F1.3YT2 Assessment - Feb 7th 2008.

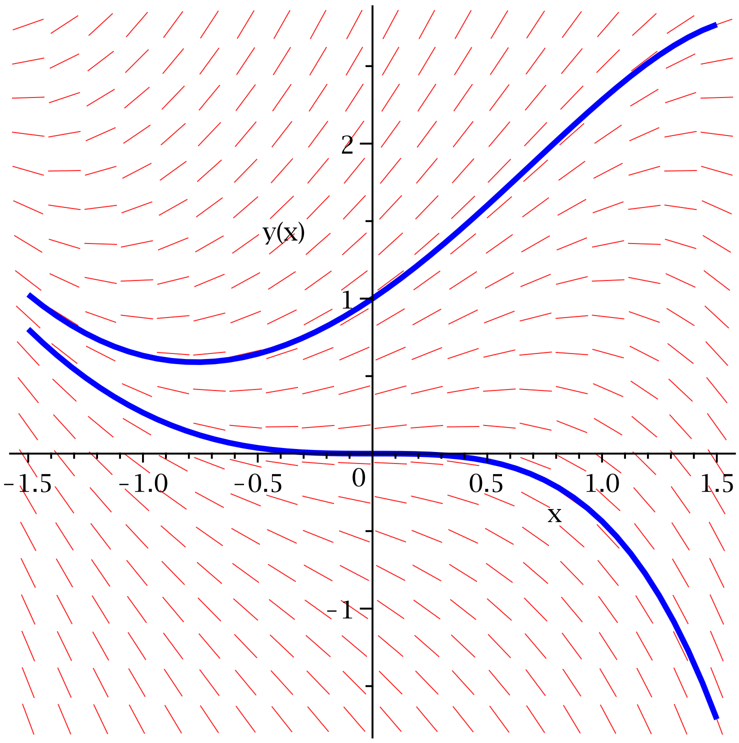
Attempt all 4 questions

1. (i) Sketch the direction field for the ODE

$$\frac{dy}{dx} = y - x^2.$$

(ii) Sketch the two solutions satisfying $y(0) = 1$ and $y(0) = 0$. **[5 marks]**

Solution: The direction field, and solutions satisfying the initial condition are



2. (i) Write the ODE

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 7x^2 = 0$$

as a system of 2 first order ODEs.

(ii) Write the following as a single system of 4 first order ODEs

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} = x, \quad \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - y = 0$$

[3 marks]

Solution: (i) Under the identification $x = x_1$ the equation may be written as the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ -7x_1^2 + 3x_2 \end{pmatrix}.$$

(ii) Under the identification $y = z_1$ and $x = z_3$ the system of equations may be written as

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ 3z_2 + z_3 \\ z_4 \\ z_1 + z_4^2 \end{pmatrix}.$$

3. Find the general solution of the following system of equations by finding the eigenvalues and eigenvectors of an appropriate matrix

$$\dot{x} = x + 3y, \quad \dot{y} = -3x + y$$

Solution: The equation may be written as

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \boldsymbol{x}.$$

The matrix has eigenvalues $(1 \pm 3i)$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. A complex solution is

$$\begin{aligned} \boldsymbol{y}(t) &= e^{(1+3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t\,(\cos(3t) + i\sin(3t))\begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix} + i\,e^t \begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}. \end{aligned}$$

Thus we obtain two real solutions

$$\boldsymbol{y}^{(1)}(t) = \operatorname{Re}\boldsymbol{y}(t) = e^t \begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix}, \quad \boldsymbol{y}^{(2)}(t) = \operatorname{Im}\boldsymbol{y}(t) = e^t \begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}.$$

These form a FSS.

The general solution is

$$\boldsymbol{x}(t) = c_1\,e^t \begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix} + c_2\,e^t \begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}$$

[6 marks]

4. Suppose that A is 2×2 constant matrix with eigenvalues -1 and 3 and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Use the method of variation of parameters to find a particular solution of the ODE $\dot{\boldsymbol{x}}(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}(t)$ where

$$\boldsymbol{b}(t) = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$$

[6 marks]

Solution: The fundamental matrix is

$$Y(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \quad \text{with} \quad \det(Y(t)) = 2\,e^{2t}.$$

Its inverse is

$$Y^{-1}(t) = \frac{e^{-2t}}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix}$$

and

$$Y^{-1}(t)\boldsymbol{b}(t) = \frac{e^{-2t}}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{4t} \\ 1 \end{pmatrix}$$

Thus $\int Y^{-1}(t)\boldsymbol{b}(t)\,dt = \frac{1}{8} \begin{pmatrix} e^{4t} \\ 4t \end{pmatrix}$ and a particular solution of the ODE is

$$\boldsymbol{x}_p(t) = \frac{1}{8} Y(t) \begin{pmatrix} e^{4t} \\ 4t \end{pmatrix} = \frac{1}{8} \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} e^{4t} \\ 4t \end{pmatrix} = \frac{e^{3t}}{8} \begin{pmatrix} 1 + 4t \\ -1 + 4t \end{pmatrix}$$