3 Solvable 2nd order ODEs

3.1 General remarks

So far, we have developed aspects of a general theory of systems of ODEs, which we have then specialised to higher order ODES. Most of the results we have obtained have been for linear systems or linear higher order ODES, first homogeneous and then inhomogeneous. In this section, we will specialise one step further to consider specifically 2nd order ODEs, i.e., equations of the form

$$y'' = f(x, y, y'). \tag{3.1}$$

The theory of second order ODEs is an extensive and highly developed part of mathematics, not least because of its numerous applications in the physical sciences. For example, Newton told us, with impressive foresight, that the radial motion of the space shuttle is governed by the equation

$$\frac{d^2r}{dt^2} = -\frac{MG}{r^2}, \tag{3.2}$$

where M is the mass of the earth and G is the gravitational constant. As another sample, a vibrating spring of mass m obeys the equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F(t),$$

where γ is a damping coefficient, k the spring constant, and $F(t)$ the applied force.

In the previous chapter, we have developed an arsenal of techniques, and shown how some of them specialise to 2nd order equations; most importantly, we have shown how the method of variation of parameters gives a practical method of solving linear inhomogeneous 2nd order equations. In this section, we shall try not to repeat ourselves, but consider instead a range of new techniques that work best specifically for 2nd order ODEs.

First of all though let us indeed repeat ourselves once, and consider the specialisation of Picard's Theorem to the initial value problem

$$\frac{d^2y}{dx^2} = f(x, y, y'), \qquad y(x_0) = \alpha, \quad y'(x_0) = \beta. \tag{3.3}$$

Theorem 3.1. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous in some cube $|x - x_0| \le a, |y - \alpha| \le b, |y' - \beta| \le c$ and that the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous there. Then there is some interval $|x - x_0| \le h \le a$ in which the initial value problem (3.3) has a unique solution.*

As very elementary example, consider the simple harmonic oscillator problem

$$y'' = -y. \tag{3.4}$$

The function $f(x, y, y') = -y$ is clearly continuous in x, y, y' and the conditions of Picard's Theorem are met. You may recall (or simply check) that both $\sin x$ and $\cos x$ solve this equation. More generally, the superposition

$$y(x) = c_1 \cos x + c_2 \sin x, \tag{3.5}$$

where c_1 and c_2 are two arbitrary constants, also solves (3.4). The constants, and hence the solution, are determined uniquely if we specify

$$y(x_0) = \alpha, \qquad y'(x_0) = \beta. \tag{3.6}$$

Now we shall go on to consider the range of promised new techniques that are most useful for 2nd order systems.

3.2 Reduction of order

Sometimes a second order equation can be reduced to a first order equation and can then hopefully be solved:

3.2.1 Equations that do not depend on y

For equations of the form

$$y'' = f(x, y') \tag{3.7}$$

we define $z = y'$. Then $y'' = z'$ and the equations becomes first order:

$$z' = f(x, z). \tag{3.8}$$

Example 3.2. *Find the general solution of $y'' + 2y' = e^{-x}$.*

Solution: Letting $z = y'$, we have $z' + 2z = e^{-x}$, which is linear and can be solved by the integrating factor e^{2x}, to give

$$z = e^{-x} + Ae^{-2x}.$$

Integrating once more gives the general solution

$$y = -e^{-x} + Be^{-2x} + C.$$

3.2.2 Equations that do not depend on x

Consider equations of the form

$$y'' = f(y, y') \tag{3.9}$$

In that case, let $z = y'$ and regard z as a function of y. Then

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z\frac{dz}{dy}. \tag{3.10}$$

Hence equation (3.9) becomes

$$z\frac{dz}{dy} = f(y, z) \tag{3.11}$$

which is again of first order.

Example 3.3. *Find the general solution of*

$$yy'' + (y')^2 = 0 \tag{3.12}$$

Solution: Letting $z = y'$ gives

$$yz\frac{dz}{dy} + z^2 = 0 \tag{3.13}$$

which is separable and solved by

$$z = \frac{A}{y} \tag{3.14}$$

for some constant A. Recalling that $z = \dfrac{dy}{dx}$ and integrating once again we obtain the general solution of (3.12):

$$\frac{1}{2}y^2 = Ax + B. \tag{3.15}$$

3.2.3 The general linear homogeneous equation

Consider the equation

$$y'' + a(x)y' + b(x)y = 0.$$

Suppose we know one solution $u(x)$ of this equation. Then letting $y = uw$, we obtain

$$y' = u'w + uw', \quad y'' = u''w + 2u'w' + uw''.$$

Substituting into the equation, we obtain

$$u''w + 2u'w' + uw'' + a(x)(u'w + uw') + b(x)uw = 0.$$

Rearranging, gives

$$(u'' + a(x)u' + b(x)u)w + uw'' + (a(x)u + 2u')w' = 0.$$

The first term is zero, and the 2nd term gives

$$u\,w'' + (a(x)u + 2u')w' = 0.$$

Writing $z = w'$ as above, we then have a first order ODE

$$u\,z' + (a(x) + 2u')z = 0,$$

which is linear and can be solved by an integrating factor. So, in principle, we can find the 2nd solution $y(x) = u(x)w(x)$.

Example 3.4. *One solution of* $y'' - 2\alpha y' + \alpha^2 y = 0$ *is* $y(x) = e^{\alpha x}$*. Use the method of reduction of order to show that the general solution is given by* $y(x) = Ae^{\alpha x} + Bxe^{\alpha x}$*.*

Solution: Letting, $y(x) = e^{\alpha x} w(x)$, we have $y' = u'w + uw'$, $y'' = u''w + 2u'w' + uw''$ and so substituting into the equation gives

$$2u'w' + uw'' - 2\alpha uw' = 0,$$

which gives $w'' = 0$ (since $u' = \alpha u$). Integrating twice (we don't really need the $z = w'$ substitution for this trivial case) gives $w = C + Dx$. Hence, the 2nd solution is of the form $y(x) = (C + Dx)e^{\alpha x}$, and the general solution is as specified.

Example 3.5. *One solution of* $x^2 y'' - (x^2 + 2x)y' + (x + 2)y = 0$ *is* $y(x) = x$*. Use the method of reduction of order to find a 2nd solution and hence write down the general solution.*

Solution: Letting $y = xw$, we get $y' = w + xw'$, $y'' = (2w' + xw'')$. Substituting into the equation gives $w'' - w' = 0$, which we can solve by letting $z = w'$, such that $z' - z = 0$. The solution of this equation is $z = Ae^x$. Integrating again gives $w = Ae^x + B$. Thus a 2nd solution is $y = x(Ae^x + B)$. The general solution is therefore

$$y(x) = Cx + Dxe^x.$$

3.3 Euler equations (L. Euler, 1707-1783)

These are equations of the form

$$x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = 0 \tag{3.16}$$

where a_0 and a_1 are constants. Remarkably, this equation can be transformed into a linear ODE with constant coefficients by writing it in terms of the variable $u = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du} \tag{3.17}$$

and

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right) = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d}{dx}\frac{dy}{du} \tag{3.18}$$

$$= -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{du}{dx}\frac{d^2 y}{du^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2 y}{du^2} \tag{3.19}$$

Hence, we have

$$x\frac{dy}{dx} = \frac{dy}{du}, \quad x^2\frac{dy}{dx} = \frac{d^2 y}{du^2} - \frac{dy}{du}.$$

The equation (3.16) becomes

$$\frac{d^2 y}{du^2} + (a_1 - 1)\frac{dy}{du} + a_0 y = 0, \tag{3.20}$$

which has constant coefficients, as promised.

Example 3.6. *Find a fundamental set of solutions of*

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0. \tag{3.21}$$

Solution: In terms of the variable $u = \ln x$ the equation becomes

$$\frac{d^2y}{du^2} + \frac{dy}{du} + y = 0, \tag{3.22}$$

which we can solve with the techniques of the previous section. The characteristic equation is

$$\lambda^2 + \lambda + 1 = 0 \tag{3.23}$$

with solutions $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Hence a fundamental system of (3.22) is

$$y^{(1)} = e^{-\frac{u}{2}}\cos(\frac{\sqrt{3}}{2}u), \qquad y^{(2)} = e^{-\frac{u}{2}}\sin(\frac{\sqrt{3}}{2}u). \tag{3.24}$$

Transforming back to x we obtain a fundamental system for (3.16):

$$y^{(1)}(x) = x^{-\frac{1}{2}}\cos(\frac{\sqrt{3}}{2}\ln x), \qquad y^{(2)}(x) = x^{-\frac{1}{2}}\sin(\frac{\sqrt{3}}{2}\ln x). \tag{3.25}$$

3.4 Method of undetermined coefficients

The method of variation of parameters works in principle for any linear 2nd order ODE of the form

$$\frac{d^2x}{dt^2} + a_2(t)\frac{dx}{dt} + \ldots + a_1(t)x = b(t) \tag{3.26}$$

as well as for higher order ODEs and general linear inhomogeneous systems.

When a_1 and a_2 are constants, and $b(t)$ is a polynomial, exp, sin or cos, there is a simpler method to obtain a particular solution that you have met in previous courses, and which we here call the **method of undetermined coefficients**. The idea is to choose a particular solution that has a ‘similar form’ to $b(t)$.. The following table gives recipes involving unknown coefficients which one can determine by substituting into the equation. There is no deep reason for these recipes other than that they work. In the table I have abbreviated homogeneous equation by HE.

$b(t)$	particular solution
$b_0 + b_1 t + ...b_n t^n$	$c_0 + c_1 t + ...c_n t^n$
$e^{\lambda t}$	$e^{\lambda t}$ is not a solution of HE $\Rightarrow$ try $ce^{\lambda t}$ $e^{\lambda t}$ is a solution of HE $\Rightarrow$ try $ct\exp(\lambda t)$ $e^{\lambda t}$ and $te^{\lambda t}$ solutions of HE $\Rightarrow$ try $ct^2\exp(\lambda t)$
$\sin(\omega t)$ or $\cos(\omega t)$	$\sin(\omega t)$, $\cos(\omega t)$ are not solutions of the HE $\Rightarrow$ try $c_1\sin(\omega t) + c_2\cos(\omega t)$ $\sin(\omega t)$, $\cos(\omega t)$ are solutions of the HE $\Rightarrow$ try $c_1 t\sin(\omega t) + c_2 t\cos(\omega t)$

Table 2

To illustrate the recipes given in the table, we consider some examples, beginning with the **polynomial case**.

$$\ddot{y} + y = t^2. \tag{3.27}$$

We try

$$y_p(t) = c_0 + c_1 t + c_2 t^2, \tag{3.28}$$

and find by inserting into (3.27)

$$(2c_2 + c_0) + c_1 t + c_2 t^2 = t^2 \tag{3.29}$$

Comparing coefficients yields

$$c_2 = 1, \quad c_1 = 0, \quad c_0 = -2 \tag{3.30}$$

so that

$$y_p(t) = t^2 - 2. \tag{3.31}$$

Continuing with the **exponential case**, consider

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 2e^{-t} \tag{3.32}$$

We have already found a particular solution of this equation using the method of variation of parameters. Let us repeat the exercise using the current method of undetermined coefficients. The roots of the characteristic polynomial are $\lambda = -1$ and $\lambda = 2$. Thus the table tells us to try

$$x_p(t) = cte^{-t}.$$

Substituting into the equation, we find

$$c = -\frac{2}{3}.$$

Thus, a particular solution is $x_p(t) = -\frac{2}{3}te^{-t}$. This is the same result as before, but you can see that the method of undetermined coefficients is somewhat simpler in this case.

Finally we turn to the recipe for the **oscillatory case** $f(t) = \cos\omega t$ given in the last row of table 2. Suppose we want to solve

$$\ddot{y} + a_1\dot{y} + a_0 y = \cos(\omega t). \tag{3.33}$$

This equation is that of the damped, driven, simple harmonic oscillator (if you didn't take the 2nd year Oscillations and Waves course don't worry, it doesn't matter). We could just use the recipe from the table, but it perhaps interesting to consider how it arises from the exponential case. Since $\cos\omega t$ is the real part of $\exp(i\omega t)$ we could try to solve

$$\ddot{y} + a_1\dot{y} + a_0 y = e^{i\omega t}. \tag{3.34}$$

first and then take the real part of the solution we obtain. This turns out to be an efficient method. Suppose that $i\omega$ is not a solution of the characteristic equation. Then try $y(t) = C\exp(i\omega t)$. Inserting into (3.34) yields

$$Ce^{i\omega t}(-\omega^2 + ia_1\omega + a_0) = e^{i\omega t}. \tag{3.35}$$

Dividing by $\exp(i\omega t)$ and solving for C we find

$$C = \frac{1}{-\omega^2 + ia_1\omega + a_0} = \frac{e^{i\phi}}{\sqrt{(a_0 - \omega^2)^2 + a_1^2\omega^2}} \tag{3.36}$$

where

$$\tan\phi = \frac{-a_1\omega}{a_0 - \omega^2} \tag{3.37}$$

Taking the real part of $y(t) = C\exp(i\omega t)$ we therfore find the solution

$$y_p(t) = \frac{1}{\sqrt{(a_0 - \omega^2)^2 + a_1^2\omega^2}}\cos(\omega t + \phi). \tag{3.38}$$

Since $\cos(\omega t + \phi) = \cos\phi\cos(\omega t) - \sin\phi\sin(\omega t)$ this solution can also be written as

$$y_p(t) = \frac{\cos\phi}{\sqrt{(a_0 - \omega^2)^2 + a_1^2\omega^2}}\cos(\omega t) - \frac{\sin\phi}{\sqrt{(a_0 - \omega^2)^2 + a_1^2\omega^2}}\sin(\omega t) \qquad (3.39)$$

which is of the form given in the last row of table 2. You can read of the coefficients c_1 and c_2 - note that they depend on the frequency ω! One can repeat this exercise in the case that $i\omega$ is a solution of the characteristic equation (try it!).

In summary, in the cases when the coefficients $a_2(t), a_1(t)$ in (3.32) are independent of t, and $b(t)$ is one of the forms given in Table 2, it usually easier to use the method of undetermined coefficients. Otherwise we must resort to the more general method of variation of parameters.