2 Systems of 1st order and higher order ODEs

2.1 General remarks

In many applications we need several real-valued functions of one real variable to describe a system. Examples are the temperature and pressure as a function of time, or the concentrations of two chemicals in a solution as a function of time, or the three position coordinates of a particle as a function of time. In this section we are going to change notation: the independent variable will be called t, and the independent functions $x_1, \ldots, x_n$. Each of these functions obeys a differential equation, which may in turn depend on the other functions, e.g.

$$\begin{aligned} \frac{dx_1}{dt} &= tx_1 + x_2 \\ \frac{dx_2}{dt} &= x_1^2 - 4t^2 x_2. \end{aligned} \tag{2.1}$$

More generally, we might have

$$\dot{x}_1(t) = F_1\left(t, x_1(t), x_2(t), \cdots, x_n(t)\right) \tag{2.2}$$

$$\dot{x}_2(t) = F_2(t, x_1(t), x_2(t), \cdots, x_n(t)) \tag{2.3}$$

$$\vdots \tag{2.4}$$

$$\dot{x}_n(t) = F_n(t, x_1(t), x_2(t), \cdots, x_n(t)) \tag{2.5}$$

$$\tag{2.6}$$

Such a set of coupled equations is called a **system of differential equations**. Clearly, it makes sense to use vector notation. Defining

$$\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n, \quad \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \tag{2.7}$$

$$\boldsymbol{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad \boldsymbol{F}(t, \boldsymbol{x}) = \begin{pmatrix} F_1(t, x_1, \cdots, x_n) \\ \vdots \\ F_n(t, x_1, \cdots, x_n) \end{pmatrix} \tag{2.8}$$

we can write the system of differential equations as one equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{F}(t, \boldsymbol{x}(t)) \tag{2.9}$$

For example, the equation (2.1) can be written

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{F}(t, \boldsymbol{x}(t)), \tag{2.10}$$

with

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{F}(t, \boldsymbol{x}) = \begin{pmatrix} tx_1 + x_2 \\ x_1^2 - 4t^2 x_2 \end{pmatrix}. \tag{2.11}$$

Although the general form (2.9) seems to involve only a first derivative, it includes differential equations of higher order as a special case. An nth order differential equation of the form

$$\frac{d^n x}{dt^n} = f(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}) \tag{2.12}$$

can be written as a first order equation for the vector-valued function $\boldsymbol{x}$ in (7.1) as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ f(t, x_1, \ldots, x_n) \end{pmatrix}. \tag{2.13}$$

We recover the differential equation (2.12) by identifying $x_1 = x$. Then the first $n-1$ components of (2.13) tell us that

$$x_2 = \frac{dx}{dt}, \; x_3 = \frac{dx_2}{dt} = \frac{d^2x}{dt^2}, \; \ldots, x_n = \frac{d^{n-1}x}{dt^{n-1}} \tag{2.14}$$

so that the last component is precisely the equation (2.12).

Example 2.1. *Write the following differential equations as first order systems.*

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 4x = t^2$

(b) $x\ddot{x} + (\dot{x})^2 = 0.$

(c) $\dfrac{d^2x}{dt^2} = -x + y, \qquad \dfrac{d^2y}{dt^2} = x + 2y.$

For (a) we need a two-component vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We identify x_1 with x and want our equation to ensure that $x_2 = \dot{x} = \dot{x_1}$. The required equation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -4x_1 - x_2 + t^2 \end{pmatrix} \tag{2.15}$$

(b) can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{x_2^2}{x_1} \end{pmatrix}. \tag{2.16}$$

For (c) we need a four component vector. We identify x_1 with x and x_3 with y. Our equation should include $\dot{x} = \dot{x_1} = x_2$ and $\dot{y} = \dot{x_3} = x_4$. The required equation is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_3 \\ x_4 \\ x_1 + 2x_3 \end{pmatrix} \tag{2.17}$$

When $\boldsymbol{F}(t,\boldsymbol{x})$ is a linear function of $x_1, x_2, \cdots, x_n$, we can write the rhs of (2.9) in the form $A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)$, where $A(t) : \mathbb{R} \to \mathbb{R}^{n^2}$ is a $n \times n$ matrix valued function and $\boldsymbol{b}(t) : \mathbb{R} \to \mathbb{R}^n$ is a vector valued function. For example, we can write (2.15) as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}. \tag{2.18}$$

2.2 Existence and uniqueness of solutions of systems of ODEs

Picard's theorem generalises to systems $\dfrac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{F}(t,\boldsymbol{x})$ as follows

Theorem 2.2. *Suppose $\boldsymbol{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous in some region $I \times U$, where $I = (t_1, t_2)$ is an open interval and $U \subset \mathbb{R}^n$ is an open set, and that the partial derivatives in the derivative matrix*

$$D\boldsymbol{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \tag{2.19}$$

are also continuous there. Then, for every $t_0 \in I$ and $\boldsymbol{x}_0 \in U$, the initial value problem

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{F}(t,\boldsymbol{x}(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{2.20}$$

has a unique solution in some open interval containing t_0

Example 2.3. *Compute the derivate matrix, and hence show the existence and uniqueness of a solution for the initial value problem consisting of (2.1) and the initial condition $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*

The derivative matrix is given by

$$D\boldsymbol{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} t & 1 \\ 2x_1 & -4t^2 \end{pmatrix} \tag{2.21}$$

which is continuous in an open set in $\mathbb{R}^3$ that contains $t = 0$, $\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We will not prove this version of Picard's theorem either, but note a corollary for linear systems of the form

$$\frac{d\boldsymbol{x}(t)}{dt} = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t). \tag{2.22}$$

The derivative matrix is now simple $D\boldsymbol{F}(t) = A(t)$, and so if A and $\boldsymbol{b}$ are continuous functions of t, the conditions of Picard's theorem are satisfied for all initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Hence we have

Corollary 2.4. *If $A : \mathbb{R} \to \mathbb{R}^{n^2}$ is a continuous matrix-valued function and and $\boldsymbol{b} : \mathbb{R} \to \mathbb{R}^n$ is a continuous vector-valued function then the linear differential equation* (2.22) *has a unique solution for all initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and this solution exists for all $t \in \mathbb{R}$.*

2.3 Linear systems of ODEs

In this section we develop a general theory of linear systems of the form

$$\frac{d\boldsymbol{x}}{dt}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t), \tag{2.23}$$

where A is a $n \times n$ matrix-valued function and $\boldsymbol{b}$ is $\mathbb{R}^n$-valued function of t. Both A and $\boldsymbol{b}$ are assumed to be continuous functions on all of $\mathbb{R}$, so that, according to Corollary 2.4, there is unique solution of (2.23) satisfying a given initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

2.3.1 Homogeneous linear systems

For a given continuous $n \times n$ matrix-valued function A of t we define the differential operator

$$L[\boldsymbol{x}] = \frac{d\boldsymbol{x}}{dt} - A\boldsymbol{x} \tag{2.24}$$

and note that L is a linear operator:

$$L(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha\frac{d\boldsymbol{x}}{dt} + \beta\frac{d\boldsymbol{y}}{dt} - \alpha A\boldsymbol{x} - \beta A\boldsymbol{y} = \alpha L[\boldsymbol{x}] + \beta L[\boldsymbol{y}]. \tag{2.25}$$

The equation

$$L[\boldsymbol{x}] = \frac{d\boldsymbol{x}}{dt} - A\boldsymbol{x} = 0 \tag{2.26}$$

is called a homogeneous linear equation. The space of solutions of (2.26) is characterised by the following theorem (you may find it helpful to consult the revision notes on vector spaces given in Appendix A):

Theorem 2.5. *(***Solution space for homogeneous linear systems***) The space*

$$S = \{\boldsymbol{x} \in C^1(\mathbb{R}, \mathbb{R}^n) | L[\boldsymbol{x}] = 0\} \tag{2.27}$$

of solutions of the homogeneous equation (2.23) *is a vector space of dimension n since the evaluation map*

$$\mathrm{Ev}_{t_0} : S \to \mathbb{R}^n, \quad \boldsymbol{x} \mapsto \boldsymbol{x}(t_0) \tag{2.28}$$

is a vector space isomorphism.

Proof: The fact that S is a vector space follows from the linearity of L (i.e. $\boldsymbol{x}, \boldsymbol{y} \in S \implies \alpha\boldsymbol{x} + \beta\boldsymbol{y} \in S$). To show that Ev_{t_0} is a vector space isomorphism, we first note that it is a linear map:

$$\mathrm{Ev}_{t_0}[\alpha\boldsymbol{x} + \beta\boldsymbol{y}] = \alpha\boldsymbol{x}(t_0) + \beta\boldsymbol{y}(t_0) = \alpha\mathrm{Ev}_{t_0}[\boldsymbol{x}] + \beta\mathrm{Ev}_{t_0}[\boldsymbol{y}].$$

Picard's theorem implies that it is bijective: it is surjective since, for any given $\boldsymbol{x}_0 \in \mathbb{R}^n$, there exists solution $\boldsymbol{x} \in S$ so that $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. It is injective since this solution is unique. Hence Ev_{t_0} is a bijective linear map, i.e. a vector space isomorphism. $\square$

Definition 2.6. (**Fundamental sets solutions for homogeneous systems**) *A basis* $\{\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(n)}\}$ *of the set* S *defined in* (2.27) *is called a fundamental set of solutions of* (2.26).

The following lemma is an immediate consequence of Theorem 2.5.

Lemma 2.7. (**Determinant for fundamental sets**) *A set of* n *solutions* $\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(n)}$ *of the homogeneous linear equation* (2.26) *forms a fundamental set of solutions if and only if the matrix*

$$Y(t) = \begin{pmatrix} y_1^{(1)}(t) & \ldots & y_1^{(n)}(t) \\ \vdots & & \vdots \\ y_n^{(1)}(t) & \ldots & y_n^{(n)}(t) \end{pmatrix} \tag{2.29}$$

has a non-zero determinant $\det(Y(t))$ *at one value* $t = t_0$ *(and hence for all values of* t*).*

Note, that if the determinant is non-zero at one value $t = t_0$, then using $Ev_{t_0}^{-1}$, this establishes that $\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(n)}$ are a basis of S. But we can then use Ev_t, to obtain another set of basis vectors of $\mathbb{R}^n$ with determinant non-zero. Hence, if the determinant is non zero for one value t_0, it is also non-zero for all values of t.

Example 2.8. *Find a fundamental set of solutions for the homogeneous equation*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.30}$$

To solve (2.30) we try a solution of the form

$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}. \tag{2.31}$$

Inserting into (2.30) we find that this is a solution if

$$\lambda \boldsymbol{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\boldsymbol{v}, \tag{2.32}$$

so that (2.31) is a solution if λ is an eigenvalue of and $\boldsymbol{v}$ an eigenvector of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A short calculation shows that the eigenvalues are 1 and -1, with eigenvectors $\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence we obtain the solutions

$$\boldsymbol{y}^{(1)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \boldsymbol{y}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}. \tag{2.33}$$

Clearly, $\boldsymbol{y}^{(1)}(0)$ and $\boldsymbol{y}^{(2)}(0)$ are linearly independent since

$$\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

Hence, $\{\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\}$ form a fundamental set. Note that fundamental sets are not unique: we can take linear combinations to obtain other fundamental sets, e.g.

$$\frac{1}{2}(\boldsymbol{y}^{(1)}(t) + \boldsymbol{y}^{(2)}(t)) = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}, \quad \frac{1}{2}(\boldsymbol{y}^{(1)}(t) - \boldsymbol{y}^{(2)}(t)) = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}. \tag{2.34}$$

As we have seen, homogeneous linear n-th order equations

$$\frac{d^n x}{dt^n} + a_n \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1 x = 0 \tag{2.35}$$

are a special case of linear homogeneous systems (2.26). Here we define

$$s = \{x \in C^n(\mathbb{R}, \mathbb{R}) | \frac{d^n x}{dt^n} + a_n \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1 x = 0\} \tag{2.36}$$

In this case, the evaluation map takes the form

$$\mathrm{ev}_{t_0} : s \longrightarrow \mathbb{R}^n, \quad x \mapsto \begin{pmatrix} x(t_0) \\ \frac{dx}{dt}(t_0) \\ \vdots \\ \frac{d^{n-1}x}{dt^{n-1}}(t_0) \end{pmatrix} \tag{2.37}$$

which assigns to each solutions the vector made out of the values of the function x and its first $n-1$ derivatives at t_0. Theorem 2.5 then states that this map is a vector space isomorphism. This is sometimes useful for checking linear independence of solutions of an n-th order homogeneous linear equation: two solutions are independent if and only if their images under ev_{t_0} are independent in $\mathbb{R}^n$. For the two solutions $y^{(1)}(t) = \cos t$ and $y^{(2)}(t) = \sin t$ of $\ddot{x} + x = 0$, the evaluation map at $t_0 = 0$ gives

$$\mathrm{ev}_0(y^{(1)}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathrm{ev}_0(y^{(2)}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.38}$$

These vectors, and hence $y^{(1)}(t)$ and $y^{(2)}(t)$ are linearly independent since

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

The FSS $\{\boldsymbol{y}^{(1)}, \cdots, \boldsymbol{y}^{(n)}\}$ of a homogeneous linear system

$$\frac{d\boldsymbol{x}(t)}{dx} = A(t)\boldsymbol{x}(t) \tag{2.39}$$

has three main uses. Firstly, we can write the general solution of (2.39) as $\boldsymbol{y}(t) = c_1\boldsymbol{y}^{(1)}(t) + \cdots + c_n\boldsymbol{y}^{(n)}(t)$ for constants $c_1, \cdots, c_n$. Secondly, we can find the particular values of $c_1, \cdots, c_n$ that give the unique solution of the initial value problem specified by (2.39) and the condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Thirdly, as we shall see in Section 2.3.4, we can use the FSS in order to construct a particular solution of the inhomogeneous problem $\dfrac{d\boldsymbol{x}(t)}{dt} = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)$.

Example 2.9. *Find the general solution of*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.40}$$

and hence find the solution of the initial value problem specified by (2.40) and the initial condition $\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We have already found the FSS for this problem; hence the general solution is

$$\boldsymbol{x}(t) = c_1\begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}. \tag{2.41}$$

The solution of the initial value problem is given by the choice

$$\begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.42}$$

with solution $c_1 = 1/2$, $c_2 = -\frac{1}{2}$. Hence the solution is

$$\boldsymbol{x}(t) = \begin{pmatrix} \sinh(t) \\ \cosh(t) \end{pmatrix}. \tag{2.43}$$

2.3.2 Matrix methods for finding the FSS

In practice, the FSS can only be found explicitly in rare happy cases. An example is the case where the matrix A is constant, and this is the case we study in some detail this section. Consider the system of linear equations

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t), \tag{2.44}$$

where $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n$ and A is a constant, real $n \times n$ matrix. To find solutions we make the *ansatz* (=inspired guess)

$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}. \tag{2.45}$$

Inserting into (2.44) we obtain

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \tag{2.46}$$

so that (2.45) is a solution iff $\boldsymbol{v}$ is an eigenvector. If A has n real linearly independent eigenvectors $\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}$ with real eigenvalues $\lambda_1, \ldots, \lambda_n$ (which need not be distinct), a fundamental set of solutions is given by

$$\boldsymbol{y}^{(1)}(t) = e^{\lambda_1 t}\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{y}^{(n)}(t) = e^{\lambda_n t}\boldsymbol{v}^{(n)}. \tag{2.47}$$

However, as discussed in Appendix B, not all real $n \times n$ matrices have a basis of n real linearly independent eigenvectors. Let us consider the different situations which can arise:
(i) n distinct real eigenvalues

In this case, we do get n linearly independent eigenvectors and the FSS is given by (2.47)

Example 2.10. *Consider the equation* (2.44) *for* $n = 2$ *and*

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \tag{2.48}$$

This matrix has eigenvalue λ if $\det(A - \lambda\mathbb{I}) = (1-\lambda)^2 - 4 = 0$ i.e. if $\lambda = -1$ or $\lambda = -3$. The vector $\boldsymbol{v}^{(1)} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector for eigenvalue -1 if

$$v_1 + 4v_2 = -v_1, \quad \text{and} \quad v_1 + v_2 = -v_2. \tag{2.49}$$

Hence $\boldsymbol{v}^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is eigenvector for eigenvalue -1. Similarly we find that $\boldsymbol{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is eigenvector for eigenvalue 3. The solutions

$$\boldsymbol{y}^{(1)}(t) = e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \boldsymbol{y}^{(2)}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.50}$$

are independent and form a fundamental set.
(ii) Repeated real eigenvalues.

If a given eigenvalue has algebraic multiplicity m (i.e. it is repeated m times) it will have $1 \le q \le m$ linearly independent eigenvalues. Let us consider the cases $q = m$, and $q < m$ separately.

- $q = m$

Example 2.11. *Find the FSS of* (2.44) *for*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.51}$$

A has $\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as eigenvectors for the repeated eigenvalue 2. In this case we find a fundamental set of solutions:

$$\boldsymbol{y}^{(1)}(t) = e^{2t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{y}^{(2)}(t) = e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.52}$$

• $q < m$. As we have said, a matrix may fail to have a basis of eigenvectors if an eigenvalue is repeated. Suppose for simplicity that a real eigenvalue λ_1 has multiplicity $m = 2$ (i.e. the characteristic polynomial contains a factor $(\lambda - \lambda_1)^2$) and that $\boldsymbol{v}$ is the only eigenvector for this eigenvalue (i.e. $q = 1$). Then $\boldsymbol{y}^{(1)}(t) = e^{\lambda_1 t}\boldsymbol{v}$ is a solution of (2.44). To find a second solution we try

$$\boldsymbol{y}^{(2)}(t) = te^{\lambda_1 t}\boldsymbol{v} + e^{\lambda_1 t}\boldsymbol{w}. \tag{2.53}$$

Inserting into (2.44) we find that this is a solution if

$$e^{\lambda t}(\lambda_1 t\boldsymbol{v} + \boldsymbol{v} + \lambda_1\boldsymbol{w}) = e^{\lambda_1} A(t\boldsymbol{v} + \boldsymbol{w}) \tag{2.54}$$

i.e. if

$$(A - \lambda_1\mathbb{I})\boldsymbol{w} = \boldsymbol{v}. \tag{2.55}$$

One can show that this equation can always be solved for $\boldsymbol{w}$. Since $(A - \lambda_1\mathbb{I})\boldsymbol{w} \neq 0$ but $(A - \lambda_1\mathbb{I})^2\boldsymbol{w} = 0$, $\boldsymbol{w}$ is sometimes called a generalised eigenvector of A.

Example 2.12.

$$A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}. \tag{2.56}$$

We find that λ is an eigenvalue if $(\lambda + 2)^2 = 0$. Hence -2 is a repeated eigenvalue. The only eigenvector is

$$\boldsymbol{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.57}$$

so that one solution of (2.44) is

$$\boldsymbol{y}^{(1)}(t) = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}. \tag{2.58}$$

To find the second solution we need to solve

$$(A + 2\mathbb{I})\boldsymbol{w} = \boldsymbol{v} \Leftrightarrow \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\boldsymbol{w} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}. \tag{2.59}$$

With elementary row operations we find

$$\boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{2.60}$$

so that a second solution is given by

$$\boldsymbol{y}^{(2)}(t) = e^{-2t}t\begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = e^{-2t}\begin{pmatrix} -1 - 3t \\ t \end{pmatrix}. \tag{2.61}$$

(*iii*) Complex eigenvalues

If the real matrix A in (2.44) has a complex eigenvalue $\lambda = \alpha + i\beta$ with corresponding complex eigenvector $\boldsymbol{v} = \boldsymbol{v}^{(1)} + i\boldsymbol{v}^{(2)}$, then $\boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{v}$ is a complex solution (note that $\boldsymbol{y}^*(t) = e^{\lambda^* t}\boldsymbol{v}^*$ is another solution). However, since A is real we have

$$\begin{aligned}\frac{d\mathrm{Re}(\boldsymbol{y}(t))}{dt} &= \mathrm{Re}(\frac{d\boldsymbol{y}(t)}{dt}) = \mathrm{Re}(A\boldsymbol{y}(t)) = A\,\mathrm{Re}(\boldsymbol{y}(t)) \\ \frac{d\mathrm{Im}(\boldsymbol{y}(t))}{dt} &= \mathrm{Im}(\frac{d\boldsymbol{y}(t)}{dt}) = \mathrm{Im}(A\boldsymbol{y}(t)) = A\,\mathrm{Im}(\boldsymbol{y}(t))\end{aligned}$$

So $\mathrm{Re}(\boldsymbol{y}(t))$ and $\mathrm{Im}(\boldsymbol{y}(t))$ are both real solutions. Thus we obtain two real solutions by taking the real and imaginary parts of $\boldsymbol{y}(t)$.

Example 2.13. *Find the FSS of* (2.44) *with*

$$A = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}. \tag{2.62}$$

The eigenvalues are $1 \pm i$, with eigenvectors $\begin{pmatrix} -2 \pm i \\ 1 \end{pmatrix}$. Thus we have the complex solution

$$\begin{aligned}\boldsymbol{y}(t) &= e^{(1+i)t}\begin{pmatrix} -2+i \\ 1 \end{pmatrix} \\ &= e^t(\cos t + i\sin t)\begin{pmatrix} -2+i \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} -2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t\begin{pmatrix} \cos t - 2\sin t \\ \sin t \end{pmatrix}\end{aligned} \tag{2.63}$$

so that a fundamental set of solutions is

$$\boldsymbol{y}^{(1)}(t) = e^t\begin{pmatrix} -2\cos t - \sin t \\ \cos t \end{pmatrix}, \qquad \boldsymbol{y}^{(2)}(t) = e^t\begin{pmatrix} \cos t - 2\sin t \\ \sin t \end{pmatrix}. \tag{2.64}$$

Note, that the other complex solution is

$$\boldsymbol{y}^*(t) = e^{(1-i)t}\begin{pmatrix} -2-i \\ 1 \end{pmatrix}$$

and we have $\mathrm{Re}(\boldsymbol{y}^*(t)) = y^{(1)}(t)$, $\mathrm{Im}(\boldsymbol{y}^*(t)) = -y^{(2)}(t)$. Thus, in order to obtain 2 real solution, we only need to deal with one of the pair of complex conjugate solutions, and it doesn't matter which one.

2.3.3 The Fundamental Matrix

IF $\{y^{(1)}, \cdots, y^{(n)}\}$ is a FSS, the non-singular matrix

$$Y(t) = \begin{pmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ \vdots & \vdots & & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{pmatrix}$$

is called the **fundamental matrix**. The general solution of the homogeneous linear ODE $\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t)$ is then

$$\boldsymbol{x}(t) = c_1\boldsymbol{y}^{(1)}(t) + c_2\boldsymbol{y}^{(2)}(t) + \cdots c_n\boldsymbol{y}^{(n)}(t), \tag{2.65}$$

where $c_1, \cdots, c_n$ are t-independent constants. (2.65) can be written in the form

$$\boldsymbol{x}(t) = Y(t)\,\boldsymbol{c}, \quad \text{where} \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}. \tag{2.66}$$

The solution to the initial value problem consisting of $\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t)$ and the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is given by (2.66) with $\boldsymbol{c}$ specified by

$$\boldsymbol{x}_0 = Y(t_0)\boldsymbol{c}, \quad \text{such that } \boldsymbol{c} = Y^{-1}(t_0)\boldsymbol{x}_0. \tag{2.67}$$

Thus we can write

$$\boldsymbol{x}(t) = Y(t)Y^{-1}(t_0)\boldsymbol{x}_0. \tag{2.68}$$

Note that $Y^{-1}(t)$ exists since $\det(Y(t)) \neq 0$. (2.68) is not of much practical use as a way to compute the solution - it is usually easier just to find $\boldsymbol{c}$ from $\boldsymbol{x}_0 = Y(t_0)\boldsymbol{c}$ by row reduction of $Y(t_0)$. It is however conceptually important and of interest in the next section.

Another useful property of $Y(t)$ is that it satisfies

$$\dot{Y}(t) = A(t)Y(t). \tag{2.69}$$

This follows since the i'th column of both sides is simply

$$\begin{pmatrix} \dot{y}_1^{(i)} \\ \vdots \\ \dot{y}_n^{(i)} \end{pmatrix} = A(t) \begin{pmatrix} y_1^{(i)} \\ \vdots \\ y_n^{(i)}. \end{pmatrix}$$

2.3.4 Inhomogeneous linear systems

We return to the more general form

$$\frac{d\boldsymbol{x}}{dt}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t), \tag{2.70}$$

of linear system with $\boldsymbol{b} \neq 0$. Such systems are called inhomogeneous linear systems. Using again the linear operator L defined in (2.24) we have the following analogue of Theorem 2.5.

Theorem 2.14. *(***Solution space for inhomogeneous linear systems***) Let* $\{\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(n)}\}$ *be a fundamental set of solution for the homogeneous equation* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$*. Then any solution of the inhomogeneous equation* (2.70) *is of the form*

$$\boldsymbol{x}(t) = \sum_{i=1}^{n} c_i \boldsymbol{y}^{(i)} + \boldsymbol{x}_p, \tag{2.71}$$

where $\boldsymbol{x}_p$ *is a particular solution of* (2.70) *and* $c_1, \ldots, c_n$ *are real constants.*

Proof: It is easy to check that (2.71) satisfies the inhomogeneous equation (2.70), using the fact that each of the $\boldsymbol{y}^{(i)}$ satisfy $\dot{\boldsymbol{y}}^{(i)} = A\boldsymbol{y}^{(i)}$. To show that every solution can be written in this way, suppose that $\boldsymbol{x}$ is a solution of (2.70). Then $\boldsymbol{x} - \boldsymbol{x}_p$ satisfies the homogeneous equation $\frac{d}{dt}(\boldsymbol{x} - \boldsymbol{x}_p) = A(\boldsymbol{x} - \boldsymbol{x}_p)$ and therefore can be expanded

$$\boldsymbol{x} - \boldsymbol{x}_p = \sum_{i=1}^{n} c_i \boldsymbol{y}^{(i)} \tag{2.72}$$

for some real constants $c_1, \ldots, c_n$. □

The most systematic way of finding a particular solution of an inhomogeneous linear equation is called the **method of variation of the parameters**. The idea is to look for a particular solution of the form

$$\boldsymbol{x}_p(t) = \sum_{i=1}^{n} c_i(t) \boldsymbol{y}^{(i)}(t), \tag{2.73}$$

where $\{\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(n)}\}$ is a fundamental set of the homogeneous equations as before, but, crucially, the c_i are now functions of t. As above, we can write (2.73) in terms of the fundamental matrix Y as

$$\boldsymbol{x}_p = Y\boldsymbol{c}. \tag{2.74}$$

Using the product rule and the fact that $\dot{Y} = AY$ we deduce

$$\dot{\boldsymbol{x}}_p = \dot{Y}\boldsymbol{c} + Y\dot{\boldsymbol{c}} = AY\boldsymbol{c} + Y\dot{\boldsymbol{c}}. \tag{2.75}$$

Hence

$$\begin{aligned} \dot{\boldsymbol{x}}_p = A\boldsymbol{x}_p + \boldsymbol{b} \quad &\Leftrightarrow \quad AY\boldsymbol{c} + Y\dot{\boldsymbol{c}} = AY\boldsymbol{c} + \boldsymbol{b} \\ &\Leftrightarrow \quad \dot{\boldsymbol{c}} = Y^{-1}\boldsymbol{b}. \end{aligned} \tag{2.76}$$

We can now compute $\boldsymbol{c}$ by integration - at least in principle. We summarise this result as follows.

Theorem 2.15. *(***Method of variation of the parameters***) Let* $\{\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(n)}\}$ *be a fundamental set of solutions of* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ *and let* Y *be the matrix constructed from* $\{\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(n)}\}$ *according to* (2.29)*. Then*

$$\boldsymbol{x}_p(t) = Y(t)\int_{t_0}^{t} Y^{-1}(\tau)\boldsymbol{b}(\tau)d\tau \tag{2.77}$$

is a particular solution of the inhomogeneous equation $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$

The formula (2.77) is very general, and therefore very powerful. To get a feeling for how it works we need to study further examples - see Problem Sheet 4.

Example 2.16. *Find a particular solution of the equation*

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

The fundamental matrix is

$$Y(t) = \begin{pmatrix} e^t & e^{4t} \\ -e^t & 2e^{4t} \end{pmatrix} \quad \text{and} \quad Y^{-1}(t) = \frac{1}{3}\begin{pmatrix} 2e^{-t} & -e^{-t} \\ e^{-4t} & e^{-4t} \end{pmatrix}$$

Thus (2.77) tells us that a particular solution is given by

$$\boldsymbol{x}_p(t) = Y(t)\int Y^{-1}(t)\begin{pmatrix} 0 \\ e^t \end{pmatrix} dt.$$

Note, since changing the value of t_0 in (2.77) just changes the particular solution by a complementary function (i.e., a solution of the homogeneous equation), we can just use the indefinite integral as we do here. Evaluating the integral, we find

$$\boldsymbol{x}_p(t) = -\frac{1}{3}e^t\begin{pmatrix} t + \frac{4}{3} \\ -t - \frac{1}{3} \end{pmatrix}.$$

Let us consider the $n = 1$ case of (2.77) in more detail and show that equation (2.77) reduces to the formula for the solution of a linear first order equations in terms of the integrating factor. With the conventions of this section, we consider the first order differential equation

$$\dot{x}(t) = a(t)x(t) + b(t) \Leftrightarrow \dot{x}(t) - a(t)x(t) = b(t) \tag{2.78}$$

and note that the homogeneous equation

$$\dot{x}(t) = a(t)x(t) \tag{2.79}$$

has the solution

$$y(t) = \exp(\int_{t_0}^{t} a(\tau)d\tau), \tag{2.80}$$

which is the inverse of the integrating factor

$$I(t) = \exp(-\int_{t_0}^{t} a(\tau)d\tau) \tag{2.81}$$

for (2.78). Thus the formula

$$x_p(t) = y(t)\int_{t_0}^{t} y^{-1}(\tau)b(\tau)d\tau = I(t)^{-1}\int_{t_0}^{t} I(\tau)b(\tau)d\tau \tag{2.82}$$

for the particular solution is precisely the solution we would obtain using the integrating factor $y^{-1}(t)$.

2.4 Higher order ODEs

Recall, that an nth order differential equation of the form

$$\frac{d^n x}{dt^n} = f(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}) \tag{2.83}$$

can be viewed as a system of systems of ODEs by identifying

$$x_1 = x,\ x_2 = \frac{dx}{dt},\ x_3 = \frac{dx_2}{dt} = \frac{d^2x}{dt^2},\ \ldots, x_n = \frac{d^{n-1}x}{dt^{n-1}} \tag{2.84}$$

Then we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ f(t, x_1, \ldots, x_n) \end{pmatrix}. \tag{2.85}$$

where the last component is precisely the equation (2.83). The initial value problem corresponds to (2.85) and a condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. For the equation (2.83) this is equivalent to specifying

$$x(t_0), \frac{dx(t_0)}{dt}, \frac{d^2x(t_0)}{dt^2}, \ldots, \frac{d^{n-1}x(t_0)}{dt^{n-1}}, \tag{2.86}$$

i.e. the function and its first $(n-1)$ derivatives.

All of the results of Section 2 specialise to the case of n'th order ODES.In this Section we consider some of these specialisations.

2.4.1 Picard's Theorem

The derivative matrix for (2.85) is

$$DF = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \cdots & \frac{\partial f}{\partial x_{n-1}} & \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Thus (2.85) satisfies the conditions of Picard's theorem if the function f and the partial derivatives

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \tag{2.87}$$

are continuous. In that case there will be a unique solution, at least locally, if we specify $\boldsymbol{x}(t_0)$, i.e., i.e. the function $x(t_0)$ and its first $(n-1)$ derivatives.

Example 2.17. *Show that $x(t) = t^2$ can not be a solution of the initial value problem*

$$\frac{d^2x}{dt^2} = t\left(\frac{dx}{dt}\right)^2 + x^3, \quad x(0) = 0, \quad \frac{dx(0)}{dt} = 0.$$

The function $f(t, x, \frac{dx}{dt})$ on the right-hand side and the two partial derivatives $\frac{\partial f}{\partial x} = 3x^2$, $\frac{\partial f}{\partial \dot{x}} = 2\dot{x}$, are continuous at $t = 0$, $x = 0$, $\dot{x} = 0$. The initial value problem therefore has a unique solution from Picard's Theorem. One solution is $x(t) = 0$ (such that we also have $\dot{x(t)} = 0$). Therefore, there can not be another solution of the form given.

2.4.2 The FSS and the Wronskian

A linear n'th order ODE is one of the form system

$$\frac{d^nx}{dt^n} + a_n(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1(t)x = b(t). \tag{2.88}$$

The equation is said to be homogeneous when $b(t) = 0$.
n linearly independent $y^{(1)}, y^{(2)}(t), .., y^{(n)}(t)$ solutions of the homogeneous linear equation constitute a Fundamental Solution Set.

The counterpart of Lemma 2.7 has a special name for n-th order equations:

Lemma 2.18. *(***Wronskian for n-th order linear homogeneous equations** *) A set of n solutions $y^{(1)}, \ldots, y^{(n)}$ of the n-th order homogeneous linear equation* (2.35) *forms a fundamental set of solutions if and only if the matrix*

$$Y = \begin{pmatrix} y^{(1)} & \ldots & y^{(n)} \\ \frac{dy^{(1)}}{dt} & \ldots & \frac{dy^{(n)}}{dt} \\ \vdots & & \vdots \\ \frac{d^{n-1}y^{(1)}}{dt^{n-1}} & \ldots & \frac{d^{n-1}y^{(1)}}{dt^{n-1}} \end{pmatrix} \tag{2.89}$$

has a non-zero determinant W=det(Y) at one value $t = t_0$ (and hence for all values of t). The determinant W of Y is called the Wronskian of $y^{(1)}, \ldots, y^{(n)}$.

Example 2.19. *The Wronskian of two solutions $y^{(1)}, y^{(2)}$ of the second order equation $\ddot{x} + a_2\dot{x} + a_1x = 0$ is $W = y^{(1)}\dot{y}^{(2)} - y^{(2)}\dot{y}^{(1)}$.*

If we have n solutions of a homogeneous, linear n'th order ODE, and we want to check for linearly independence, we simply have to evaluate the Wronskian at any value of t (or whatever the variable is called).

Example 2.20. *Two solutions of the equation* $\frac{d^2y}{dx^2} + 4y = 0$ *are* $\cos(2x)$ *and* $\sin(2x)$. *Show that these two solutions constitute a FSS.*

The Wronskian is $W = \cos(2x)\cos(2x) + \sin(2x)\sin(2x) = 1$, hence the solutions are linearly independent and are a FSS.

2.4.3 The FSS for equations with constant coefficients

Consider the linear homogeneous equation

$$\frac{dx^n}{dt} + a_n \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1 x = 0 \tag{2.90}$$

in the case when $a_n, \cdots, a_1$ are constant coefficients. Then, we may write the equation in matrix form (after identifying $x = x_1$ as)

$$\frac{d\boldsymbol{x}}{dt} = A\,\boldsymbol{x}(t), \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \end{pmatrix}. \tag{2.91}$$

As above, we can press on a solve this equation by finding the eigenvalues λ that satisfy $\det(A - \lambda\mathbb{I}) = 0$. It is simple exercise (see Problem Sheet 3) to show that this equation reduces to the **characteristic equation**

$$\lambda^n + a_n\lambda^{n-1} + \cdots a_2\lambda + a_1 = 0. \tag{2.92}$$

For n real, distinct solutions $\lambda_1, \cdots, \lambda_n$, n linearly independent solutions of (2.91) are given by by

$$\begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \cdots & \lambda_n e^{\lambda_n t} \\ \vdots & & \vdots & \\ \lambda_1^{n-1} e^{\lambda_1 t} & \lambda_2^{n-1} e^{\lambda_2 t} & \cdots & \lambda_n^{n-1} e^{\lambda_n t} \end{pmatrix}$$

The corresponding solutions of (2.90) are just $e^{\lambda_1 t}, \cdots, e^{\lambda_n t}$.

Of course there is a far easier way of deriving these results for the (2.90). We just try a solution of the form $e^{\lambda t}$ and substitute into (2.90) to get the equation (2.92) as we would do for the $n = 2$ case. The solutions are then $e^{\lambda_i t}$ for distinct eigenvalues. However, it is hopefully illuminating to see how these results can be found by specialising from the matrix methods discussed earlier.

Example 2.21. *Find the general solution of*

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} - 7\frac{d^2x}{dt^2} - \frac{dx}{dt} + 6x = 0.$$

Hence, find the solution that satisfies the initial conditions

$$x(0) = 1, \quad \frac{dx}{dt}(0) = 0, \quad \frac{d^2x}{dt^2}(0) = -2, \quad \frac{d^3x}{dt^3}(0) = -1.$$

The characteristic equation is $\lambda^4 + \lambda^3 - 7\lambda^2 + 6 = 0$, with roots $\lambda = 1, -1, 2, -3$. Hence the general solution is

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

The solution to the initial value problem is given by the choice of the c_i such that satisfies

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2 \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1, \end{aligned}$$

which we solve to give

$$c_1 = \frac{11}{8}, \quad c_2 = \frac{5}{12}, \quad c_3 = -\frac{2}{3}, \quad c_4 = -\frac{1}{8}.$$

Thus the solution is

$$x(t) = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}.$$

Repeated eigenvalues: If real λ is repeated m times, we just take $e^{\lambda t}, te^{\lambda t}, \cdots, t^{m-1}e^{\lambda t}$ as solutions. See Problem Sheet 4.

Complex eigenvalues: If a root $\lambda = \alpha + i\beta$ is complex, we get 2 real solutions by taking the real and imaginary parts of $e^{\lambda t}$. i.e. two solutions are given by $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin(\beta t)$.

2.4.4 Inhomogeneous linear n'th order ODES - variation of parameters

We now return to the linear inhomogeneous equation (2.88), which we can write in the form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ -a_1(t)x_1 - a_2(t)x_2 \cdots - a_n(t)x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix}. \tag{2.93}$$

If we have a FSS $y^{(1)}, \cdots, y^{(n)}$, we can immediately write down a particular solution of the system (2.93) using the variation of parameters method as

$$\boldsymbol{x}_p(t) = Y(t) \int_{t_0}^{t} Y^{-1}(\tau) \boldsymbol{b}(\tau) d\tau \tag{2.94}$$

where $Y(t)$ is the fundamental matrix given by given (2.89), and $\boldsymbol{b}$ is the vector

$$\boldsymbol{b}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix}.$$

The particular solution $x_p(t)$ is just given by the 1st component of this vector.

Formula (2.94) is a rather formal result for general n, but simplifies and becomes a practical method when $n = 2$. The starting point is the formula for the inverse of a 2×2 matrix:

$$\text{If} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a. \end{pmatrix} \tag{2.95}$$

Thus we have

$$Y^{-1} = \frac{1}{W(y^{(1)}, y^{(2)})} \begin{pmatrix} \dot{y}^{(2)} & -y^{(2)} \\ -\dot{y}^{(1)} & y^{(1)} \end{pmatrix}, \quad \text{and} \quad Y^{-1}\boldsymbol{b} = \frac{b}{W(y^{(1)}, y^{(2)})} \begin{pmatrix} -y^{(2)} \\ y^{(1)} \end{pmatrix}$$

where the Wronskian is given by $W(y^{(1)}, y^{(2)}) = y^{(1)}\dot{y}^{(2)} - \dot{y}^{(1)}y^{(2)}$. Thus we arrive at the formula

$$x_p(t) = \int_{t_0}^{t} \frac{y^{(1)}(\tau)y^{(2)}(t) - y^{(1)}(t)y^{(2)}(\tau)}{W(y^{(1)}(\tau), y^{(2)}(\tau))}\, b(\tau)\, d\tau. \tag{2.96}$$

Example 2.22. *Use the method of variation of parameters to find a particular solution of the equation*

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 2e^{-t}$$

First, we must find a FSS. The characteristic eqn is $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ with roots $\lambda = 2, -1$. Thus a FSS is $y^{(1)}(t) = e^{2t}$, $y^{(2)}(t) = e^{-t}$. From above, a particular solution is given (with the arbitrary choice $t_0 = 0$) by

$$\begin{aligned} x(t) &= c_1(t)y^{(1)}(t) + c_2(t)y^{(2)}(t), \\ c_1(t) &= -\int_0^t \frac{y^{(2)}(\tau)}{W(y^{(1)}(\tau), y^{(2)}(\tau))}\, 2e^{-\tau}\, d\tau, \\ c_2(t) &= \int_0^t \frac{y^{(1)}(\tau)}{W(y^{(1)}(\tau), y^{(2)}(\tau))}\, 2e^{-\tau}\, d\tau. \end{aligned}$$

The Wronskian is given by

$$W(y^{(1)}(\tau), y^{(2)}(\tau)) = e^t(-1 - 2) = -3e^t$$

and so

$$\begin{aligned} c_1(t) &= \int_0^t \frac{2}{3}e^{-3t} d\tau = -\frac{2}{9}(e^{-3t} - 1) \\ c_2(t) &= -\int_0^t \frac{2}{3} d\tau = -\frac{2}{3}t. \end{aligned}$$

Thus the solution is

$$x(t) = \frac{2}{9}e^{2t} - \frac{2}{9}e^{-t}(1+3t).$$

Note that this solution is not unique, $-\frac{2}{3}te^{-t}$ is another particular solution (that we could obtain more simply by the method of 'undetermined coefficients' that we discuss in Section 3). In fact we can always add arbitrary constants to c_1 and c_2 obtained from (2.96) and still obtain a particular integral. Put another way we can in general obtain c_1 and c_2 from the indefinite integrals. Thus the final statement of the method of variation of parameters for $n = 2$, is that a particular solution is given by

$$\begin{aligned}
x(t) &= c_1(t)y^{(1)}(t) + c_2(t)y^{(2)}(t), \quad \text{with} \\
c_1(t) &= -\int \frac{y^{(2)}(t)}{W(y^{(1)}(t), y^{(2)}(t))}\, b(t)\, dt, \\
c_2(t) &= \int \frac{y^{(1)}(t)}{W(y^{(1)}(t), y^{(2)}(t))}\, b(t)\, dt.
\end{aligned}$$

Example 2.23. *Find the general solution of*

$$y'' + y = \tan x. \tag{2.97}$$

A fundamental system of $y'' + y = 0$ is given by $y^{(1)}(x) = \cos x$ and $y^{(2)}(x) = \sin x$. The Wronskian is $W = 1$. We find

$$c_1(x) = -\int \frac{\sin^2 x}{\cos x} dx = \int (\cos x - \sec x) dx \tag{2.98}$$

$$c_2(x) = \int \sin x \, dx \tag{2.99}$$

In computing c_1 we encounter the integral $\int \sec x \, dx$ - slightly tricky, but standard with the substitution $t = \tan(x/2)$. The answer is

$$c_1(x) = \sin x - \ln(\sec x + \tan x) \tag{2.100}$$

The integration for c_2 is easier:

$$c_2(x) = -\cos x \tag{2.101}$$

The general solution of (2.97) is therefore

$$y(x) = A\cos x + B\sin x - \ln(\sec x + \tan x) \cdot \cos x. \tag{2.102}$$