0 \varphi(\cdot,0) + \int_0^T \int_\Omega \Big\{ d_1(y) u \varphi_{xx} + d_2(y) u \varphi_{yy} \Big\} \tag{2.1}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,T))$ which are such that $(d_1(y)\varphi_x, d_2(y)\varphi_y) \cdot \nu = 0$ on $\partial\Omega \times (0,T)$. In the case $T = \infty$ we also call u a global very weak solution of (1.4).

2.1 Conservation properties of arbitrary very weak solutions

Let us state some useful mass conservation properties of arbitrary very weak solutions of (1.4), regardless of whether they can be approximated by solutions of regularised u_ε-problem (1.9).

Lemma 2.1 *Let $T \in (0,\infty]$ and u be a very weak solution of (1.4) in $\Omega \times (0,T)$. Then for all $y_\star \in [a,b]$ there exists a null set $N \subset (0,T)$ such that*

$$\int_0^{y_\star} \int_0^{L_x} u(x,y,t)dxdy = \int_0^{y_\star} \int_0^{L_x} u_0(x,y)dxdy \qquad \text{for all } t \in (0,T) \setminus N \tag{2.2}$$

and

$$\int_{y_\star}^{L_y} \int_0^{L_x} u(x,y,t)dxdy = \int_{y_\star}^{L_y} \int_0^{L_x} u_0(x,y)dxdy \qquad \text{for all } t \in (0,T) \setminus N. \tag{2.3}$$

Proof. Given $y_\star \in [a,b]$, we introduce

$$z(t) := \int_0^{y_\star} \int_0^{L_x} u(x,y,t)dxdy, \qquad t \in (0,T). \tag{2.4}$$

Then clearly $z \in L^1_{loc}([0,T))$, and hence almost every point in $(0,T)$ is a Lebesgue point of z. Therefore there exists a null set $N \subset (0,T)$ such that

$$\frac{1}{h}\int_{t_0}^{t_0+h} z(t)dt \to z(t_0) \quad \text{as } h \searrow 0 \qquad \text{for all } t_0 \in (0,T) \setminus N. \tag{2.5}$$

In order to use this in an appropriate way, we fix $t_0 \in (0,T) \setminus N$ and $h > 0$ such that $t_0 + h < T$, and pick sequences $(\chi_j)_{j\in\mathbb{N}} \subset C_0^\infty([0,L_y))$ and $(\psi_j)_{j\in\mathbb{N}} \subset C_0^\infty([0,T))$ such that

$$\chi_j \equiv 1 \quad \text{in } [0,a] \qquad \text{and} \qquad \chi_j \equiv 0 \quad \text{in } [b,L_y] \qquad \text{for all } j \in \mathbb{N},$$

and such that

$$\chi_j \to \chi \quad \text{in } L^2((0,L_y)) \qquad \text{and} \qquad \psi_j \to \psi \quad \text{in } W^{1,2}((0,T)) \qquad \text{as } j \to \infty, \tag{2.6}$$

where

$$\chi(y) := \begin{cases} 1 & \text{if } y \in [0,y_\star], \\ 0 & \text{if } y \in (y_\star, L_y], \end{cases} \tag{2.7}$$

and

$$\psi(t) := \begin{cases} 1 & \text{if } t \in [0, t_0], \\ -\frac{1}{h}(t - t_0) + 1 & \text{if } t \in (t_0, t_0 + h], \\ 0 & \text{if } t \in (t_0 + h, T). \end{cases} \tag{2.8}$$

We now choose φ_j with $\varphi_j(x, y, t) := \chi_j(y)\psi_j(t)$, $(x, y, t) \in \bar{\Omega} \times [0, T)$, as a test function in (2.1), which is possible since evidently $\frac{\partial \varphi_j}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$. We thus obtain

$$\begin{aligned} -\int_0^T \int_\Omega u(x, y, t)\chi_j(y)\psi_{jt} d(x, y)dt &= \int_\Omega u_0(x, y)\chi_j(y)d(x, y) \\ &\quad + \int_0^T \int_\Omega d_2(y)u(x, y, t) \cdot \chi_{jyy}(y) \cdot \psi_j(t)d(x, y)dt, \end{aligned}$$

because $\varphi_{jxx} \equiv 0$. Here the last term vanishes due to the fact that $\operatorname{supp} \chi_{jyy} \subset (a, b)$ and $d_2 \equiv 0$ in (a, b). But thereupon (2.6) allows us to take $j \to \infty$ to gain

$$-\int_0^T \int_\Omega u(x, y, t)\chi(y)\psi_t(t)d(x, y)dt = \int_\Omega u_0(x, y)\chi(y)d(x, y),$$

which in view of (2.7), (2.8) and (2.4) is equivalent to saying that

$$\begin{aligned} \frac{1}{h}\int_{t_0}^{t_0+h} z(t)dt &= \frac{1}{h}\int_{t_0}^{t_0+h} \int_0^{y_\star} \int_0^{L_x} u(x, y, t)dxdydt \\ &= -\int_0^T \int_\Omega u(x, y, t)\chi(y)\psi_t(t)d(x, y)dt \\ &= \int_0^{y_\star} \int_0^{L_x} u_0(x, y)dxdy \end{aligned}$$

holds for all $h > 0$ with $t_0 + h < T$. In view of (2.5), this shows that indeed (2.2) is valid for all $t_0 \in (0, T) \setminus N$. The proof of (2.3) can be run in quite a similar way. $\square$

An immediate consequence is that the total mass is conserved in the following sense.

Corollary 2.2 *Let $T \in (0, \infty]$ and u be any very weak solution of (1.4) in $\Omega \times (0, T)$. Then*

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \qquad \textit{for a.e. } t \in (0, T). \tag{2.9}$$

Proof. We only need to fix an arbitrary $y_\star \in [a, b]$ and add the resulting identities (2.2) and (2.3). $\square$

In the alignment domain $\Omega_{ab} = (0, L_x) \times (a, b)$ where formally no diffusion occurs in the direction of the variable y, more detailed information is available:

Lemma 2.3 *Let u be a very weak solution of (1.4) in $\Omega \times (0, T)$ for some $T \in (0, \infty]$. Then there exists a null set $N_\star \subset (a, b)$ with the property that for all $y \in (a, b) \setminus N_\star$ one can find a null set $N_\star(y) \subset (0, T)$ such that*

$$\int_0^{L_x} u(x, y, t)dx = \int_0^{L_x} u_0(x, y)dx \qquad \textit{for all } t \in (0, T) \setminus N_\star(y). \tag{2.10}$$

Proof. Since $u \in L^1_{loc}(\bar{\Omega} \times [0,T))$, the function z defined on $(0, L_y)$ by $z(y) := \int_0^{L_x} u(x,y,t)dx$, $y \in (0, L_y)$, belongs to $L^1((0,L_y))$ for all $t \in (0,T) \setminus N_1$, where $|N_1| = 0$. Taking $N \subset (0,T)$ as provided by Lemma 2.1 and letting $N_2 := N_1 \cup N$, twice applying (2.2) we infer that

$$\frac{1}{h}\int_{y_\star}^{y_\star+h}\int_0^{L_x} u(x,y,t)dxdy = \frac{1}{h}\int_{y_\star}^{y_\star+h}\int_0^{L_x} u_0(x,y)dxdy \quad \text{for all } t \in (0,T) \setminus N_2, \text{ any } y_\star \in [a,b] \text{ and each } h \in (0, b-y_\star). \tag{2.11}$$

Now by definition of N_1, for each $t \in (0,T) \setminus N_2 \subset (0,T) \setminus N_1$ we know that almost every point in $(0, L_y)$ is a Lebesgue point of z as defined above; that is, for any such t we can find a null set $N_2(t) \subset (a,b)$ such that

$$\frac{1}{h}\int_{y_\star}^{y_\star+h}\int_0^{L_x} u(x,y,t)dxdy \to \int_0^{L_x} u(x,y_\star,t)dx \quad \text{as } h \searrow 0 \quad \text{for all } y_\star \in (a,b) \setminus N_2(t).$$

Since by continuity we have

$$\frac{1}{h}\int_{y_\star}^{y_\star+h}\int_0^{L_x} u_0(x,y)dxdy \to \int_0^{L_x} u_0(x,y_\star)dx \qquad \text{for all } y_\star \in (0, L_y),$$

we thus obtain from (2.11) that

$$\int_0^{L_x} u(x,y,t)dx = \int_0^{L_x} u_0(x,y)dx \qquad \text{for all } t \in (0,T) \setminus N_2 \text{ and any } y \in (a,b) \setminus N_2(t). \tag{2.12}$$

Now by the Fubini-Tonelli theorem, the exceptional set

$$\hat{N} := \Big((0,L_y) \times N_2\Big) \cup \Big\{(y,t) \in (a,b) \times (0,T) \ \Big| \ t \in (0,T) \setminus N_2 \text{ and } y \in N_2(t)\Big\}$$

has measure zero in $(a,b) \times (0,T)$ and can be rewritten in the form

$$\hat{N} = \Big(N_\star \times (0,T)\Big) \cup \Big\{(y,t) \in (a,b) \times (0,T) \ \Big| \ y \in (a,b) \setminus N_\star \text{ and } t \in N_\star(y)\Big\}$$

with certain null sets $N_\star \subset (a,b)$ and $N_\star(y) \subset (0,T)$ for $y \in (a,b) \setminus N_\star$. Therefore (2.12) is equivalent to (2.10). □

For later use, let us state the following immediate consequence of the above lemma.

Corollary 2.4 *Let $T \in (0,\infty]$ and u be a very weak solution of (1.4) in $\Omega \times (0,T)$. Then there exists a null set $N_\star \subset (a,b)$ such that whenever $t_0 \in (0,T)$ is such that $t_0 + 1 < T$, we have*

$$\int_{t_0}^{t_0+1}\int_0^{L_x} u(x,y,t)dxdt = \int_0^{L_x} u_0(x,y)dx \qquad \textit{for all } y \in (a,b) \setminus N_\star. \tag{2.13}$$

Proof. We only need to integrate (2.10) over $t \in (t_0, t_0+1)$. □

3 A priori estimates

In our arguments below it will be convenient to deal with smooth solutions, which (1.9) does not necessarily possess in view of the fact that the rectangle Ω only has Lipschitz boundary. We therefore approximate solutions of u_ε-problem (1.9) as follows: As shown in Figure 1 (right) we fix a sequence $(\Omega_k)_{k\in\mathbb{N}}$ of bounded domains $\Omega_k \subset \mathbb{R}^2$ with smooth boundary such that

$$(0, L_x) \times \Big(\frac{a}{k}, L_y - \frac{L_y - b}{k}\Big) \subset \Omega_k \subset \Omega \qquad \text{for all } k \in \mathbb{N}, \tag{3.1}$$

and consider the problems

$$\begin{cases} u_{\varepsilon kt} = \Big(d_{1\varepsilon}(y)u_{\varepsilon k}\Big)_{xx} + \Big(d_{2\varepsilon}(y)u_{\varepsilon k}\Big)_{yy} & (x,y) \in \Omega_k,\ t > 0, \\ \Big((d_{1\varepsilon}(y)u_{\varepsilon k})_x, (d_{2\varepsilon}(y)u_{\varepsilon k})_y)\Big) \cdot \nu = 0 & (x,y) \in \partial\Omega_k,\ t > 0, \\ u_{\varepsilon k}(x,y,0) = u_0(x,y), & (x,y) \in \Omega_k. \end{cases} \tag{3.2}$$

For later reference we call (3.2) the $u_{\varepsilon k}$-problem. Parabolic theory ([16]) asserts that for each fixed $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$, (3.2) admits a global classical solution $u_{\varepsilon k} \in C^0(\bar{\Omega}_k \times [0,\infty)) \cap C^{2,1}(\bar{\Omega}_k \times (0,\infty))$. We shall see in Lemma 4.1 below that these solutions approach a weak solution of the u_ε-problem (1.9) for $k \to \infty$. In order to prepare this, and to collect some useful properties that will be inherited by u_ε and eventually also by u, let us collect some a priori estimates for the solutions of the $u_{\varepsilon k}$-problem (3.2).

3.1 Pointwise a priori estimates

In this section we will apply parabolic comparison arguments to derive some pointwise estimates for the solutions of (3.2). For convenience in presentation, let us introduce the functions $v_{\varepsilon k}$ defined by

$$v_{\varepsilon k}(x,y,t) := d_{2\varepsilon}(y) \cdot u_{\varepsilon k}(x,y,t), \qquad (x,y,t) \in \Omega_k \times (0,\infty). \tag{3.3}$$

It can then easily be checked that $v_{\varepsilon k}$ satisfies

$$v_{\varepsilon kt} = d_{1\varepsilon}(y) v_{\varepsilon kxx} + d_{2\varepsilon}(y) v_{\varepsilon kyy}, \qquad (x,y,t) \in \Omega_k \times (0,\infty), \tag{3.4}$$

along with the boundary conditions

$$\Big(\frac{d_{1\varepsilon}(y)}{d_{2\varepsilon}(y)}\, v_{\varepsilon kx}, v_{\varepsilon ky}\Big) \cdot \nu = 0 \qquad \text{on } \partial\Omega_k. \tag{3.5}$$

The first observation that can readily be made is that $v_{\varepsilon k}$ is nonnegative and bounded from above by an ε-dependent constant. Restated in the original variable this reads as follows.

Lemma 3.1 *For all $\varepsilon \in (0,1)$ and each $k \in \mathbb{N}$, the solution of (3.2) satisfies*

$$0 \le u_{\varepsilon k}(x,y,t) \le \frac{(\|d_2\|_{L^\infty((0,L_y))} + 1) \cdot \|u_0\|_{L^\infty(\Omega)}}{d_{2\varepsilon}(y)} \qquad \text{for all } (x,y,t) \in \Omega_k \times (0,\infty). \tag{3.6}$$

PROOF. First, since $v_{\varepsilon k}(\cdot,0)=u_0$ is nonnegative by assumption (1.7), in view of (3.4) and (3.5) the parabolic maximum principle ensures that $v_{\varepsilon k}\ge 0$ in $\Omega_k\times(0,\infty)$. Since (1.8) entails that $d_{2\varepsilon}>0$ in $(0,L_y)$ for each fixed $\varepsilon\in(0,1)$, this implies the left inequality in (3.6). To see the right one, we let $\overline{v}(x,y,t):=(\|d_2\|_{L^\infty((0,L_y))}+1)\cdot\|u_0\|_{L^\infty(\Omega)}$ for $(x,y,t)\in\Omega_k\times(0,\infty)$. Then clearly $\overline{v}$ is a solution of (3.4)-(3.5) which dominates $v_{\varepsilon k}$ initially, because thanks to (1.8) we have

$$v_{\varepsilon k}(x,y,0)=d_{2\varepsilon}(y)u_0(x,y)\le(d_2(y)+1)u_0(x,y)\le\overline{v}(x,y,0)\qquad\text{for all }(x,y)\in\Omega_k.$$

Therefore the comparison principle states that $v_{\varepsilon k}\le\overline{v}$ in $\Omega_k\times(0,\infty)$, which is equivalent to the right inequality in (3.6). □

At the points where d_2 vanishes, the upper estimate in (3.6) breaks down in the limit $\varepsilon\searrow 0$. An ε-independent bound can be derived by using more complicated comparison functions.

Lemma 3.2 *There exist $C>0$, $\lambda>0$ and $k_0\in\mathbb{N}$ such that for any $\varepsilon\in(0,1)$ and all $k\ge k_0$, the solution of (3.2) satisfies*

$$u_{\varepsilon k}(x,y,t)\le Ce^{\lambda t}\qquad\textit{for all }(x,y,t)\in\Omega_k\times(0,\infty).\tag{3.7}$$

PROOF. We fix an arbitrary number $\eta\in(0,\min\{a,L_y-b\})$ and then can choose a nonnegative $\chi\in C^2([0,L_y])$ with the properties $0\le\chi\le 1$ in $[0,L_y]$, $\chi\equiv 1$ in $K:=[0,\eta]\cup[L_y-\eta,L_y]$ and $\chi\equiv 0$ in $[a,b]$. Then

$$\phi_\varepsilon(y):=\chi(y)+(1-\chi(y))\cdot d_{2\varepsilon}(y)\quad\text{and}\quad\tilde{\phi}_\varepsilon(y):=\frac{\phi_\varepsilon(y)}{d_{2\varepsilon}(y)},\qquad y\in[0,L_y],$$

define two nonnegative functions belonging to $C^2([0,L_y])$. Clearly, $\phi_\varepsilon\equiv 1$ in K, so that in particular

$$\nabla\phi_\varepsilon\equiv 0\quad\text{in}\quad K.\tag{3.8}$$

Moreover, using that $0\le\chi\le 1$ we see that

$$\tilde{\phi}_\varepsilon(y)\le c_1:=\sup_{\varepsilon\in(0,1)}\Big\|\frac{1}{d_{2\varepsilon}}\Big\|_{L^\infty(\operatorname{supp}\chi)}+1\qquad\text{for all }y\in[0,L_y]\text{ and }\varepsilon\in(0,1),\tag{3.9}$$

where c_1 is finite because d_2 is positive in $\operatorname{supp}\chi$ and $d_{2\varepsilon}\to d_2$ uniformly in $\operatorname{supp}\chi$ by (1.8). On the other hand, being a convex combination of $\frac{1}{d_{2\varepsilon}(y)}$ and 1, $\tilde{\phi}_\varepsilon(y)$ satisfies

$$\tilde{\phi}_\varepsilon(y)\ge\min\Big\{\frac{1}{d_{2\varepsilon}(y)},1\Big\}\ge c_2:=\frac{1}{\|d_2\|_{L^\infty((0,L_y))}+1}\qquad\text{for all }y\in[0,L_y]\text{ and }\varepsilon\in(0,1),\tag{3.10}$$

again in view of (1.6). As a final preparation, we observe that for all $\varepsilon\in(0,1)$,

$$\begin{aligned}|\phi_{\varepsilon yy}| &= |(1-d_{2\varepsilon})\chi_{yy}-2\chi_y d_{2\varepsilon y}+(1-\chi)d_{2\varepsilon yy}|\\ &\le c_3:=(\|d_2\|_{L^\infty((0,L_y))}+1)\|\chi_{yy}\|_{L^\infty((0,L_y))}\\ &\quad+2\sup_{\varepsilon\in(0,1)}\|d_{2\varepsilon y}\|_{L^\infty((0,L_y))}\|\chi_y\|_{L^\infty((0,L_y))}+\sup_{\varepsilon\in(0,1)}\|d_{2\varepsilon yy}\|_{L^\infty((0,L_y))}\end{aligned}\tag{3.11}$$

holds throughout $(0, L_y)$, where we note that c_3 is finite due to the fact that $(d_{2\varepsilon})_{\varepsilon\in(0,1)}$ is bounded in $C^2([0, L_y])$ by (1.8).
We now introduce

$$\overline{v}_\varepsilon(x, y, t) := B\phi_\varepsilon(y) \cdot e^{\lambda t}, \qquad (x, y, t) \in \bar{\Omega}_k \times [0, \infty),$$

with

$$B := \frac{\|u_0\|_{L^\infty(\Omega)}}{c_2} \quad \text{and} \quad \lambda := \frac{c_3}{c_2}.$$

Then at $t = 0$ we have

$$\overline{v}_\varepsilon(x, y, 0) = Bd_{2\varepsilon}(y)\tilde{\phi}_\varepsilon(y) \geq c_2 Bd_{2\varepsilon}(y) \geq \|u_0\|_{L^\infty(\Omega)} \cdot d_{2\varepsilon}(y) \geq d_{2\varepsilon}(y) \cdot u_0(x, y) = v_{\varepsilon k}(x, y, 0)$$

for all $(x, y) \in \Omega_k$ by (3.10), whereas (3.8) and (3.1) entail that $\nabla\overline{v}_\varepsilon \equiv 0$ on $\partial\Omega_k \times (0, \infty)$ for all $k > k_1 := \max\{\frac{a}{\eta}, \frac{L_y - b}{\eta}\}$. Furthermore,

$$\begin{aligned} I &:= \overline{v}_{\varepsilon t} - d_1(y)\overline{v}_{\varepsilon xx} - d_{2\varepsilon}(y)\overline{v}_{\varepsilon yy} \\ &= \lambda B\phi_\varepsilon e^{\lambda t} - d_{2\varepsilon} \cdot Be^{\lambda t} \cdot \phi_{\varepsilon yy} \\ &= Bd_{2\varepsilon}e^{\lambda t} \cdot \{\lambda\tilde{\phi}_\varepsilon - \phi_{\varepsilon yy}\} \qquad \text{in } \Omega_k \times (0, \infty), \end{aligned}$$

so that using (3.10), (3.11) and the definition of λ we obtain

$$I \geq Bd_{2\varepsilon}e^{\lambda t} \cdot \{\lambda c_2 - c_3\} \geq 0 \qquad \text{in } \Omega_k \times (0, \infty).$$

The comparison principle thus entails that $\overline{v}_\varepsilon \geq v_{\varepsilon k}$ in $\Omega_k \times (0, \infty)$, which after division by $d_{2\varepsilon}$ means that

$$u_{\varepsilon k}(x, y, t) \leq B\tilde{\phi}_\varepsilon(y)e^{\lambda t} \qquad \text{for all } (x, y, t) \in \Omega_k \times (0, \infty).$$

In light of (3.9), (3.7) thus holds if we let $C := Bc_1$ and take any integer $k_0 \geq k_1$. $\square$

3.2 Entropy estimates

We next derive appropriate integral estimates, the first family of which will involve powers of $u_{\varepsilon k}$, whereas the second will be related to spatial derivatives thereof. Following common practice in PDE analysis, we will call the former entropy estimates and the latter energy estimates, without having a particular physical concept in mind.

The following basic statement will be applied twice in the sequel: First, it will be the source of the entropy estimate in Lemma 3.4 and thereby entail the space-time integrability property (5.10) which will be useful in deriving the stabilisation result *outside* Ω_{ab} in Theorem 1.1 i). On the other hand, an appropriate choice of the function φ_ε appearing below will enable us to obtain bounds for u_ε and $u_{\varepsilon x}$ *inside* Ω_{ab} (cf. Lemma 3.6 and Lemma 5.4).

Lemma 3.3 *Let $\varepsilon \in (0,1)$ and suppose that $\varphi_\varepsilon \in C^1([0,L_y])$ is positive and such that*

$$\varphi_\varepsilon(y) = d_{2\varepsilon}(y) \qquad \textit{for all } y \in [0,L_y] \setminus (a,b). \tag{3.12}$$

Then for any $p>1$ and each $k \in \mathbb{N}$, the solution of the $u_{\varepsilon k}$-problem (3.2) satisfies

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_k} \varphi_\varepsilon^{p-1}(y) u_{\varepsilon k}^p + p(p-1)\int_{\Omega_k} d_{1\varepsilon}(y)\varphi_\varepsilon^{p-1}(y) u_{\varepsilon k}^{p-2} u_{\varepsilon kx}^2 \\
&\qquad + p(p-1)\varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon^{p-1}(y) u_{\varepsilon k}^{p-2} u_{\varepsilon ky}^2 + p(p-1)\int_{\Omega_k\setminus\Omega_{ab}} (d_{2\varepsilon}(y)u_{\varepsilon k})^{p-2}(d_{2\varepsilon}(y)u_{\varepsilon k})_y^2 \\
&= -p(p-1)\varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon^{p-2}(y)\varphi_{\varepsilon y}(y) u_{\varepsilon k}^{p-1} u_{\varepsilon ky} \qquad \textit{for all } t>0.
\end{aligned} \tag{3.13}$$

Proof. Since $\varphi_\varepsilon \in C^1([a,b])$ and $\varphi_\varepsilon > 0$, we may choose $(\varphi_\varepsilon u_{\varepsilon k})^{p-1}$ as a test function in (3.2) to obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\Omega_k} \varphi_\varepsilon^{p-1} u_{\varepsilon k}^p &= \int_{\Omega_k} \varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-1} \cdot \Big\{ (d_{1\varepsilon} u_{\varepsilon k})_{xx} + (d_{2\varepsilon} u_{\varepsilon k})_{yy} \Big\} \\
&= -\int_{\Omega_k} (\varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-1})_x (d_{1\varepsilon} u_{\varepsilon k})_x - \int_{\Omega_k} (\varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-1})_y (d_{2\varepsilon} u_{\varepsilon k})_y \\
&=: -I_1 - I_2 \qquad \text{for all } t>0.
\end{aligned} \tag{3.14}$$

Since both φ_ε and $d_{1\varepsilon}$ depend on y only, we compute

$$I_1 = \int_{\Omega_k} d_{1\varepsilon}\varphi_\varepsilon^{p-1} (u_{\varepsilon k}^{p-1})_x u_{\varepsilon kx} = (p-1)\int_{\Omega_k} d_{1\varepsilon}\varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-2} u_{\varepsilon kx}^2. \tag{3.15}$$

As for I_2, we use (3.12) and our assumption that $d_{2\varepsilon} \equiv \varepsilon$ in (a,b) in splitting

$$I_2 = \int_{\Omega_k\setminus\Omega_{ab}} (d_{2\varepsilon}^{p-1} u_{\varepsilon k}^{p-1})_y (d_{2\varepsilon} u_{\varepsilon k})_y + \varepsilon \int_{\Omega_{ab}} (\varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-1})_y u_{\varepsilon ky} =: I_{21} + I_{22}, \tag{3.16}$$

where clearly

$$I_{21} = (p-1)\int_{\Omega_k\setminus\Omega ab} (d_{2\varepsilon} u_{\varepsilon k})^{p-2} (d_{2\varepsilon} u_{\varepsilon k})_y^2 \tag{3.17}$$

and

$$I_{22} = (p-1)\varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon^{p-1} u_{\varepsilon k}^{p-2} u_{\varepsilon ky}^2 + (p-1)\varepsilon\int_{\Omega_{ab}} \varphi_\varepsilon^{p-2}\varphi_{\varepsilon y} u_{\varepsilon k}^{p-1} u_{\varepsilon ky}.$$

Combining this with (3.14)-(3.17) and multiplying by p, we arrive at (3.13). □

Lemma 3.4 *For all $\varepsilon \in (0,1)$ and any $k \in \mathbb{N}$, the solution of (1.9) satisfies*

$$\begin{aligned}
&\int_{\Omega_k} d_{2\varepsilon}(y) u_{\varepsilon k}^2(x,y,t) d(x,y) + 2\int_0^t \int_{\Omega_k} d_{1\varepsilon}(y) d_{2\varepsilon}(y) u_{\varepsilon kx}^2 + 2\int_0^t\int_{\Omega_k} (d_{2\varepsilon}(y)u_{\varepsilon k})_y^2 \\
&\qquad \le \int_{\Omega_k} d_{2\varepsilon}(y) u_0^2(x,y) d(x,y) \qquad \textit{for all } t>0.
\end{aligned} \tag{3.18}$$

In particular, writing $C := \sup_{\varepsilon\in(0,1)} \int_\Omega d_{2\varepsilon}(y)u_0(x,y)d(x,y) < \infty$ *we have*

$$\int_0^\infty \int_{\Omega_k} d_{1\varepsilon}(y)d_{2\varepsilon}(y)u_{\varepsilon kx}^2 + \int_0^\infty \int_{\Omega_k} (d_{2\varepsilon}(y)u_{\varepsilon k})_y^2 \le C \qquad \text{for all } \varepsilon \in (0,1) \text{ and } k \in \mathbb{N}. \tag{3.19}$$

PROOF. We apply Lemma 3.3 to $p := 2$ and $\varphi_\varepsilon \equiv d_{2\varepsilon}$. Then $\varphi_{\varepsilon y} \equiv 0$ in $[a,b]$, and hence (3.18) directly results upon integrating (3.13) in time. The consequence (3.19) is immediate, where C indeed is finite according to (1.8). $\square$

We next plan to apply Lemma 3.3 to differently chosen φ_ε. To this end, we let

$$\Theta(y) := \sin\frac{\pi(y-a)}{b-a}, \qquad y \in (a,b). \tag{3.20}$$

Then a simple but useful observation is the following.

Lemma 3.5 *Let* Θ *be as in (3.20). Then for all* $\delta > 0$ *we have the inequality*

$$\frac{\Theta^2(y)\Theta_y^2(y)}{(\Theta^2(y)+\delta)^2} \le \frac{\pi^2}{4(b-a)^2\delta} \qquad \text{for all } y \in (a,b). \tag{3.21}$$

PROOF. Obviously, $|\Theta_y(y)| = |\frac{\pi}{b-a}\cos\frac{\pi(y-a)}{b-a}| \le \frac{\pi}{b-a}$ for all $y \in (a,b)$. Next, it can easily be checked that $\psi(\xi) := \frac{\xi}{\xi^2+\delta}$, $\xi \ge 0$, attains its maximum at $\xi_0 := \sqrt{\delta}$ with $\psi(\xi_0) = \frac{1}{2\sqrt{\delta}}$. Therefore,

$$\frac{\Theta^2(y)\Theta_y^2(y)}{(\Theta^2(y)+\delta)^2} \le \Big(\frac{\pi}{b-a}\Big)^2 \cdot \Big(\frac{1}{2\sqrt{\delta}}\Big)^2,$$

which implies (3.21). $\square$

Using this function appropriately, we shall obtain another entropy-like estimate as follows.

Lemma 3.6 *Let* Θ *be as in (3.20). Then there exists* $\beta > 0$ *such that for any* $p \ge 2$ *and each* $\varepsilon \in (0,1)$ *and* $k \in \mathbb{N}$*, the solution of (3.2) satisfies*

$$\begin{aligned}
&\int_{\Omega_{ab}} \Big(\Theta^2(y) + \varepsilon^{\frac{p-1}{p}}\Big)^p u_{\varepsilon k}^p(x,y,t)d(x,y) \\
&\qquad + p(p-1)\int_0^t \int_{\Omega_{ab}} d_{1\varepsilon}(y)\Big(\Theta^2(y) + \varepsilon^{\frac{p-1}{p}}\Big)^p u_{\varepsilon k}^{p-2}u_{\varepsilon kx}^2 \\
&\qquad + \frac{p(p-1)\varepsilon}{2}\int_0^t \int_{\Omega_{ab}} \Big(\Theta^2(y) + \varepsilon^{\frac{p-1}{p}}\Big)^p u_{\varepsilon k}^{p-2}u_{\varepsilon ky}^2 \\
&\le \Big\{\int_{\Omega\setminus\Omega_{ab}} d_{2\varepsilon}^{p-1}(y)u_0^p(x,y)d(x,y) + \int_{\Omega_{ab}} \Big(\Theta^2(y) + \varepsilon^{\frac{p-1}{p}}\Big)^p u_0^p(x,y)d(x,y)\Big\} \cdot e^{p^2\beta\varepsilon^{\frac{1}{p}}t}
\end{aligned} \tag{3.22}$$

for all $t > 0$.

Proof. We let

$$\varphi_\varepsilon(y) := \begin{cases} d_{2\varepsilon}(y) & \text{if } y\in[0,L_y]\setminus(a,b),\\ (\Theta^2(y)+\varepsilon^{\frac{p-1}{p}})^{\frac{p}{p-1}} & \text{if } y\in(a,b). \end{cases}$$

Then φ_ε is positive and both φ_ε and $\varphi_{\varepsilon y}$ are continuous in $[0,L_y]$, and hence Lemma 3.3 applies to yield the identity (3.13). Its right-hand side can be estimated using Young's inequality according to

$$-p(p-1)\varepsilon\int_{\Omega_{ab}}\varphi_\varepsilon^{p-2}\varphi_{\varepsilon y}u_{\varepsilon k}^{p-1}u_{\varepsilon ky} \le \frac{p(p-1)\varepsilon}{2}\int_{\Omega_{ab}}\varphi_\varepsilon^{p-1}u_{\varepsilon k}^{p-2}u_{\varepsilon ky}^2+\frac{p(p-1)\varepsilon}{2}\int_{\Omega_{ab}}\varphi_\varepsilon^{p-3}\varphi_{\varepsilon y}^2u_{\varepsilon k}^p. \tag{3.23}$$

Here we split $\varphi_\varepsilon^{p-3}\varphi_{\varepsilon y}^2u_{\varepsilon k}^p=\frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon^2}\cdot(\varphi_\varepsilon^{p-1}u_{\varepsilon k}^p)$ and compute in (a,b)

$$\frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon^2}=\frac{\Big\{\frac{p}{p-1}\cdot 2\Theta\Theta_y\cdot(\Theta^2+\varepsilon^{\frac{p-1}{p}})^{\frac{p}{p-1}-1}\Big\}^2}{(\Theta^2+\varepsilon^{\frac{p-1}{p}})^{\frac{2p}{p-1}}}=\frac{4p^2}{(p-1)^2}\cdot\frac{\Theta^2\Theta_y^2}{(\Theta^2+\varepsilon^{\frac{p-1}{p}})^2}.$$

Using Lemma 3.5 and our assumption $p\ge 2$ we therefore find that

$$\frac{p(p-1)\varepsilon}{2}\cdot\frac{\varphi_{\varepsilon y}^2(y)}{\varphi_\varepsilon^2(y)}\le\frac{p(p-1)\varepsilon}{2}\cdot\frac{4p^2}{(p-1)^2}\cdot\frac{\pi^2}{4(b-a)^2\varepsilon^{\frac{p-1}{p}}}=\frac{\pi^2p^3\varepsilon^{\frac1p}}{2(b-a)^2(p-1)}\le p^2\beta\varepsilon^{\frac1p}$$

for all $y\in(a,b)$, where $\beta:=\frac{\pi^2}{(b-a)^2}$. Thus, inserting (3.23) into (3.13) and dropping nonnegative terms we obtain the inequality

$$\frac{d}{dt}\int_{\Omega_k}\varphi_\varepsilon^{p-1}(y)u_{\varepsilon k}^p+p(p-1)\int_{\Omega_k}d_{1\varepsilon}(y)\varphi_\varepsilon^{p-1}(y)u_{\varepsilon k}^{p-2}u_{\varepsilon kx}^2+\frac{p(p-1)\varepsilon}{2}\int_{\Omega_{ab}}\varphi_\varepsilon^{p-1}(y)u_{\varepsilon k}^{p-2}u_{\varepsilon ky}^2$$
$$\le p^2\beta\varepsilon^{\frac1p}\int_{\Omega_{ab}}\varphi_\varepsilon^{p-1}u_{\varepsilon k}^p\qquad\text{for all } t>0.$$

This says that writing $z(t):=\int_{\Omega_k}\varphi_\varepsilon^{p-1}u_{\varepsilon k}^p$ and

$$f(t):=p(p-1)\int_{\Omega_k}d_{1\varepsilon}(y)\varphi_\varepsilon^{p-1}(y)u_{\varepsilon k}^{p-2}u_{\varepsilon kx}^2+\frac{p(p-1)\varepsilon}{2}\int_{\Omega_{ab}}\varphi_\varepsilon^{p-1}(y)u_{\varepsilon k}^{p-2}u_{\varepsilon ky}^2,$$

we have $z'(t)+f(t)\le\gamma z(t)$ for all $t>0$ with $\gamma:=p^2\beta\varepsilon^{\frac1p}$. Since $f\ge0$, this first yields $z(t)\le z(0)\cdot e^{\gamma t}$ and then upon another integration

$$z(t)-z(0)+\int_0^t f(s)ds\le\gamma\int_0^t z(0)e^{\gamma s}ds=z(0)e^{\gamma t}-z(0)\qquad\text{for all } t>0.$$

Upon splitting the integrals into $\Omega_k\backslash\Omega_{ab}$ and Ω_{ab} and dropping two positive integrals on $\Omega_k\backslash\Omega_{ab}$, and in view of the definition of φ_ε, we readily obtain (3.22). □

3.3 Energy estimates

We proceed to derive integral estimates of energy-type, involving time derivatives of $u_{\varepsilon k}$. Parallel to the previous section, we begin with a basic estimate containing a weight function which is at our disposal and will be chosen in two different ways below.

Lemma 3.7 *Given $\varepsilon \in (0,1)$, let $\varphi_\varepsilon \in C^1([0,L_y])$ be positive in $[0,L_y]$ and such that*

$$\varphi_\varepsilon(y) = d_{2\varepsilon}(y) \qquad \text{for all } y \in [0,L_y] \setminus (a,b). \tag{3.24}$$

Then for all $k \in \mathbb{N}$, the solution of (3.2) satisfies

$$\begin{aligned} \int_{\Omega_k} \varphi_\varepsilon u_{\varepsilon kt}^2 \quad &+ \quad \frac{d}{dt}\left\{ \int_{\Omega_k} d_{1\varepsilon}\varphi_\varepsilon u_{\varepsilon kx}^2 + \varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon u_{\varepsilon ky}^2 + \int_{\Omega_k\setminus\Omega_{ab}} (d_{2\varepsilon}u_{\varepsilon k})_y^2 \right\} \\ &\le \quad \frac{\varepsilon^2}{2}\int_{\Omega_k} \frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon} u_{\varepsilon ky}^2 \end{aligned} \tag{3.25}$$

Proof. Using $\varphi_\varepsilon u_{\varepsilon kt}$ as a test function for (1.9), we obtain

$$\begin{aligned} \int_{\Omega_k} \varphi_\varepsilon u_{\varepsilon kt}^2 \quad &= \quad -\int_{\Omega_k} (d_{1\varepsilon}u_{\varepsilon k})_x(\varphi_\varepsilon u_{\varepsilon kt})_x - \int_{\Omega_k} (d_{2\varepsilon}u_{\varepsilon k})_y(\varphi_\varepsilon u_{\varepsilon kt})_y \\ &= \quad -\int_{\Omega_k} d_{1\varepsilon}\varphi_\varepsilon u_{\varepsilon kx}u_{\varepsilon kxt} - \varepsilon\int_{\Omega_{ab}} u_{\varepsilon ky}(\varphi_\varepsilon u_{\varepsilon kt})_y - \int_{\Omega_k\setminus\Omega_{ab}} (d_{2\varepsilon}u_{\varepsilon k})_y(d_{2\varepsilon}u_{\varepsilon k})_{yt} \\ &=: \quad I_1 + I_2 + I_3 \qquad \text{for all } t > 0, \end{aligned} \tag{3.26}$$

where we have used (3.24) and our assumption that $d_{2\varepsilon} \equiv \varepsilon$ in (a,b). Clearly,

$$I_1 = -\frac{1}{2}\frac{d}{dt}\int_{\Omega_k} d_{1\varepsilon}\varphi_\varepsilon u_{\varepsilon kx}^2 \qquad \text{and} \qquad I_3 = -\frac{1}{2}\frac{d}{dt}\int_{\Omega_k\setminus\Omega_{ab}} (d_{2\varepsilon}u_{\varepsilon k})_y^2, \tag{3.27}$$

and

$$\begin{aligned} I_2 \quad &= \quad -\varepsilon\int_{\Omega_{ab}} \varphi_\varepsilon u_{\varepsilon ky}u_{\varepsilon kyt} - \varepsilon\int_{\Omega_{ab}} \varphi_{\varepsilon y}u_{\varepsilon ky}u_{\varepsilon kt} \\ &= \quad -\frac{\varepsilon}{2}\frac{d}{dt}\int_{\Omega_{ab}} \varphi_\varepsilon u_{\varepsilon ky}^2 - \varepsilon\int_{\Omega_{ab}} \varphi_{\varepsilon y}u_{\varepsilon ky}u_{\varepsilon kt}. \end{aligned} \tag{3.28}$$

Here we use Young's inequality to estimate

$$-\varepsilon\int_{\Omega_{ab}} \varphi_{\varepsilon y}u_{\varepsilon ky}u_{\varepsilon kt} \le \frac{1}{2}\int_{\Omega_{ab}} \varphi_\varepsilon u_{\varepsilon kt}^2 + \frac{\varepsilon^2}{2}\int_{\Omega_{ab}} \frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon}u_{\varepsilon ky}^2, \tag{3.29}$$

and collect (3.26)-(3.29) to complete the proof. □

A straightforward consequence is the natural energy inequality associated with (3.2).

Lemma 3.8 *For any $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$, the solution of (3.2) fulfils*

$$\int_{t_0}^t \int_{\Omega_k} d_{2\varepsilon}(y) u_{\varepsilon kt}^2 + E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t)) \le E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t_0)) \qquad \textit{whenever } 0 < t_0 < t, \tag{3.30}$$

where we have set

$$E_{\varepsilon k}(z) := \int_{\Omega_k} d_{1\varepsilon}(y) d_{2\varepsilon}(y) z_x^2 + \int_{\Omega_k} (d_{2\varepsilon}(y) z)_y^2 \qquad \textit{for } z \in W^{1,2}(\Omega_k). \tag{3.31}$$

PROOF. Choosing $\varphi_\varepsilon \equiv d_{2\varepsilon}$ in Lemma 3.7, from (3.25) we obtain

$$\int_{\Omega_k} d_{2\varepsilon} u_{\varepsilon kt}^2 + \frac{d}{dt} E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t)) \le 0 \qquad \text{for all } t > 0,$$

because $d_{2\varepsilon} \equiv \varepsilon$ and hence $\varphi_{\varepsilon y} \equiv 0$ in (a,b). An integration in time yields (3.30). □

A different and less standard energy-like estimate can be obtained by choosing φ_ε in a way similar to that in Lemma 3.6.

Lemma 3.9 *Let $q > 2$ and*

$$\varphi_\varepsilon(y) := \begin{cases} d_{2\varepsilon}(y) & \textit{if } y \in [0, L_y] \setminus (a,b), \\ \left(\Theta^2(y) + \varepsilon^{\frac{2}{q}}\right)^{\frac{q}{2}} & \textit{if } y \in (a,b), \end{cases} \tag{3.32}$$

for $\varepsilon \in (0,1)$, with Θ as defined in (3.20). Then for all $k \in \mathbb{N}$ we have

$$\int_{t_0}^t \int_{\Omega_k} \varphi_\varepsilon(y) u_{\varepsilon kt}^2 + F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t)) \le F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t_0)) \cdot e^{\frac{\pi^2 q^2}{8(b-a)^2} \cdot \varepsilon^{1-\frac{2}{q}} \cdot (t-t_0)} \qquad \textit{whenever } 0 < t_0 < t, \tag{3.33}$$

where

$$F_{\varepsilon k}(z) := \int_{\Omega_k} d_{1\varepsilon}(y) \varphi_\varepsilon(y) z_x^2 + \int_{\Omega_k \setminus \Omega_{ab}} (d_{2\varepsilon}(y) z)_y^2 + \varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon(y) z_y^2 \qquad \textit{for } z \in W^{1,2}(\Omega_k). \tag{3.34}$$

PROOF. By (3.32) and Lemma 3.5,

$$\frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon^2} = \frac{q^2 \Theta^2 \Theta_y^2}{(\Theta^2 + \varepsilon^{\frac{2}{q}})^2} \le q^2 \cdot \frac{\pi^2}{4(b-a)^2 \varepsilon^{\frac{2}{q}}} \qquad \text{in } (a,b).$$

Therefore, the right-hand side in (3.25) can be estimated according to

$$\begin{aligned} \frac{\varepsilon^2}{2} \int_{\Omega_{ab}} \frac{\varphi_{\varepsilon y}^2}{\varphi_\varepsilon} u_{\varepsilon ky}^2 &\le \frac{\varepsilon}{2} \cdot q^2 \cdot \frac{\pi^2}{4(b-a)^2 \varepsilon^{\frac{2}{q}}} \cdot \left(\varepsilon \int_{\Omega_{ab}} \varphi_\varepsilon u_{\varepsilon ky}^2\right) \\ &\le \frac{\pi^2 q^2 \varepsilon^{1-\frac{2}{q}}}{8(b-a)^2} \cdot F_{\varepsilon k}(u_{\varepsilon k}). \end{aligned}$$

Thus, (3.33) results upon an integration of (3.25). □

4 Global existence

Let us now make sure that along suitable subsequences, the global solutions of the $u_{\varepsilon k}$-problem (3.2) converge to global weak solutions of the u_ε-problem (1.9), and that the latter approach some global very weak solution of the degenerate problem (1.4) as $\varepsilon \searrow 0$.

4.1 Global weak solutions in the non-degenerate problem

Lemma 4.1 *Let $\varepsilon \in (0,1)$. Then there exist a sequence $(k_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $k_l \to \infty$ as $l \to \infty$, and a nonnegative function*

$$u_\varepsilon \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \cap C^{2,1}(\Omega \times (0,\infty)) \tag{4.1}$$

such that

$$\begin{aligned} &u_{\varepsilon k} \rightharpoonup u_\varepsilon \qquad \text{in } L^2_{loc}([0,\infty); L^2_{loc}(\Omega)) \qquad \text{and} \\ &u_{\varepsilon k} \to u_\varepsilon \in C^{2,1}_{loc}(\Omega \times (0,\infty)) \end{aligned} \tag{4.2}$$

as $k = k_l \to \infty$. The limit function u_ε is a global weak solution of (1.9) in the sense that for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$, the identity

$$-\int_0^\infty \int_\Omega u_\varepsilon \varphi_t + \int_0^\infty \int_\Omega \Big\{ \Big(d_{1\varepsilon}(y) u_\varepsilon\Big)_x \varphi_x + \Big(d_{2\varepsilon}(y) u_\varepsilon\Big)_y \varphi_y \Big\} = \int_\Omega u_0 \varphi(\cdot,0) \tag{4.3}$$

is valid.

Proof. We let χ_{Ω_k} denote the characteristic function of Ω_k in Ω. Then from Lemma 3.1 we know that $(\chi_{\Omega_k} u_{\varepsilon k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega \times [0,\infty))$, whereas Lemma 3.4 in conjunction with the positivity of both $d_{1\varepsilon}$ and $d_{2\varepsilon}$ in $[0, L_y]$ entails that $(\chi_{\Omega_k} \nabla u_{\varepsilon k})_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega \times (0,\infty))$. We therefore can pick a sequence of integers $k_l \to \infty$ and two functions $u_\varepsilon \geq 0$ and z such that

$$\begin{aligned} &\chi_{\Omega_k} u_{\varepsilon k} \rightharpoonup u_\varepsilon \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{and} \\ &\chi_{\Omega_k} \nabla u_{\varepsilon k} \rightharpoonup z \qquad \text{in } L^2(\Omega \times (0,\infty)) \end{aligned} \tag{4.4}$$

as $k = k_l \to \infty$. In order to identify $z = \nabla u_\varepsilon$, given $\psi \in C_0^\infty(\Omega \times (0,\infty))$ we recall (3.1) to find $k_1 \in \mathbb{N}$ such that $\operatorname{supp} \psi(\cdot,t) \subset \Omega_k$ for all $k \geq k_1$ and $t \geq 0$, and hence in fact we have

$$\begin{aligned} \int_0^\infty \int_\Omega z\psi &= \lim_{k=k_l\to\infty} \int_0^\infty \int_\Omega \chi_{\Omega_k} \nabla u_{\varepsilon k} \psi = \lim_{k=k_l\to\infty} \int_0^\infty \int_\Omega \nabla u_{\varepsilon k} \psi \\ &= -\lim_{k=k_l\to\infty} \int_0^\infty \int_\Omega u_{\varepsilon k} \nabla\psi = -\lim_{k=k_l\to\infty} \int_0^\infty \int_\Omega \chi_{\Omega_k} u_{\varepsilon k} \nabla \psi = -\int_0^\infty \int_\Omega u_\varepsilon \nabla\psi \end{aligned}$$

for any such ψ.

Moreover, since (3.2) is non-degenerate, interior parabolic Schauder estimates ([16]) show that $(u_{\varepsilon k})_{k\in\mathbb{N}}$ is also bounded in $C^{2+\theta,1+\frac{\theta}{2}}(\Omega \times (0,\infty))$ for some $\theta > 0$. Thanks to the Arzelà-Ascoli theorem we thus may pass to a subsequence if necessary to conclude that

indeed (4.2) and (4.1) hold.
To verify the claimed solution property of u_ε, we fix $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and then have $\varphi \in C_0^\infty(\bar\Omega_k \times [0,\infty))$ according to (3.1). We therefore may integrate by parts in (3.2) to find

$$\int_0^\infty \int_\Omega \chi_{\Omega_k} u_{\varepsilon k} \varphi_t + \int_{\Omega_k} u_0 \varphi(\cdot,0)$$
$$= \int_0^\infty \int_\Omega \Big\{ d_{1\varepsilon}(y) \cdot (\chi_{\Omega_k} u_{\varepsilon k x}) \cdot \varphi_x + d_{2\varepsilon}(y) \cdot (\chi_{\Omega_k} u_{\varepsilon k y}) \cdot \varphi_y + d_{2\varepsilon y}(y) \cdot (\chi_{\Omega_k} u_\varepsilon) \cdot \varphi_y \Big\}$$

for all $k \in \mathbb{N}$. Here we use (4.4) to see that in the limit $k = k_l \to \infty$, the first term on the left and each of the integrals on the right approach the respective terms containing u_ε instead of $u_{\varepsilon k}$, whereas clearly $\int_{\Omega_k} u_0\varphi(\cdot,0) \to \int_\Omega u_0 \varphi(\cdot,0)$ as $k \to \infty$. This establishes (3.2). □

4.2 Global very weak solutions in the degenerate problem

Based on the pointwise estimate in Lemma 3.2, we can proceed to assert that indeed the above solutions u_ε approach a limit which satisfies (1.4) in the sense specified in Definition 2.1.

Theorem 4.2 *There exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that for the weak solutions u_ε of (1.9) constructed in Lemma 4.1 we have*

$$u_\varepsilon \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{4.5}$$

with some nonnegative global very weak solution u of (1.4). This solution satisfies

$$u(x,y,t) \le C e^{\lambda t} \qquad \text{for a.e. } (x,y,t) \in \Omega \times (0,\infty) \tag{4.6}$$

with some $C > 0$ and $\lambda > 0$. Moreover, u has the additional property

$$u \in C^0_{w\text{-}\star}([0,\infty); L^\infty(\Omega)); \tag{4.7}$$

that is, upon a modification on a null set of times we can achieve that u is continuous on $[0,\infty)$ as an $L^\infty(\Omega)$-valued function with respect to the weak-$\star$ topology on $L^\infty(\Omega)$.

Proof. Thanks to the Banach-Alaoglu theorem, the statement (4.5) is an immediate consequence of (4.2) and the estimate in Lemma 3.2, whereupon (4.6) easily follows from Lemma 3.2 and (4.2). Upon another integration by parts in (4.3), the integral identity (2.1) results in a straightforward manner from (4.5).
To see (4.7), we fix $T > 0$ and observe that given $\varphi \in C_0^\infty(\Omega)$, $z^{(\varphi)}_{\varepsilon k}(t) := \int_{\Omega_k} u_{\varepsilon k}(\cdot,t)\varphi$, $t \in [0,T]$, satisfies

$$\Big|(z^{(\varphi)}_{\varepsilon k})'(t)\Big| = \Big|\frac{d}{dt}\int_{\Omega_k} u_{\varepsilon k}\varphi\Big| = \Big|\int_{\Omega_k} d_{1\varepsilon}(y) u_{\varepsilon k}\varphi_{xx} + \int_{\Omega_k} d_{2\varepsilon}(y) u_{\varepsilon k}\varphi_{yy}\Big| \le c_1(T)\|\varphi\|_{W^{2,\infty}(\Omega)}$$

for all $t \in (0,T)$ with some $c_1(T) > 0$ independent of ε and k, so that the family $(z_{\varepsilon k}^{(\varphi)})_{\varepsilon\in(0,1),k\in\mathbb{N}}$ is bounded in $C^1([0,T])$. Thus, using the Arzelà-Ascoli theorem, for any such φ we may extract subsequences $(k_{l_m})_{m\in\mathbb{N}}$ of $(k_l)_{l\in\mathbb{N}}$ and $(\varepsilon_{j_i})_{i\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ along which $z_{\varepsilon k}^{(\varphi)}$ converges in $C^0([0,T])$, and we can thereby introduce a time-dependent functional $f(t)$ on $C_0^\infty(\Omega)$ by defining $f(t)[\varphi] := \lim_{\varepsilon=\varepsilon_{j_i}\searrow 0}\lim_{k=k_{l_m}\to\infty} z_{\varepsilon k}^{(\varphi)}(t)$, $t\in[0,T], \varphi\in C_0^\infty(\Omega)$. Moreover, since we may use Lemma 3.2 to find $c_2(T) > 0$ such that $|z_{\varepsilon k}^{(\varphi)}(t)| \le c_2(T)\|\varphi\|_{L^1(\Omega)}$ for all $\varphi \in C_0^\infty(\Omega)$, it is obvious upon a density argument that $f(t)$ can be extended to an element $\tilde{u}(t)$ of $(L^1(\Omega))^\star \cong L^\infty(\Omega)$, where it is easy to see that still $t \mapsto \int_\Omega \tilde{u}(\cdot,t)\varphi$ is continuous on $[0,T]$ for all $\varphi \in L^1(\Omega)$. Now since for any $\varphi \in C_0^\infty(\Omega)$ and each $\psi \in C_0^\infty((0,T))$ we have

$$\begin{aligned}\int_0^T f(t)[\varphi]\cdot\psi(t)dt &= \lim_{\varepsilon=\varepsilon_{j_k}\searrow 0}\lim_{k=k_{l_m}\to\infty}\int_0^T\int_{\Omega_k} u_{\varepsilon k}(x,y,t)\varphi(x,y)\psi(t)d(x,y)dt \\ &= \int_0^T\int_{\Omega_k} u(x,y,t)\varphi(x,y)\psi(t)d(x,y)dt,\end{aligned}$$

it follows that actually $\int_\Omega \tilde{u}(\cdot,t)\varphi = f(t)[\varphi] = \int_\Omega u(\cdot,t)\varphi$ for all $\varphi \in C_0^\infty(\Omega)$ and a.e. $t \in (0,T)$, which entails that $\tilde{u} = u$ a.e. in $\Omega\times(0,T)$. This implies that rearranging u on a null set of times we may indeed assume that (4.7) is valid. □

Using (4.7), for the particular solution constructed above we can sharpen the assertions in (2.9) and (2.2) so as to hold for all times without any exceptional set. Inter alia, this will be helpful in the proof of Theorem 1.2.

Corollary 4.3 *The solution u defined in (4.5) satisfies*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t>0, \tag{4.8}$$

and moreover for each $y_\star \in [a,b]$ we have

$$\int_0^{y_\star}\int_0^{L_x} u(x,y,t)dxdy = \int_0^{y_\star}\int_0^{L_x} u_0(x,y)dxdy \qquad \textit{for all } t>0. \tag{4.9}$$

Proof. Using $\Omega \ni (x,y) \mapsto \psi(x,y) = 1$ as a test function in the weak continuity statement (4.7), we see that (2.9) immediately implies (4.8), because the complement of a null set in $(0,\infty)$ clearly is dense in $(0,\infty)$. Similarly, the choice $\psi(x,y) = \chi_{(0,y_\star)}(y)$ with the characteristic function $\chi_{(0,y_\star)}$ of $(0,y_\star)$ shows that (4.9) is a consequence of (2.2). □

4.3 Solution properties of U

In this section we briefly address the one-dimensional problem (1.18) and discuss in how far the function U defined by (1.13) indeed is a solution thereof. Our main results say that indeed, U is a global very weak solution of (1.18), and that according to the above regularity properties of u, this solution in fact belongs to a function class within such solutions are uniquely determined.

Proposition 4.4 *Let U and U_0 be as defined in (1.13) and (1.14), respectively. Then U is a global very weak solution of (1.18) in the sense that U belongs to $L^1_{loc}([0,L_y]\times[0,\infty))$ and satisfies*

$$-\int_0^\infty \int_0^{L_y} U\Phi_t = \int_0^{L_y} U_0\Phi(\cdot,0) + \int_0^\infty \int_0^{L_y} d_2(y)U\Phi_{yy} \tag{4.10}$$

for all $\Phi \in C_0^\infty([0,L_y]\times[0,\infty))$ fulfilling $\Phi_y = 0$ on $\{0,L_y\}\times(0,\infty)$. Moreover, we have

$$U \in C^0_{w-\star}([0,\infty);L^\infty((0,L_y))). \tag{4.11}$$

Proof. Since u is a global very weak solution of (1.4) and belongs to $C^0_{w-\star}([0,\infty);L^\infty(\Omega))$, the claim is immediate upon choosing $\varphi(x,y,t) := \Phi(y,t)$ in Definition 2.1. $\Box$

Proposition 4.5 *There exists at most one function in $C^0_{w-\star}([0,\infty);L^\infty((0,L_y)))$ which solves (1.18) in the very weak sense specified in Proposition 4.4.*

Proof. We follow a duality argument which is well-established in the analysis of degenerate parabolic equations, also involving nonlinear diffusion ([1]). Here, due to linearity we only need to show that if $U_0 \equiv 0$ in $(0,L_y)$ then $U \equiv 0$ in $(0,L_y)\times(0,\infty)$, and this will be accomplished in two steps.

Step 1. We first claim that for any $t_0 > 0$ and each $\Phi \in C^\infty([0,L_y]\times[0,t_0]$ satisfying $\Phi_y = 0$ on $\{0,L_y\}\times(0,t_0)$ we have

$$\int_0^{L_y} U(\cdot,t_0)\Phi(\cdot,t_0) = \int_0^{t_0}\int_0^{L_y} \Big\{\Phi_t + d_2(y)\Phi_{yy}\Big\}\cdot U. \tag{4.12}$$

Indeed, given $t_0 > 0$ and any such Φ, it is clear upon a standard approximation argument that (4.10) continues to hold with Φ replaced by

$$\Phi_\delta(y,t) := \chi_\delta(t)\cdot\Phi(y,t), \qquad y\in[0,L_y],\ t\ge 0,$$

where

$$\chi_\delta(t) := \begin{cases} 1, & t\in[0,t_0], \\ \frac{t_0+\delta-t}{\delta}, & t\in(t_0,t_0+\delta), \\ 0, & t\ge t_0+\delta, \end{cases}$$

for $\delta\in(0,1)$. Therefore, (4.10) yields

$$-\int_0^\infty\int_0^{L_y} \chi_\delta U\Phi_t + \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_0^{L_y} U\Phi = \int_0^\infty\int_0^{L_y} \chi_\delta\cdot d_2(y)U\Phi_{yy}, \tag{4.13}$$

because $U_0\equiv 0$. Here we have

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_0^{L_y} U\Phi \to \int_0^{L_y} U(\cdot,t_0)\Phi(\cdot,t_0) \qquad \text{as } \delta\searrow 0,$$

for U belongs to $C^0_{w-\star}([0,\infty);L^\infty((0,L_y)))$. Therefore, (4.12) is a consequence of (4.13).

Step 2. We are now in the position to make sure that for any $t_0>0$ and each $\phi \in C_0^\infty((0,L_y))$, the identity

$$\int_0^{L_y} U(y,t_0)\phi(y)dy = 0 \tag{4.14}$$

holds, which will evidently yield $U\equiv 0$.
For this purpose, we first fix a sequence $(d_{2j})_{j\in\mathbb{N}} \subset C^\infty([0,L_y])$ such that $d_{2j}>0$ in $[0,L_y]$ for each $j\in\mathbb{N}$ and

$$\frac{d_{2j}-d_2}{\sqrt{d_{2j}}} \to 0 \quad \text{in } L^2((0,L_y)) \qquad \text{as } j\to\infty, \tag{4.15}$$

which can easily be seen to be possible, because $d_2\ge 0$. Now given $\phi\in C_0^\infty((0,L_y))$, for $j\in\mathbb{N}$ we consider

$$\begin{cases} \Psi_{jt} = d_{2j}(y)\Psi_{jyy}, \qquad y\in(0,L_y),\ t>0,\\ \Psi_{jy}=0, \qquad y\in\{0,L_y\},\ t>0,\\ \Psi_j(y,0)=\phi(y), \qquad y\in(0,L_y), \end{cases} \tag{4.16}$$

which possesses a solution $\Psi_j\in C^\infty([0,L_y]\times[0,\infty))$ according to standard parabolic theory, because $d_{2j}>0$ in $[0,L_y]$ and ϕ vanishes near the boundary of $(0,L_y)$. Consequently, $\Phi_j(y,t):=\Psi_j(y,t_0-t)$, $(y,t)\in[0,L_y]\times[0,t_0]$, satisfies the backward problem

$$\begin{cases} \Phi_{jt} = -d_{2j}(y)\Phi_{jyy}, \qquad y\in(0,L_y),\ t\in(0,t_0),\\ \Phi_{jy}=0, \qquad y\in\{0,L_y\},\ t\in(0,t_0),\\ \Phi_j(y,t_0)=\phi(y), \qquad y\in(0,L_y), \end{cases}$$

and using Φ_j as a test function in (4.12) yields

$$\int_0^{L_y} U(y,t_0)\phi(y)dy = \int_0^{t_0}\int_0^{L_y} \Big\{(d_2(y)-d_{2j}(y))\Phi_{jyy}\Big\}\cdot U. \tag{4.17}$$

In order to estimate the integral on the right, we multiply (4.16) by Ψ_{jyy} and integrate by parts to obtain the energy inequality

$$\frac12\frac{d}{dt}\int_0^{L_y}\Psi_{jy}^2 = -\int_0^{L_y} d_{2j}(y)\Psi_{jyy}^2 \qquad \text{for all } t>0,$$

which upon a time integration shows that

$$\int_0^{t_0}\int_0^{L_y} d_{2j}(y)\Psi_{jyy}^2 \le \int_0^{L_y}\phi_y^2 \tag{4.18}$$

for all $j\in\mathbb{N}$. Therefore, using the Cauchy-Schwarz inequality we find that

$$\begin{aligned} &\left|\int_0^{t_0}\int_0^{L_y}\Big\{(d_2(y)-d_{2j}(y))\Phi_{jyy}\Big\}\cdot U\right| \\ &\le \|U\|_{L^\infty((0,L_y)\times(0,t_0))}\cdot\left(\int_0^{t_0}\int_0^{L_y} d_{2j}(y)\Phi_{jyy}^2\right)^{\frac12}\cdot\left(\int_0^{t_0}\int_0^{L_y}\frac{(d_{2j}(y)-d_2(y))^2}{d_{2j}(y)}\right)^{\frac12} \\ &\to 0 \qquad \text{as } j\to\infty. \end{aligned}$$

Accordingly, (4.14) results from (4.17), (4.18) and (4.15) in the limit $j \to \infty$. $\square$

Remark. Unfortunately, the corresponding uniqueness question for the two-dimensional degenerate problem (1.4) has to be left open here. Indeed, it would be interesting to see whether effects stemming from either the mere presence of a second independent variable, or from the non-smoothness of the spatial rectangle Ω may result in nonuniqueness within our very weak solution concept in appropriate cases.

5 Large time behaviour

We now address the main topic of this paper by turning our attention to the large time behaviour of our solutions to (1.4) and (1.9). As a general functional analytic ingredient, let us recall a known observation which provides an elementary but highly useful tool not only in several places in this paper, but also in the description of large time behaviour in many related evolution problems with dissipative structure (cf. [1] or [25], for instance).

Lemma 5.1 *Let $n \geq 1$, $G \subset \mathbb{R}^n$ be measurable and $\phi = \phi(\xi, t) \in C^0([0, \infty); L^2(G))$ be such that*

$$\int_1^\infty \int_G \phi_t^2(\xi, t) d\xi dt < \infty. \tag{5.1}$$

Then for any sequence $(t_j)_{j \in \mathbb{N}} \subset (1, \infty)$ such that $t_j \to \infty$ as $j \to \infty$ we have

$$\int_{t_j}^{t_j+1} \int_G |\phi(\xi, t) - \phi(\xi, t_j)|^2 d\xi dt \to 0 \qquad \text{as } j \to \infty. \tag{5.2}$$

PROOF. Since $\phi(\cdot, t) - \phi(\cdot, t_j) = \int_{t_j}^t \phi_t(\cdot, s) ds$, using the Cauchy-Schwarz inequality we can estimate

$$\begin{aligned}
\int_{t_j}^{t_j+1} \int_G |\phi(\xi, t) - \phi(\xi, t_j)|^2 d\xi dt &\leq \int_{t_j}^{t_j+1} \int_G \left(\int_{t_j}^t \phi_t^2(\xi, s) ds \right) \cdot (t - t_j) d\xi dt \\
&\leq \left(\int_{t_j}^\infty \int_G \phi_t^2(\xi, s) d\xi ds \right) \cdot \left(\int_{t_j}^{t_j+1} (t - t_j) dt \right) \\
&= \frac{1}{2} \int_{t_j}^\infty \int_G \phi_t^2.
\end{aligned}$$

Therefore, (5.2) results from (5.1). $\square$

The following statement on large time behaviour is basically well-known and implicitly contained in several arguments in the literature (see [1], for instance). We include a proof for the sake of completeness.

Lemma 5.2 *Let $n \geq 1$, $G \subset \mathbb{R}^n$ be a bounded domain and $\phi = \phi(\xi, t) \in C^0([0, \infty); L^2(G))$ satisfy*

$$\int_1^\infty \int_G \phi_t^2 < \infty \qquad \text{and} \qquad \int_1^\infty \int_G |\nabla \phi|^2 < \infty. \tag{5.3}$$

Then the ω-limit set of ϕ in $L^2(G)$ exclusively consists of constants; that is, if $(t_j)_{j\in\mathbb{N}} \subset (1,\infty)$ and $w \in L^2(G)$ are such that $t_j \to \infty$ and $\phi(\cdot,t_j) \to w$ in $L^2(G)$ as $j \to \infty$, then there exists $c \in \mathbb{R}$ such that $w \equiv c$ a.e. in G.

Proof. We let $(t_j)_{j\in\mathbb{N}}$ and w be as in the above hypothesis and abbreviate $\overline{\phi}(t) := \frac{1}{|G|}\int_G \phi(\cdot,t)$ for $t>1$. Then the Poincaré inequality provides $C_P > 0$ such that

$$\int_G |\phi(\cdot,t) - \overline{\phi}(t)|^2 \le C_P \int_G |\nabla\phi(\cdot,t)|^2 \qquad \text{for all } t>1,$$

so that using (5.3) we find that

$$I_1(j) := \int_{t_j}^{t_j+1} \int_G |\phi(\cdot,t) - \overline{\phi}(t)|^2 \le C_P \int_{t_j}^{\infty} \int_G |\nabla\phi|^2 \to 0 \qquad \text{as } j \to \infty. \tag{5.4}$$

We now apply Lemma 5.1 twice to ϕ and $\overline{\phi}$ to see that

$$I_2(j) := \int_{t_j}^{t_j+1} \int_G |\phi(\cdot,t) - \phi(\cdot,t_j)|^2 \to 0 \qquad \text{as } j \to \infty \tag{5.5}$$

and

$$I_3(j) := \int_{t_j}^{t_j+1} \int_G |\overline{\phi}(t) - \overline{\phi}(t_j)|^2 \to 0 \qquad \text{as } j \to \infty. \tag{5.6}$$

Combining (5.4)-(5.6) and using our assumption that $\phi(\cdot,t_j) \to w$ in $L^2(G)$ as $j\to\infty$, we infer that

$$\begin{aligned}
\|\overline{\phi}(t_j) - w\|_{L^2(G)} &= \left(\int_{t_j}^{t_j+1}\int_G |\overline{\phi}(t_j) - w|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{I_3(j)} + \sqrt{I_1(j)} + \sqrt{I_2(j)} + \left(\int_{t_j}^{t_j+1}\int_G |\phi(\cdot,t_j) - w|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{I_3(j)} + \sqrt{I_1(j)} + \sqrt{I_2(j)} + \|\phi(\cdot,t_j) - w\|_{L^2(G)} \\
&\to 0 \qquad \text{as } j \to \infty,
\end{aligned}$$

which combined with the fact that G is connected means that indeed w must coincide with a constant a.e. in G. □

5.1 Large time behaviour in the regularised problem. Proof of Proposition 1.3

Without any further preparation, we can immediately pass to the proof of Proposition 1.3.

Proof of Proposition 1.3. We apply Lemma 3.4 and recall (4.2) to obtain

$$\int_0^\infty \int_\Omega \left\{ d_{1\varepsilon}(y) d_{2\varepsilon}(y) u_{\varepsilon x}^2 + \Big(d_{2\varepsilon}(y) u_\varepsilon\Big)_y \right\} < \infty.$$

Since $\frac{d_{1\varepsilon}}{d_{2\varepsilon}}$ is bounded in $[0, L_y]$ for each fixed $\varepsilon \in (0,1)$, this entails that the function v_ε defined by $v_\varepsilon(x,y,t) := d_{2\varepsilon}(y) \cdot u_\varepsilon(x,y,t)$, $(x,y,t) \in \Omega \times (0,\infty)$, satisfies

$$\int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 < \infty. \tag{5.7}$$

Similarly, Lemma 3.8 and (4.2) yield

$$\int_1^\infty \int_\Omega d_{2\varepsilon}(y) u_{\varepsilon t}^2 + \sup_{t>1} \int_\Omega \Big\{ d_{1\varepsilon}(y) d_{2\varepsilon}(y) u_{\varepsilon x}^2 + \Big(d_{2\varepsilon}(y) u_\varepsilon \Big)_y \Big\} < \infty$$

and thereby imply that

$$\int_1^\infty \int_\Omega v_{\varepsilon t}^2 + \sup_{t>1} \int_\Omega |\nabla u_\varepsilon(\cdot,t)\| < \infty. \tag{5.8}$$

In particular, the latter in conjunction with (3.6) shows that $(u_\varepsilon(\cdot,t))_{t>1}$ is bounded in $W^{1,2}(\Omega)$ and hence relatively compact in $L^2(\Omega)$. Now in order to show that

$$v_\varepsilon(\cdot,t) \to A_\varepsilon \quad \text{in } L^2(\Omega) \qquad \text{as } t \to \infty, \tag{5.9}$$

let us assume on the contrary that this was false. Then we could find a sequence $(t_j)_{j\in\mathbb{N}} \subset (1,\infty)$ such that $t_j \to \infty$ as $j \to \infty$ but $\liminf_{j\to\infty} \|v_\varepsilon(\cdot,t_j) - A_\varepsilon\|_{L^2(\Omega)} > 0$. Extracting a subsequence if necessary, we may assume that $v_\varepsilon(\cdot,t_j) \to w$ in $L^2(\Omega)$ as $j \to \infty$, whereupon in view of (5.7) and (5.8), Lemma 5.2 applies to ensure that $w \equiv A$ in Ω for some constant $A \in \mathbb{R}$. However, since by integration of (3.2) in space and using (4.2) we know that $\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0$ for all $t > 0$, this implies that

$$A \int_\Omega \frac{1}{d_{2\varepsilon}(y)} = \int_\Omega \frac{w}{d_{2\varepsilon}(y)} = \lim_{j\to\infty} \int_\Omega \frac{v_\varepsilon(\cdot,t_j)}{d_{2\varepsilon}(y)} = \lim_{j\to\infty} \int_\Omega u_\varepsilon(\cdot,t_j) = \int_\Omega u_0$$

and thereby identifies $A = A_\varepsilon$. This contradicts our assumption on the distance of $v_\varepsilon(\cdot,t_j)$ to A_ε and hence proves that actually (5.9) must be valid. Since $d_{2\varepsilon}$ is bounded from above and below by positive constants throughout $[0, L_y]$, (1.16) now results as a consequence of (5.9). $\Box$

5.2 Estimates for u

In order to determine the large time behaviour in the degenerate problem (1.4) on the basis of the estimates obtained so far, let us draw some consequences of the integral estimates gained in Section 3 for the limit function u. We first exploit Lemma 3.4 in a way convenient for our purposes.

Lemma 5.3 *Let u denote the function defined in (4.5). Then $(x,y,t) \mapsto d_2(y)u(x,y,t)$ belongs to $L^2_{loc}([0,\infty); W^{1,2}(\Omega))$, and we have*

$$\int_0^\infty \int_\Omega |\nabla (d_2(y)u)|^2 < \infty. \tag{5.10}$$

Proof. According to (3.19) and the fact that $\frac{d_{2\varepsilon}}{d_{1\varepsilon}}$ is bounded in $(0, L_y)$ uniformly with respect to $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$ by (1.8) and (1.6), (3.19) and (4.2) entail that $(\nabla(d_{2\varepsilon}u_\varepsilon))_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega\times(0,\infty))$. Hence, passing to a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ in (4.5) we can achieve that $\nabla(d_{2\varepsilon}u_\varepsilon)$ converges weakly in this space to some limit which can readily be identified to coincide with $\nabla(d_2 u)$ a.e. in $\Omega\times(0,\infty)$. The pointwise bound in Lemma 3.1 along with (4.2) thus completes the proof. □

Taking $\varepsilon \searrow 0$ appropriately, we next obtain from Lemma 3.6 a bound for u_x inside the domain Ω_{ab} of degenerate diffusion. This will enable us to conclude that u becomes homogeneous with respect to x as $t\to\infty$ in Theorem 1.1 ii). Apart from that, as a by-product the above entropy estimate will also provide a weighted pointwise bound for u in Ω_{ab} which might be of independent interest.

Lemma 5.4 *The limit function defined in (4.5) lies in $L^\infty_{loc}(\Omega_{ab}\times[0,\infty))$ and satisfies*

$$u(x,y,t) \le \frac{C}{(y-a)^2(b-y)^2} \qquad \text{for a.e. } (x,y,t)\in\Omega_{ab}\times(0,\infty) \tag{5.11}$$

with some $C>0$. Moreover, u_x belongs to $L^2((0,\infty);L^2_{loc}(\Omega_{ab}))$ with

$$\int_0^\infty\int_{\Omega_{ab}} (y-a)^4(b-y)^4u_x^2 < \infty. \tag{5.12}$$

Proof. For fixed $p\ge 2$, (3.22) combined with (4.2) asserts that $((\Theta^2(y)+\varepsilon^{\frac{p-1}{p}})u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty_{loc}([0,\infty);L^p(\Omega_{ab}))$, and hence along a subsequence of the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ in (4.5) we have

$$\Big(\Theta^2(y)+\varepsilon^{\frac{p-1}{p}}\Big)u_\varepsilon \stackrel{\star}{\rightharpoonup} \Theta^2(y)u \qquad \text{in } L^\infty_{loc}([0,\infty);L^p(\Omega_{ab})$$

by the Banach-Alaoglu theorem. We now let $\varepsilon\searrow 0$ along this sequence to see from (3.22) and a well-known argument involving lower semicontinuity with respect to weak-$\star$ convergence that

$$\int_{\Omega_{ab}}(\Theta^2(y)u)^p \le \int_{\Omega\setminus\Omega_{ab}}\frac{1}{d_2(y)}(d_2(y)u_0)^p + \int_{\Omega_{ab}}(\Theta^2(y)u_0)^p \qquad \text{for a.e. } t>0.$$

Taking the p-th root on both sides and using that $(A+B)^{\frac{1}{p}} \le 2^{\frac{1}{p}}(A^{\frac{1}{p}}+B^{\frac{1}{p}})$ for $A\ge 0$ and $B\ge 0$, from this we obtain that for a.e. $t>0$,

$$\|\Theta^2(y)u(\cdot,t)\|_{L^p(\Omega_{ab})} \le 2^{\frac{1}{p}}\cdot\bigg\{\Big(\int_{\Omega\setminus\Omega_{ab}}\frac{1}{d_2(y)}(d_2(y)u_0)^p\Big)^{\frac{1}{p}} + \Big(\int_{\Omega_{ab}}(\Theta^2(y)u_0)^p\Big)^{\frac{1}{p}}\bigg\}.$$

In the limit $p\to\infty$ we thus infer that $\Theta^2(y)u\in L^\infty(\Omega_{ab})$ for a.e. $t>0$ with norm controlled according to

$$\|\Theta^2(y)u(\cdot,t)\|_{L^\infty(\Omega_{ab})} \le c_1 := \|d_2(y)u_0\|_{L^\infty(\Omega\setminus\Omega_{ab})} + \|\Theta^2(y)u_0\|_{L^\infty(\Omega_{ab})} \qquad \text{for a.e. } t>0.$$

Now in view of the pointwise estimate $\Theta(y) \ge c_2(y-a)(b-y)$, valid for all $y \in (a,b)$ with some positive constant c_2, from this we conclude that (5.11) holds if we let $C := \frac{c_1}{c_2^2}$.

To verify the statements involving the derivatives with respect to x, we fix $p := 2$ in Lemma 3.6 to infer from (3.22) and the positivity of d_1 in $[0, L_y]$ that $((\Theta^2(y) + \varepsilon^{\frac{1}{2}})^2 u_{\varepsilon x})_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega_{ab} \times (0,\infty))$ and hence weakly convergent in this space along a suitable subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$. Thus, the proof becomes complete upon the observation that the corresponding limit must evidently coincide with $\Theta^4(y)u_x$ a.e. in $\Omega_{ab} \times (0,\infty)$. $\square$

The most important outcome of the next lemma is that the time derivative of u decays in a certain sense. In view of the weight function involved in the precise version (5.13) of this statement, in conjunction with Lemma 5.3 this will be the main ingredient for the proof of the large time asymptotics outside Ω_{ab}.

Lemma 5.5 *Let $v(x,y,t) := d_2(y)u(x,y,t)$, $(x,y,t) \in \Omega\times(0,\infty)$, where u is the very weak solution of (1.4) defined in (4.5). Then v belongs to $C^0((0,\infty); L^2(\Omega)) \cap L^\infty_{loc}((0,\infty); W^{1,2}(\Omega))$ with $v_t \in L^2_{loc}((0,\infty); L^2(\Omega))$. Moreover, we have*

$$\int_1^\infty \int_\Omega (d_2(y)u)_t^2 < \infty, \tag{5.13}$$

and there exists $C > 0$ such that

$$\int_\Omega |\nabla(d_2(y)u(\cdot,t))|^2 \le C \qquad \textit{for all } t > 1. \tag{5.14}$$

Proof. We let c_1 denote the constant in (3.19) and then obtain from that inequality that with $E_{\varepsilon k}$ as in Lemma 3.8 we have

$$\int_0^1 E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t))dt \le c_1 \qquad \text{for all } \varepsilon \in (0,1) \text{ and } k \in \mathbb{N}.$$

Thus, for any $\tau \in (0,1]$, each $\varepsilon \in (0,1)$ and all $k \in \mathbb{N}$ we can pick $t_{\varepsilon k} \in (0,\tau)$ such that

$$E_{\varepsilon k}(u_{\varepsilon k}(\cdot, t_{\varepsilon k})) \le \frac{c_1}{\tau}.$$

Hence, (3.30) tells us that for all $t > \tau$, $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$,

$$\int_\tau^t \int_{\Omega_k} d_{2\varepsilon} u_{\varepsilon kt}^2 + E_{\varepsilon k}(u_{\varepsilon k}(\cdot t)) \le \int_{t_{\varepsilon k}}^t \int_{\Omega_k} d_{2\varepsilon} u_{\varepsilon kt}^2 + E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t)) \le E_{\varepsilon k}(u_{\varepsilon k}(\cdot,t_{\varepsilon k})) \le \frac{c_1}{\tau} \tag{5.15}$$

Now according to (1.6) and (1.8), $(d_{2\varepsilon})_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,L_y))$ and $(d_{1\varepsilon})_{\varepsilon\in(0,1)}$ is bounded from below by a positive constant in $[0,L_y]$, whence we can find $c_2 > 0$ and $c_3 > 0$ fulfilling

$$(d_{2\varepsilon}u_{\varepsilon k})_t^2 \le c_2 d_{2\varepsilon} u_{\varepsilon kt}^2 \quad \text{and} \quad |\nabla(d_{2\varepsilon}u_{\varepsilon k})|^2 \le c_3\Big(d_{1\varepsilon}d_{2\varepsilon}u_{\varepsilon kx}^2 + (d_{2\varepsilon}u_{\varepsilon k})_y^2\Big) \quad \text{in } \Omega_k \times (0,\infty)$$

for all $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$. Consequently, (5.15) and (4.2) show that $((d_{2\varepsilon}u_\varepsilon)_t)_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega \times (\tau,\infty))$, and that $(\nabla(d_{2\varepsilon}u_\varepsilon))_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((\tau,\infty); L^2(\Omega))$.

According to the boundedness of $(d_{2\varepsilon}u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,\infty))$ as asserted by Lemma 3.1, the Arzelà-Ascoli theorem thus ensures that $(d_{2\varepsilon}u_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact with respect to the strong topology in $C^0_{loc}([\tau,\infty);L^2(\Omega))$. Since $\tau\in(0,1]$ was arbitrary, it is therefore clear that along a suitable subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ we have

$$\begin{cases} d_{2\varepsilon}(y)u_\varepsilon \to d_2(y)u & \text{in } C^0_{loc}((0,\infty);L^2(\Omega)) \quad \text{and a.e. in } \Omega\times(0,\infty),\\ \nabla(d_{2\varepsilon}(y)u_\varepsilon) \overset{\star}{\rightharpoonup} \nabla(d_2(y)u) & \text{in } L^\infty_{loc}(0,\infty);L^2(\Omega))\\ d_{2\varepsilon}(y)u_{\varepsilon t} \rightharpoonup d_2(y)u_t & \text{in } L^2_{loc}((0,\infty);L^2(\Omega)). \end{cases}$$

Finally, choosing $\tau:=1$ and taking limits in (5.15) we also obtain the inequalities

$$\int_1^\infty\int_\Omega (d_2u)_t^2 \le c_1c_2 \qquad \text{and} \qquad \int_\Omega |\nabla(d_2u(\cdot,t))|^2 \le c_1c_3 \quad \text{for all } t>1$$

and conclude the proof. □

Next, inside the domain of degenerate diffusion we shall rely on the following.

Lemma 5.6 *The solution u of (1.4) defined through (4.5) belongs to $C^0((0,\infty);L^2_{loc}(\Omega_{ab}))$ and satisfies*

$$\int_1^\infty\int_{\Omega_{ab}} (y-a)^4(b-y)^4u_t^2 < \infty. \tag{5.16}$$

Proof. We let φ_ε and $F_{\varepsilon k}$ be as in Lemma 3.9 with $q:=4$. We then recall the estimates (3.19) from Lemma 3.4 and apply (3.22) in Lemma 3.6 to $p:=2$ to obtain $c_1>0$ and $c_2>0$ such that

$$\int_0^1\int_{\Omega_k} d_{1\varepsilon}d_{2\varepsilon}u_{\varepsilon kx}^2 + \int_0^1\int_{\Omega_k}(d_{2\varepsilon}u_{\varepsilon k})_y^2 \le c_1$$

and

$$\int_0^1\int_{\Omega_{ab}} d_{1\varepsilon}(\Theta^2+\varepsilon^{\frac12})^2u_{\varepsilon kx}^2 + \varepsilon\int_0^1\int_{\Omega_{ab}}(\Theta^2+\varepsilon^{\frac12})^2u_{\varepsilon ky}^2 \le c_2$$

for all $\varepsilon\in(0,1)$ and $k\in\mathbb{N}$, where we have used that u_0 is bounded in Ω. Therefore,

$$\begin{aligned}\int_0^1 F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t))dt &\le \int_0^1\int_{\Omega_k} d_{1\varepsilon}d_{2\varepsilon}u_{\varepsilon kx}^2 + \int_0^1\int_{\Omega_{ab}} d_{1\varepsilon}(\Theta^2+\varepsilon^{\frac12})^2u_{\varepsilon kx}^2\\ &\quad +\varepsilon\int_0^1\int_{\Omega_{ab}}(\Theta^2+\varepsilon^{\frac12})^2u_{\varepsilon ky}^2 + \int_0^1\int_{\Omega_k}(d_{2\varepsilon}u_{\varepsilon k})_y^2\\ &\le c_1+c_2 \qquad \text{for all } \varepsilon\in(0,1),\end{aligned}$$

so that for all $\tau\in(0,1]$, $\varepsilon\in(0,1)$ and $k\in\mathbb{N}$ we can pick $t_{\varepsilon k}\in(0,\tau)$ such that

$$F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t_{\varepsilon k})) \le \frac{c_1+c_2}{\tau}.$$

Hence, (3.33) tells us that writing $\gamma := \frac{2\pi^2}{(b-a)^2}$ we have

$$\begin{aligned} \int_\tau^t \int_{\Omega_k} \varphi_\varepsilon u_{\varepsilon kt}^2 &\le \int_{t_{\varepsilon k}}^t \int_{\Omega_k} \varphi_\varepsilon u_{\varepsilon kt}^2 + F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t)) \le F_{\varepsilon k}(u_{\varepsilon k}(\cdot,t_{\varepsilon k})) \cdot e^{\gamma\cdot\varepsilon^{\frac{1}{2}}(t-t_{\varepsilon k})} \\ &\le (c_1+c_2)\cdot e^{\gamma\varepsilon^{\frac{1}{2}}t} \end{aligned} \tag{5.17}$$

for all $t>1$, each $\varepsilon \in (0,1)$ and any $k \in \mathbb{N}$.

In light of (4.2), this shows that $(\varphi_\varepsilon^{\frac{1}{2}} u_{\varepsilon t})_{\varepsilon\in(0,1)}$ is relatively compact in $L^2_{loc}([\tau,\infty);L^2(\Omega))$, which implies that $(\varphi_\varepsilon^{\frac{1}{2}} u_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $C^0_{loc}([\tau,\infty);L^2(\Omega))$. Since $\tau \in (0,1]$ was arbitrary, in view of (4.5) we thus clearly have $\varphi_\varepsilon^{\frac{1}{2}} u_{\varepsilon t} \rightharpoonup \varphi^{\frac{1}{2}} u_t$ in $L^2_{loc}((0,\infty);L^2(\Omega))$ and $\varphi_\varepsilon^{\frac{1}{2}} u_\varepsilon \to \varphi^{\frac{1}{2}} u$ in $C^0_{loc}((0,\infty);L^2(\Omega))$ along some subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$, where

$$\varphi(y) := \begin{cases} d_2(y) & \text{if } y \in [0,L_y]\setminus(a,b), \\ \Theta^4(y) & \text{if } y \in (a,b). \end{cases}$$

Moreover, on specifying $\tau := 1$ and taking limits in (5.17) we also gain that

$$\int_1^\infty \int_\Omega \varphi u_t^2 \le c_1 + c_2$$

from which (5.16) follows, because $\Theta(y) \ge c_3(y-a)(b-y)$ for all $y \in (a,b)$ and some $c_3 > 0$. □

5.3 Large time behaviour in $\Omega \setminus \Omega_{ab}$. Proof of Theorem 1.1 i)

We are now in the position to clarify the large time behaviour of u outside the domain where diffusion becomes degenerate.

PROOF of Theorem 1.1 i). We write $v(x,y,t) := d_2(y)u(x,y,t)$ for $(x,y,t) \in \Omega\times(0,\infty)$ and then obtain from Lemma 5.5 that

$$\int_1^\infty \int_\Omega v_t^2 < \infty \tag{5.18}$$

and

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le c_1 \qquad \text{for all } t>1 \tag{5.19}$$

with some $c_1 > 0$. Moreover, Lemma 5.3 ensures that

$$\int_1^\infty \int_\Omega |\nabla v|^2 < \infty. \tag{5.20}$$

Now from (5.19) and (4.8) we know that the semi-orbit $(v(\cdot,t))_{t>1}$ is bounded in $W^{1,2}(\Omega)$ and hence relatively compact in $L^2(\Omega)$. In view of a standard argument, in order to show that

$$v(\cdot,t) \to 0 \qquad \text{in } L^2(\Omega) \quad \text{as } t\to\infty \tag{5.21}$$

it is thus sufficient to make sure that zero is the only element of the ω-limit set of v in $L^2(\Omega)$; that is, (5.21) will be established as soon as we have shown that

$$\begin{cases} \text{whenever } (t_j)_{j\in\mathbb{N}} \subset (1,\infty) \text{ and } w \in L^2(\Omega) \text{ are such that } t_j \to \infty \\ \text{and } v(\cdot,t_j) \to w \text{ in } L^2(\Omega) \text{ as } j \to \infty, \text{ then } w \equiv 0 \text{ a.e. in } \Omega. \end{cases} \tag{5.22}$$

To assert the latter, given such $(t_j)_{j\in\mathbb{N}}$ and w we note that due to (5.18) and (5.20) we may apply Lemma 5.2 to obtain $c \in \mathbb{R}$ such that $w \equiv c$ a.e. in Ω. Since evidently $v = d_2(y)u \equiv 0$ a.e. in $\Omega_{ab} \times (0,\infty)$, we must have $w = \lim_{j\to\infty} v(\cdot,t_j) \equiv 0$ a.e. in Ω_{ab}. This proves (5.21), which in turn yields (1.10) in view of the fact that $d_2(y) > 0$ for all $y \in [0,L_y] \setminus [a,b]$. □

5.4 Large time behaviour in Ω_{ab}. Proof of Theorem 1.1 ii)

We next determine the large time behaviour of u inside Ω_{ab}.

Proof of Theorem 1.1 ii). We need to show that for any numbers α and β such that $a < \alpha < \beta < b$ we have

$$u(\cdot,t) \rightharpoonup \overline{u}_0 \qquad \text{in } L^2(\Omega_{\alpha\beta}) \quad \text{as } t \to \infty, \tag{5.23}$$

where $\Omega_{\alpha\beta} := (0,L_x) \times (\alpha,\beta)$. To this end, we observe that as a particular consequence of Lemma 5.4, the semi-orbit $(u(\cdot,t))_{t>0}$ is bounded in $L^2(\Omega_{\alpha\beta})$ and hence relatively compact with respect to weak convergence in $L^2(\Omega_{\alpha\beta})$. Guided by the procedure in the proof of Theorem 1.1 i), we note that in order to prove (5.23) it is thus sufficient to assert that

$$\begin{cases} \text{if } w \in L^2(\Omega_{\alpha\beta}) \text{ and } (t_j)_{j\in\mathbb{N}} \subset (1,\infty) \text{ are such that } t_j \to \infty \\ \text{and } u(\cdot,t_j) \rightharpoonup w \text{ in } L^2(\Omega_{\alpha\beta}) \text{ as } j \to \infty, \text{ then } w(x,y) = \overline{u}_0(y) \text{ for a.e. } (x,y) \in \Omega_{\alpha\beta}. \end{cases} \tag{5.24}$$

To see this, given $(t_j)_{j\in\mathbb{N}}$ and w as in (5.24), let us set

$$z_j(x,y) := \int_{t_j}^{t_j+1} u(x,y,t)dt, \qquad (x,y) \in \Omega_{\alpha\beta},\ j \in \mathbb{N},$$

and carry out the rest of the proof in four steps.

Step 1. We first claim that

$$z_j \rightharpoonup w \qquad \text{in } L^2(\Omega_{\alpha\beta}) \quad \text{as } j \to \infty. \tag{5.25}$$

In fact, since $\int_1^\infty \int_{\Omega_{\alpha\beta}} u_t^2 < \infty$ by Lemma 5.6, using the Cauchy-Schwarz inequality and recalling Lemma 5.1 we obtain

$$\begin{aligned} \int_{\Omega_{\alpha\beta}} |z_j - u(\cdot,t_j)|^2 &= \int_{\Omega_{\alpha\beta}} \left| \int_{t_j}^{t_j+1} \Big(u(x,y,t) - u(x,y,t_j) \Big) dt \right|^2 d(x,y) \\ &\le \int_{\Omega_{\alpha\beta}} \int_{t_j}^{t_j+1} |u(x,y,t) - u(x,y,t_j)|^2 dt d(x,y) \\ &\to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

Along with (5.24) this implies (5.25).

Step 2. We proceed to make sure that

$$z_{jx} \to 0 \qquad \text{in } L^2(\Omega_{\alpha\beta}) \quad \text{as } j \to \infty. \tag{5.26}$$

Indeed, as a consequence of Lemma 5.4 we have $\int_0^\infty \int_{\Omega_{\alpha\beta}} u_x^2 < \infty$. Therefore, again invoking the Cauchy-Schwarz inequality, we infer that

$$\int_{\Omega_{\alpha\beta}} z_{jx}^2 = \int_{\Omega_{\alpha\beta}} \left| \int_{t_j}^{t_j+1} u_x(x,y,t)dt \right|^2 d(x,y) \le \int_{\Omega_{\alpha\beta}} \int_{t_j}^{t_j+1} u_x^2(x,y,t)dtd(x,y) \to 0$$

as $j \to \infty$, as claimed.

Step 3. Let us next show that

$$z_j \to \overline{u}_0 \qquad \text{in } L^2(\Omega_{\alpha\beta}) \quad \text{as } j \to \infty. \tag{5.27}$$

For this purpose, we recall that the Poincaré inequality on the interval $(0, L_x)$ provides $C_P > 0$ such that

$$\int_0^{L_x} \left| \phi(x) - \frac{1}{L_x} \int_0^{L_x} \phi(\xi)d\xi \right|^2 dx \le C_p \int_0^{L_x} \phi_x^2(x)dx \qquad \text{for all } \phi \in W^{1,2}((0, L_x)). \tag{5.28}$$

We apply this to $\phi(x) := z_j(x,y)$ for fixed $y \in (\alpha, \beta) \setminus N_\star$ with the null set $N_\star$ as given by Corollary 2.4. Since this choice of y ensures that

$$\frac{1}{L_x} \int_0^{L_x} z_j(x,y)dx = \overline{u}_0(y)$$

according to Corollary 2.4, (5.28) implies that

$$\int_0^{L_x} |z_j(x,y) - \overline{u}_0(y)|^2 dx \le C_P \int_0^{L_x} z_{jx}^2(x,y)dx \qquad \text{for all } y \in (\alpha, \beta) \setminus N_\star.$$

Integrating this inequality over $y \in (\alpha, \beta) \setminus N_\star$ and using that $|N_\star| = 0$, we find that

$$\int_{\Omega_{\alpha\beta}} |z_j(x,y) - \overline{u}_0(y)|^2 d(x,y) \le C_P \int_{\Omega_{\alpha\beta}} z_{jx}^2(x,y)d(x,y) \qquad \text{for all } j \in \mathbb{N},$$

which combined with the outcome of Step 2 yields (5.27).

Step 4. We conclude the proof by combining (5.25) with (5.27) to infer that indeed $w(x,y)$ must coincide with $\overline{u}_0(y)$ for a.e. $(x,y) \in \Omega_{\alpha\beta}$, as desired. □

5.5 Convergence to Dirac measures. Proof of Theorem 1.2

Let us finally prove that in the sense asserted by Theorem 1.2, in the large time limit all the mass initially present in $(0, L_x) \times (0, a)$ and $(0, L_x) \times (b, L_y)$ will concentrate on the horizontal lines $(0, L_x) \times \{a\}$ and $(0, L_x) \times \{b\}$, respectively.

PROOF of Theorem 1.2. We fix $\varphi \in C^0([0, L_y])$ and claim that then for each $\eta > 0$ we can pick $t_0 = t_0(\varphi, \eta) > 0$ such that

$$I(t) := \Big| \int_0^{L_y} U(y,t)\varphi(y)dy - \int_a^b U_0(y)\varphi(y)dy - m_1 \cdot \varphi(a) - m_2 \cdot \varphi(b) \Big| < \eta \qquad \text{for all } t > t_0. \tag{5.29}$$

To see this, given $\eta > 0$ we choose $\mu \in (0, \min\{a, L_y - b, \frac{b-a}{2}\})$ small enough such that writing $m := \|u_0\|_{L^1(\Omega)}$ we have

$$\sup_{y\in(a-\mu,a+\mu)} |\varphi(y) - \varphi(a)| < \frac{\eta}{9m} \qquad \text{and} \qquad \sup_{y\in(b-\mu,b+\mu)} |\varphi(y) - \varphi(b)| < \frac{\eta}{9m}, \tag{5.30}$$

as well as

$$\int_a^{a+\mu} \int_0^{L_x} u_0(x,y)dxdy < \frac{\eta}{9} \qquad \text{and} \qquad \int_{b-\mu}^{b} \int_0^{L_x} u_0(x,y)dxdy < \frac{\eta}{9}, \tag{5.31}$$

where we note that (5.30) is possible due to the continuity of φ. Next, thanks to Theorem 1.1 i) we know that $u(\cdot,t) \to 0$ in $L^2_{loc}(\bar\Omega \setminus \bar\Omega_{ab})$ as $t \to \infty$, whence in particular we can find $t_1 > 0$ such that

$$\int_0^{a-\mu} \int_0^{L_x} u(x,y,t)dxdy < \frac{\eta}{9M} \quad \text{and} \quad \int_{b+\mu}^{L_y} \int_0^{L_x} u(x,y,t)dxdy < \frac{\eta}{9M} \quad \text{for all } t > t_1, \tag{5.32}$$

where $M := \|\varphi\|_{L^\infty(\Omega)}$. Moreover, Theorem 1.1 ii) warrants that $\int_{a+\mu}^{b-\mu} \int_0^{L_x} (u(x,y,t) - u_0(x,y))\varphi(y)dxdy \to 0$ as $t \to \infty$, whence for some $t_2 > 0$ we have

$$\Big| \int_{a+\mu}^{b-\mu} \int_0^{L_x} u(x,y,t)\varphi(y)dxdy - \int_{a+\mu}^{b-\mu} \int_0^{L_x} u_0(x,y)\varphi(y)dxdy \Big| < \frac{\eta}{9} \qquad \text{for all } t > t_2. \tag{5.33}$$

Then the expression on the left of (5.29) can be estimated according to

$$\begin{aligned}
I(t) \;\le\; & \Big| \int_0^{a-\mu} U(y,t)\varphi(y)dy \Big| + \Big| \int_{a-\mu}^{a+\mu} U(y,t) \cdot [\varphi(y) - \varphi(a)]dy \Big| \\
& + \Big| \Big(\int_{a-\mu}^{a+\mu} U(y,t)dy - m_1 \Big) \cdot \varphi(a) \Big| + \Big| \int_{a+\mu}^{b-\mu} U(y,t)\varphi(y) - \int_{a+\mu}^{b-\mu} U_0(y)\varphi(y)dy \Big| \\
& + \Big| \int_{b-\mu}^{b+\mu} U(y,t)[\varphi(y) - \varphi(b)]dy \Big| + \Big| \Big(\int_{b-\mu}^{b+\mu} U(y,t)dy - m_2 \Big) \cdot \varphi(b) \Big| \\
& + \Big| \int_{b+\mu}^{L_y} U(y,t)\varphi(y)dy \Big| \\
=: \;& I_1(t) + ... + I_7(t) \qquad \text{for all } t > 0.
\end{aligned} \tag{5.34}$$

Here by nonnegativity of u and (5.32),

$$I_1(t) = \Big| \int_0^{a-\mu} \int_0^{L_x} u(x,y,t)\varphi(y)dxdy \Big| \le M \cdot \int_0^{a-\mu} \int_0^{L_x} u(x,y,t)dxdy < \frac{\eta}{9} \quad \text{for all } t > t_1 \tag{5.35}$$

and similarly

$$I_7(t) < \frac{\eta}{9} \qquad \text{for all } t > t_1. \tag{5.36}$$

Next, using (5.30) and (4.8) we can estimate

$$\begin{aligned} I_2(t) &= \Big| \int_{a-\mu}^{a+\mu} \int_0^{L_x} u(x,y,t) \cdot [\varphi(y) - \varphi(a)]dxdy \Big| \\ &\le \|u(\cdot,t)\|_{L^1(\Omega)} \cdot \sup_{y\in(a-\mu,a+\mu)} |\varphi(y) - \varphi(a)| < \frac{\eta}{9} \qquad \text{for all } t > 0, \end{aligned} \tag{5.37}$$

and by the same token we see that

$$I_6(t) < \frac{\eta}{9} \qquad \text{for all } t > 0. \tag{5.38}$$

We now further split

$$I_3(t) = \Big| \Big(\int_0^{a+\mu} U(y,t)dy - m_1 \Big) \cdot \varphi(a) - \Big(\int_0^{a-\mu} U(y,t) \Big) \cdot \varphi(a) \Big|$$

and recall (4.9) and (5.31) which state that $\int_0^{a+\mu} U(y,t)dy - m_1 < \frac{\eta}{9}$ for all $t > 0$. Therefore, again by (5.32) we obtain

$$I_3(t) \le \frac{\eta}{9} + \Big(\int_0^{a-\mu} \int_0^{L_x} u(x,y,t)dxdy \Big) \cdot |\varphi(a)| < \frac{2\eta}{9} \qquad \text{for all } t > t_1, \tag{5.39}$$

and in the same way it follows that

$$I_5(t) < \frac{2\eta}{9} \qquad \text{for all } t > t_1. \tag{5.40}$$

Finally, (5.33) precisely says that

$$I_4(t) \le \frac{\eta}{9} \qquad \text{for all } t > t_2.$$

In conjunction with (5.35)-(5.40), inserted into (5.32) this establishes (5.29) with $t_0 := \max\{t_1, t_2\}$, whereby the proof is completed. □

6 Numerical explorations

In this section we perform a numerical study of the initial-boundary value problem (1.4) to both validate and extend the results of previous sections. We choose the following two choices for $d_{1,2}$: a smooth step

$$d_1(y) = \frac{2 + \tanh\frac{y-a}{\varepsilon} - \tanh\frac{y-b}{\varepsilon}}{4}, \quad d_2(y) = \frac{2 - \tanh\frac{y-a}{\varepsilon} + \tanh\frac{y-b}{\varepsilon}}{4}; \tag{6.1}$$

and the piecewise constant form

$$d_1(y) = \begin{cases} 1-\varepsilon & y \in [a,b] \\ 0.5 & y \in [0,L_y] \setminus [a,b] \end{cases} , \quad d_2(y) = \begin{cases} \varepsilon & y \in [a,b] \\ 0.5 & y \in [0,L_y] \setminus [a,b] \end{cases} . \tag{6.2}$$

Both of the above approach the prototypical form (1.5) in the limit $\varepsilon \to 0$.
Numerical results were obtained via a Methods of Lines approach, with spatial terms in equations (1.4) discretised using a second order central difference scheme. The resulting ODEs were solved using a variable time-stepping stiff integrator. Spatial discretisation in both one and two dimensions has been performed with a variable mesh, positioning a larger number of lattice points at the interface between the aligned and isotropic regions to provide greater refinement in these areas. Accuracy of the scheme has been examined through comparing against analytical steady state predictions, performing simulations on both fixed and variable-spaced meshes of greater refinement and employing distinct time integration schemes.

6.1 Quasi-one-dimensional simulations

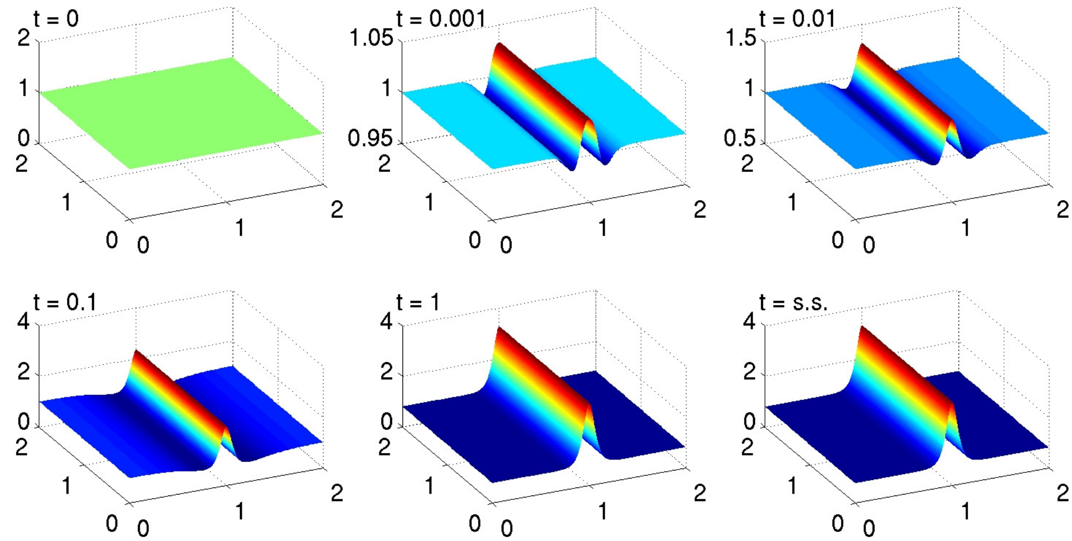
We begin by considering a quasi-one-dimensional scenario in which $u_0(x,y) = 1$. Figure 3 plots the time evolution of u for system (1.4) together with the smooth form (6.1) for $a = 0.9$, $b = 1.1$ and (a) $\varepsilon = 0.1$, (b) $\varepsilon = 0.001$. For both values of ε alignment along the strip acts to trap the population within this region, resulting in increased density.
In accordance with Proposition 1.3, we see in Figure 3 (a) that for larger values of ε the population quickly evolves to a non-uniform steady state distribution in which the maximum density lies along the centre of the aligned region. For comparison, the final frame in Figure 3 (a) plots the analytically-determined steady state solution obtained by setting $u_t = 0$: we observe negligible difference with the computed solution at $t = 10$.
For smaller ε, however, we instead see extremely steep ridges of high population density form at the interface between isotropic and aligned regions (i.e. along the lines $y = a$ and $y = b$). Subsequent dispersal of these ridges within the aligned region is extremely slow. While the expected final steady state pattern would be a single highly concentrated ridge lying along the midline this is never observed within simulation timescales. Simulations (not shown) for the piecewise linear form (6.2) show similar behaviour as $\varepsilon \to 0$.
We exploit the quasi-one-dimensional nature of the simulations to perform a refined and extended analysis. Specifically, we consider the equivalent one-dimensional model

$$u_t = (d_2(y)u)_{yy} , \quad (d_2(y)u)_y = 0 \text{ at } y = 0, L_y , \quad u(y,0) = u_0(y) , \tag{6.3}$$

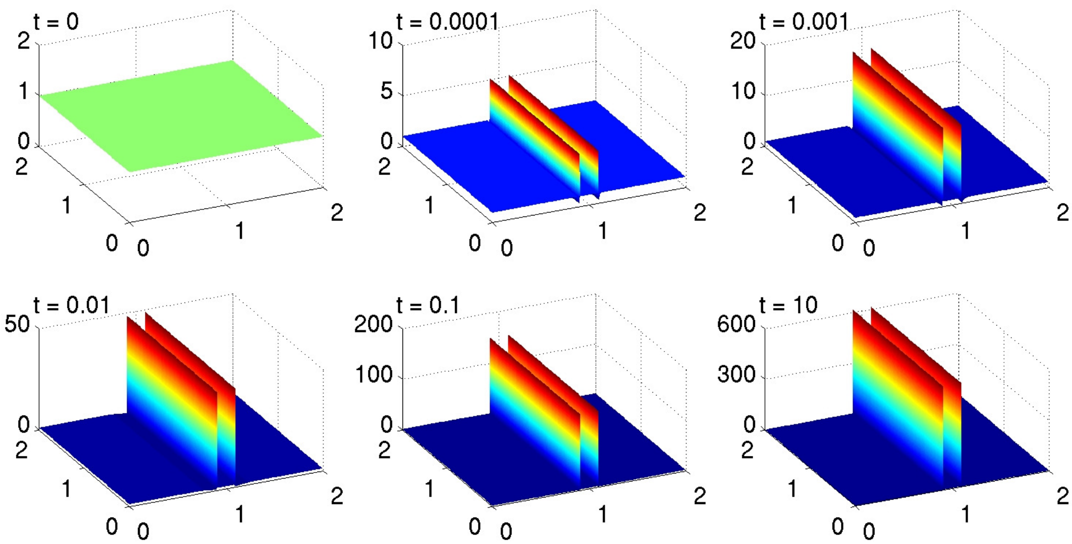
together with the previously proposed forms for $d_2(y)$. Setting $u_t = 0$ in (6.3), integrating and applying the boundary conditions gives (for nonconstant d_2) the heteregeneous steady state

$$u_{ss}(y) = \frac{c}{d_2(y)} . \tag{6.4}$$

In the above, the constant c is determined from conservation of mass and the imposed initial conditions, cf. also the corresponding brief discussion in the introduction, and Proposition 1.3. For the smooth form (6.1) we have a single minimum at $y_m = (a+b)/2$ and we expect the steady state to be composed of a single aggregate with maximum at y_m.



(a) epsilon = 0.1



(b) epsilon = 0.001

Figure 3: Simulation of the anisotropic diffusion model, system (1.4) together with $u_0(x, y) = 1$ and the smooth form (6.1). (a) Time evolution showing solutions using $\epsilon = 0.1$, $a = 0.9$ and $b = 1.1$ in (6.1). The population accumulates inside the aligned region, evolving to a heterogeneous steady state distribution. The analytically-determined solution steady state solution is shown in the final panel. (b) Time evolution showing solutions using $\epsilon = 0.001$, $a = 0.9$ and $b = 1.1$ in (6.1). The population accumulates into two extremely concentrated ridges at the interface between the isotropic and aligned regions, with little subsequent movement within simulation timescales. Simulations performed as described in the text. Here we have employed a variable 2D mesh, concentrating points at the aligned region to provide better resolution of the ridge development. Absolute and relative tolerances for the time-integration scheme were set at 10^{-8}.

An extended simulation of (6.3) with $d_2(y)$ as in equation (6.1) and $\varepsilon = 0.01, a = 0.9, b = 1.1$ is plotted in Figure 4. Initially we observe the accumulation of the population into two concentrated peaks close to $y = a$ and $y = b$. The peaks subsequently converge within the region of low d_2 on a much slower timescale, eventually accumulating into a single concentrated peak at $y = (a+b)/2$: the solution at $t = 10^5$ is, as predicted by Proposition 1.3, extremely close to the steady state solution in (6.4) shown in the final panel.

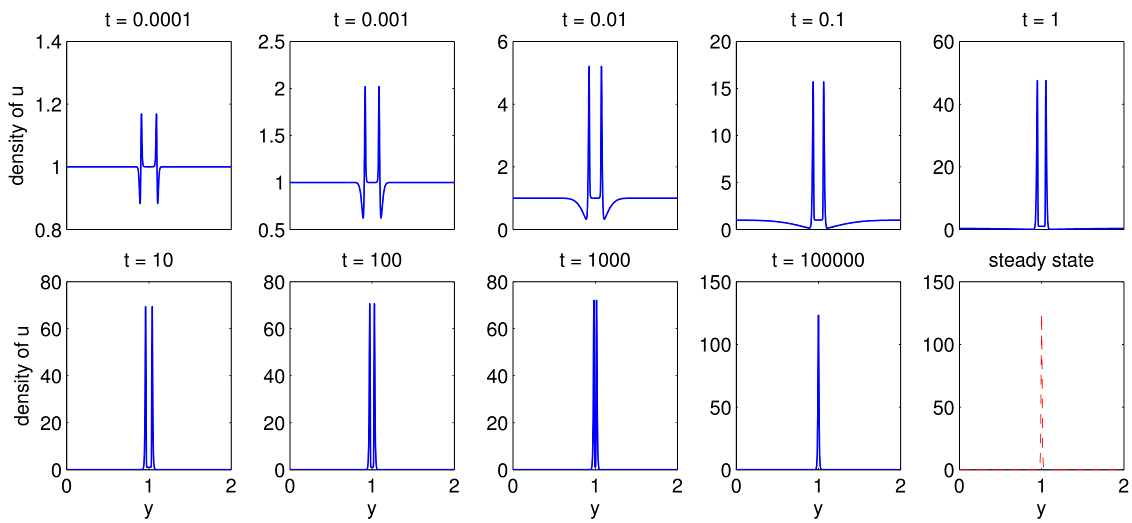


Figure 4: Simulation showing the time evolution of solutions to equation (6.3) with $u_0(y) = 1$ and $d_2(y)$ as in (6.1) with $\epsilon = 0.01$, $a = 0.9$ and $b = 1.1$. The population initially accumulates into two concentrated peaks at $y = a$ and $y = b$, which subsequently undergo slow convergence to a single peak in the centre of domain. The final panel plots the analytically determined steady state solution. Simulations performed as described in the text, where we have again employed a variable mesh. Error tolerances as in Figure 3.

As $\varepsilon \to 0$ we obtain convergence to the prototypical form for d and we might expect Dirac-type singularities to form at the interface points $y = a$ and $y = b$. Simulations presented in Figure 5 appear to support this conjecture. While for larger values of ε (top row) any peak formation at $y = a$ and $y = b$ is highly transient with solutions quickly converging to the steady state distribution, for smaller ε two peaks initially form at $y = a$ and $y = b$. Notably, these peaks become more concentrated with decreasing ε and any subsequent convergence of the peaks for the smallest values of ε (bottom two rows) is imperceptible within practical numerical timescales.

6.2 Comparison with a Dirichlet problem

Formation of the boundary peaks results from diffusion between the isotropic and anisotropic region where they become pinned due to realignment. In this section we provide *numerical* evidence indicating that this population transfer evolves according to a Dirichlet problem

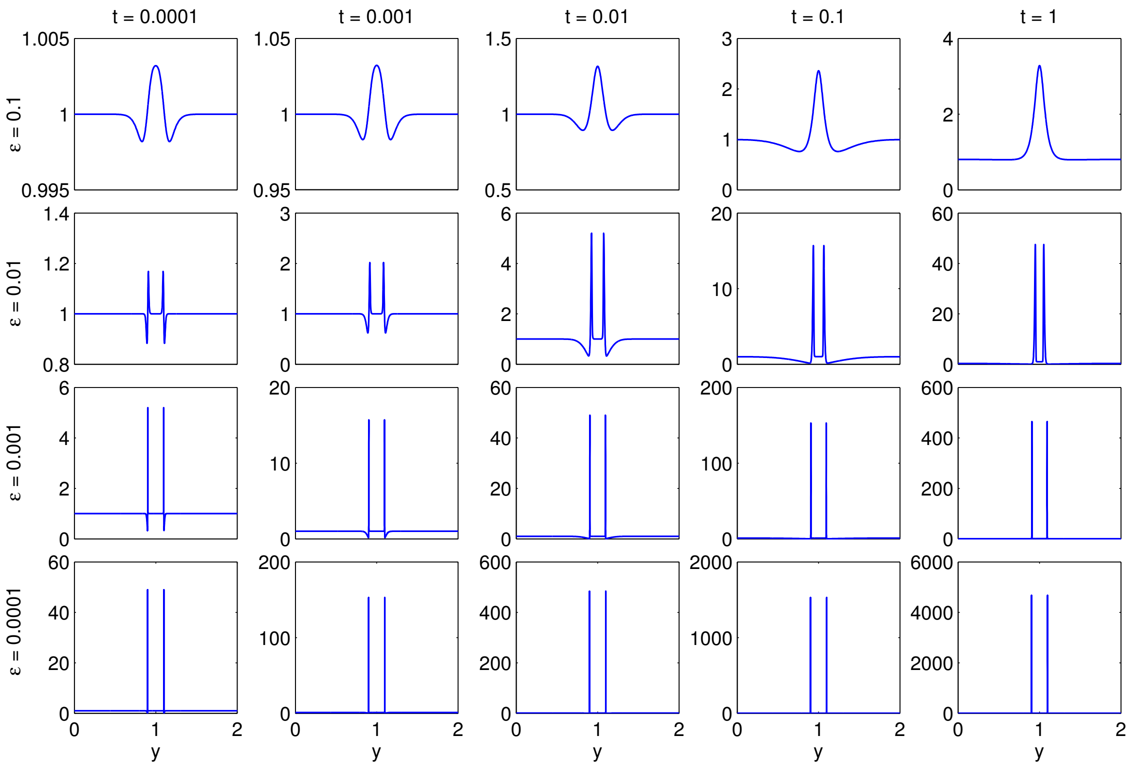


Figure 5: Simulation of the 1D diffusion model, equation (6.3), together with $u_0(y) = 1$ and $d_2(y)$ as in (6.1) for varying ε, $a = 0.9$ and $b = 1.1$. In the top to bottom rows we compare the evolution of solutions at comparable times for $\varepsilon = 0.1, 0.01, 0.001$ and 0.0001. As $\varepsilon \to 0$ we note that solutions become more sharply concentrated into aggregates at $y = a$ and $y = b$. Simulation method as described in Figure 4.

in the limit $\varepsilon \to 0$. Specifically, for the region $y \in (0, a)$ (or $y \in (b, L_y)$) we consider

$$v_t = 0.5 v_{yy}\,, \quad v_y(0,t) = 0\,, v(a,t) = 0\,, \quad v(y,0) = 1\,. \tag{6.5}$$

Thus the point a defines a sink for the population v within the isotropic region. Solved using standard methods we obtain the analytical solution

$$v_a(y,t) = \sum_{i=0}^{\infty} \frac{4}{(2i+1)\pi} \sin\left(\frac{(2i+1)\pi(x+a)}{2a}\right) \exp\left(\frac{-0.5\,(2i+1)^2\,\pi^2 t}{4a^2}\right)\,. \tag{6.6}$$

In Figure 6 we plot numerical simulations for equations (6.3) together with $d_2(y)$ as in equation (6.1) under varying ε. Here we assume $a = 1$ and $a \ll b \ll L_y$, restricting our attention to the region $y \in [0, 2]$ such that we concentrate on the single smooth step centered on $y = 1$. Further, we append these plots with the analytical solution v_a derived from the Dirichlet problem (6.5). Here we have calculated v_a by truncating at the first 100 terms, more than enough needed to generate a highly accurate solution.

As expected from previous studies, as $\varepsilon \to 0$ we observe the development of a single concentrated peak located at $y = a$. Moreover, as $\varepsilon \to 0$ the numerical solutions in the region $y \in [0, a)$ appear to converge with the analytical form $v_a(y,t)$ derived from the Dirichlet problem. This convergence becomes more apparent in the blown-up sequence of plots shown in the bottom row. We have further controlled our assertion through testing for other values of a, using the step-like form for d_2 and exploring whether the same behaviour occurs in the original two dimensional simulations.

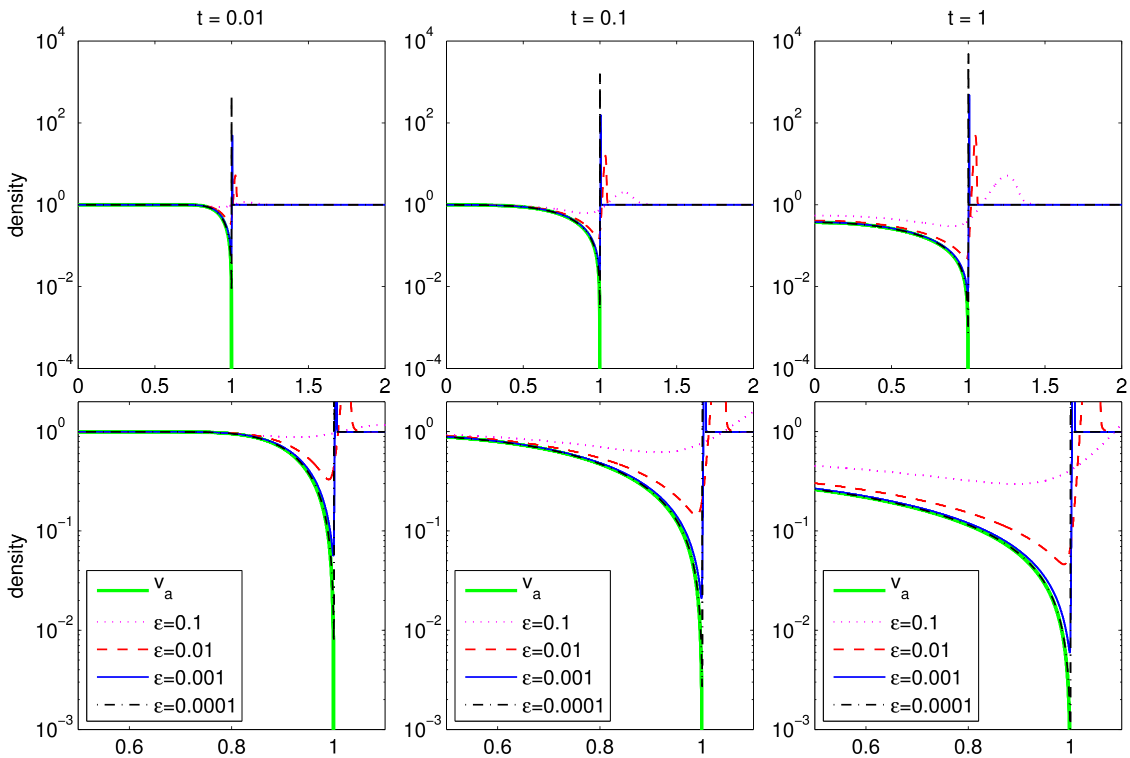


Figure 6: Comparison between solutions to the 1D diffusion model (6.3) and the analytical solution (6.6) to the Dirichlet problem. For (6.3) we consider $u_0(y) = 1$, $d_2(y)$ as in (6.1) for varying ε and $a = 1$. The top row plots the $\log_{10}$ of the population density against y for various values of ε at the times indicated. In each frame we add the analytical solution $v_a(y,t)$ for the region $y \in [0, a)$. The bottom row expands the bottom-left corner of these plots to reveal the convergence to the analytical form. Simulation method as described in Figure 4.

6.3 2D simulations: nonuniform initial conditions

In the above simulations we have concentrated either on the quasi-one dimensional case with uniform initial conditions or its equivalent 1D version. In this section we consider non-uniform initial conditions. Again we consider the smooth form (6.1), together with

$a = 0.9, b = 1.1$ and $L_x = L_y = 2$ and initially suppose the population is concentrated into two aggregates arranged as follows:

$$u_0(x, y) = 0.5e^{-100\left((x-0.5)^2+(y-0.5)^2\right)} + e^{-100\left((x-1.5)^2+(y-1.5)^2\right)}.$$

Simulations in Figure 7 reveal the population evolution for (a) $\varepsilon = 0.1$, and (b) $\varepsilon = 0.001$. For both values of ε the population diffuses into the anisotropic region where, due to realignment, the population accumulates. Initially a much larger accumulation is observed at the points closest to the initial aggregation sites. Dispersal along the direction of alignment results in the subsequent broadening of these aggregates until a quasi-one-dimensional configuration is achieved. As previously, while for the larger values of ε solutions converge to the heterogeneous steady state within the simulation timescales, for smaller ε the population remains confined to a sharp ridge. Note that according to Theorem 1.1 and Theorem 1.2, the density of these ridges corresponds to the size of the initial population within the isotropic regions from which each ridge derives.

7 Conclusion

It is surprising that a simple linear diffusion equation (1.1) can have such a rich behaviour in spatial pattern formation and blow-up. This is, of course, related to the singular nature of the diffusion tensor. It is of large interest to study other geometries and to understand and classify all singularities that can arise from equations of the form (1.1).

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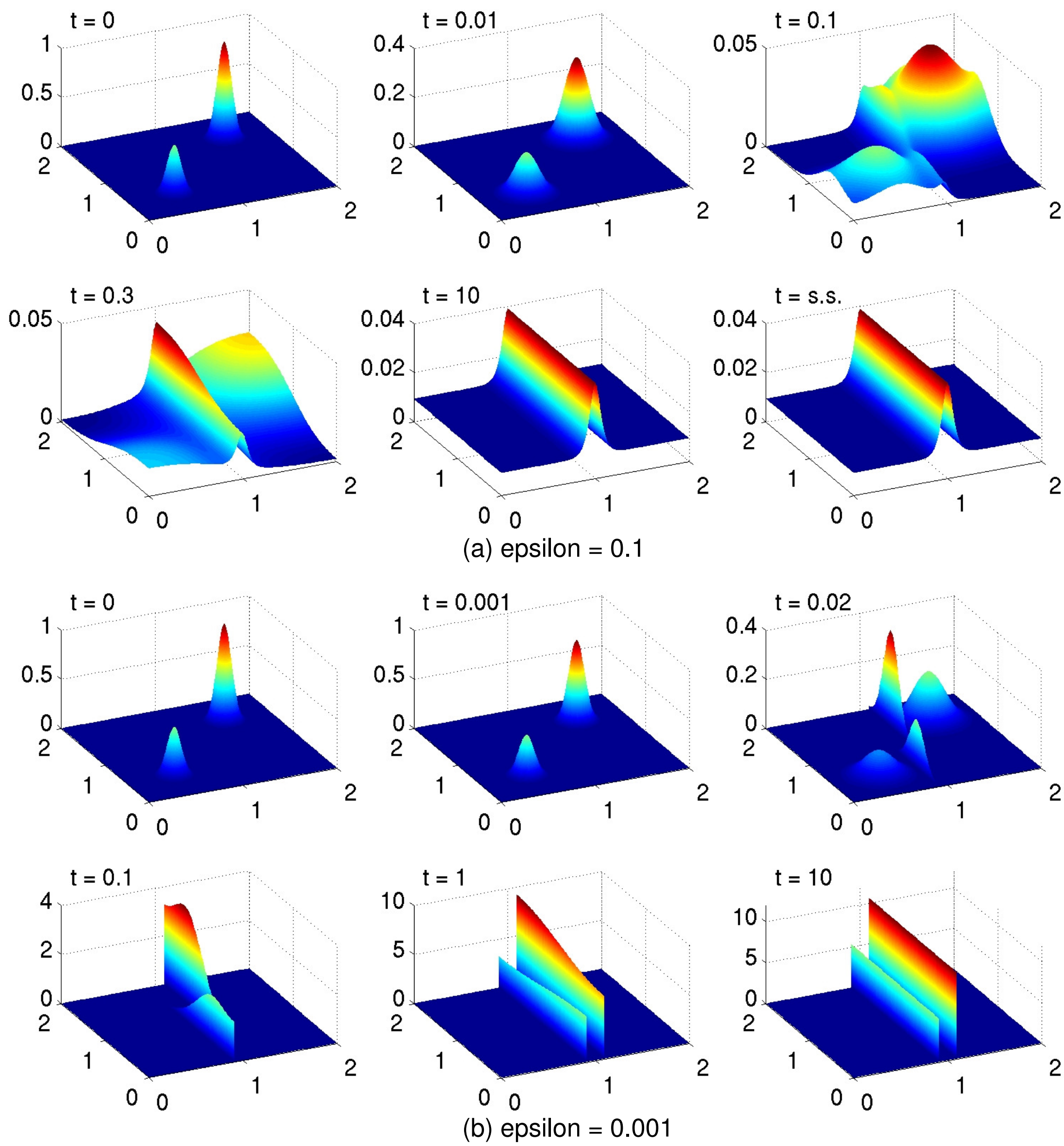

Figure 7: Simulation of the anisotropic diffusion model, system (1.4) with nonuniform initial conditions (see text for details). (a) We set $\epsilon = 0.1$, $a = 0.9$ and $b = 1.1$ in (6.1). (b) We set $\epsilon = 0.001$, $a = 0.9$ and $b = 1.1$ in (6.1). For details of numerical implementation, see Figure 3.

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