F1.3YT2 Assessment - March 13th 2008.

Attempt all 4 questions

1. Find the solution of the following initial value problem using the method of reduction of order

$$y'' - 3y^5 = 0, \quad y(0) = 1, \;\; y'(0) = 1$$

[5 marks]

Solution: This has no explicit x dependence and may be solved by reduction of order using the substitution $z = y'$. We then have

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy}z$$

and the equation becomes

$$z\frac{dz}{dy} = 3y^5$$

which is separable. Integration gives $z^2 = y^6 + c$, and $z(x = 0) = y'(0) = 1$, and $y(0) = 1$ then imply $c = 0$. Thus, we have $z^2 = y^6$ and we must have

$$z = \frac{dy}{dx} = y^3$$

i.e. the +ve square root, for compatibility with the initial conditions.
Integrating gives $-/(2y^2) = x + d$, and $y(0) = 1$ gives $d = -1/2$. Thus

$$y^2 = \frac{1}{1 - 2x} \quad \text{and} \;\; y = \frac{1}{(1 - 2x)^{1/2}}$$

2. Solve the initial value problem

$$\ddot{y} + 4\dot{y} + 5y = \delta(t - 3); \quad y(0) = 1, \;\; \dot{y}(0) = -2.$$

using Laplace transforms. **[5 marks]**

Solution: Taking the Laplace transform of both sides gives

$$(s^2 + 4s + 5)\bar{y}(s) - s - 2 = e^{-3s}.$$

The quadratic $s^2 + 4s + 5$ has complex roots, and so we complete the square and write $s^2 + 4s + 5 = (s+2)^2 + 1$. Thus we have

$$\bar{y}(s) = \frac{e^{-3s}}{(s+2)^2 + 1} + \frac{s+2}{(s+2)^2 + 1}.$$

Hence, we have

$$y(t) = u_3(t)e^{-2(t-3)}\sin(t-3) + e^{-2t}\cos(t).$$

3. Find a Green's function for the following boundary value problem

$$y'' + 9y = f(x); \quad y(0) = y'(1) = 0.$$

[You may find the identity $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ useful.]

[5 marks]

Solution: The homogeneous equation has a FSS $u_1(x) = \cos(3x)$, $u_2(x) = \sin(3x)$. The linear combinations $y^{(1)}(x) = \sin(3x)$ and $y^{(2)}(x) = \cos(3(x-1))$ then satisfy the boundary conditions $y^{(1)}(0) = 0$ and $y^{(2)\prime}(1) = 0$. The Wronskian is

$$W(y^{(1)}(x), y^{(2)}(x)) = -3\sin(3x)\sin(3(x-1)) - 3\cos(3x)\cos(3(x-1)) = -3\cos(3).$$

Thus the Green's funciton is given by

$$G(x,s) = \begin{cases} \frac{-1}{3\cos(3)}\sin(3s)\cos(3(x-1)) & \text{for } 0 \le s \le x \le 1 \\ \frac{-1}{3\cos(3)}\sin(3x)\cos(3(s-1)) & \text{for } 0 \le x \le s \le 1 \end{cases}$$

4. Find the $\lambda > 0$ eigenvalues and the associated eigenfunctions of the boundary value problem

$$-y'' = \lambda y, \quad y'(0) = 0, \quad y'(1) = 0$$

[5 marks]

Solution: Letting $\lambda = k^2$ with $k > 0$, the equation becomes $y'' = -k^2 y$ with general solution $y(x) = A\sin(kx) + B\cos(kx)$. Then we have

$$y'(0) = 0 \iff A = 0$$
$$y'(1) = 0 \iff -Bk\sin(k) = 0$$

Thus non-zero solution are possible when $k = n\pi$ with $n \in \{1, 2, 3, \cdots\}$. The eigenvalues and corresponding eigenfunctions are thus

$$\lambda = n^2\pi^2, \quad \cos(n\pi x)$$