4 Laplace transforms

4.1 Definition and basic properties

The Laplace transform is a useful tool for solving differential equations, in particular initial value problems. It also provides an example of integral transforms, which play an important role in various branches of mathematics. It is named after Pierre-Simon Laplace (1749-1827).

Definition 4.1. *Let $f : [0,\infty) \to \mathbb{R}$. We define the Laplace transform of f as*

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{4.1}$$

Sometimes we write $\mathcal{L}[f(t)]$ for $\bar{f}$.

Note that the integral defining the Laplace transform converges for $s > s_0$ provided $|f(t)| \leq Ke^{s_0 t}$ for some constant K.

4.1.1 Examples

(i) Let $f(t) \equiv 1$. Then

$$\bar{f}(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}. \tag{4.2}$$

(ii) Let $f(t) = e^{\alpha t}$. Then

$$\bar{f}(s) = \int_0^\infty e^{-(s-\alpha)t}dt = \frac{1}{s-\alpha} \tag{4.3}$$

for $s > \alpha$.

(iii) Let $f(t) = \sin(\alpha t)$. Then

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \sin(\alpha t)e^{-st}dt = \mathrm{Im}\int_0^\infty e^{i\alpha t}e^{-st}dt \\ &= \mathrm{Im}\int_0^\infty e^{-(s-i\alpha)t}dt = \mathrm{Im}\frac{1}{s-i\alpha} = \frac{\alpha}{s^2+\alpha^2}. \end{aligned} \tag{4.4}$$

Similarly one shows that if $f(t) = \cos(\alpha t)$ then $\bar{f}(s) = \dfrac{s}{s^2+\alpha^2}$.

(iv) If $f(t) = t$ then $\bar{f}(s) = \dfrac{1}{s^2}$. You should be able to prove (by induction) that if $f(t) = t^n$ then $\bar{f}(s) = \dfrac{n!}{s^{n+1}}$

4.1.2 Simple properties of Laplace transforms

It is easy to check that the Laplace transform is linear in the sense that

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] \quad \text{and} \quad \mathcal{L}[\alpha f(t)] = \alpha\mathcal{L}[f(t)]. \tag{4.5}$$

For us it is particularly important to work out the Laplace transform of the derivative of a function:

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}|_{t=0}^{t=\infty} + s\int_0^\infty f(t)e^{-st}dt$$

and hence

$$\mathcal{L}[f'(t)] = s\bar{f}(s) - f(0). \tag{4.6}$$

It follows that

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'](s) - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0), \tag{4.7}$$

and, by induction,

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0)... - f^{(n-1)}(0). \tag{4.8}$$

4.2 Solution of initial value problems

Consider the initial value problem

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1, y'(0) = 0. \tag{4.9}$$

Taking Laplace transforms we obtain

$$s^2\bar{y}(s) - sy(0) - y'(0) - 3(s\bar{y}(s) - y(0)) + 2\bar{y}(s) = \frac{1}{s-3}. \tag{4.10}$$

Inserting the initial values and re-arranging terms yields

$$(s^2 - 3s + 2)\bar{y}(s) = \frac{1}{s-3} + s - 3 \tag{4.11}$$

and hence

$$\bar{y}(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}. \tag{4.12}$$

Our next goal is to invert the Laplace transform, i.e. to find the function $y(t)$ whose Laplace transform is (4.12). To achieve this we write the right hand side of (4.12) in terms of partial fractions

$$\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}. \tag{4.13}$$

Then

$$s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2). \tag{4.14}$$

Setting in turn $s = 1$, $s = 2$ and $s = 3$ we find

$$A = \frac{5}{2}, \quad B = -2, \quad C = \frac{1}{2}. \tag{4.15}$$

Thus

$$\bar{y}(s) = \frac{5}{2}\frac{1}{(s-1)} - 2\frac{1}{(s-2)} + \frac{1}{2}\frac{1}{(s-3)}. \tag{4.16}$$

Now we find $y(t)$ by comparing (4.16) with (4.3). Using the linearity of the Laplace transform you can check that

$$y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t} \tag{4.17}$$

has the Laplace transform (4.16). One can show that that the Laplace transform of a function is (essentially) unique. It is thus possible to invert Laplace transforms by "guessing" or "inspection". There is a more systematic method based on contour integrals, but in many applications the method of inspection is the quickest. It is the one we will use in this course.

To sum up, the **three steps** in solving a differential equation using the method of Laplace transfroms are:

1. Apply the Laplace transform to the differential equation for $y(t)$.
2. Solve the resulting algebraic equation for $\bar{y}(s)$.
3. Invert the Laplace transform to find $y(t)$.

Why do this when we already have two ways of computing a particular solution to ODEs of the type considered above? Firstly, $y(s)$ is given by the solution to a simple algebraic problem rather a diffferential equation. Secondly, this technique gives us the solution to the initial value problem directly; we don't need to find a general solution in the fom $y(x) = y_p(x) + c_1 y^{(1)}(x) + c_2 y^{(2)}(x)$ and then find c_1 and c_2 that give $y(0) = \alpha$, $y'(0) = \beta$.

Of the three steps mentioned, the last step is often the trickiest. We therefore devote a special subsection to it.

4.3 Inverting Laplace transforms

Finding the inverse Laplace transform of $\bar{y}(s)$ by inspection is sometimes facilitated by expanding $\bar{y}(s)$ in **partial fractions**. The example in the previous section contained the simplest sort of partial fraction. The following example is designed to remind you of some possible additional complications, such as repeated factors.

Example 4.2. *Find $y(t)$ when* $\bar{y}(s) = \dfrac{2s^2+2s+1}{s^2(s^2+1)}$.

Solution: Remember that the theory of partial fractions allows us to rewrite proper rational functions (functions that are the ratio of two polynomials in which the degree of the denominator is strictly less than the degree of the numerator). Such a function can be rewritten as the the sum of proper rational functions; the denominators of the partial fractions are the irreducible factors of the denominator of the original function.

In the present case, we let

$$\frac{2s^2+2s+1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

and multiply by $s^2(s^2+1)$ to obtain

$$\begin{aligned} 2s^2+2s+1 &= As(s^2+1) + B(s^2+1) + (Cs+D)s^2 \\ &= B + As + (B+D)s^2 + (A+C)s^3. \end{aligned} \tag{4.18}$$

Now compare coefficients of powers of s:

$$\begin{aligned} s^0 &: \quad B = 1 \\ s^1 &: \quad A = 2 \\ s^2 &: \quad B + D = 2 \Rightarrow D = 1 \\ s^3 &: \quad A + C = 0 \Rightarrow C = -2 \end{aligned} \tag{4.19}$$

Thus

$$\bar{y}(s) = \frac{2}{s} + \frac{1}{s^2} - 2\frac{s}{s^2+1} + \frac{1}{s^2+1} \tag{4.20}$$

and hence

$$y(t) = 2 + t - 2\cos t + \sin t \tag{4.21}$$

A second important tool is provided by the following

Lemma 4.3. *Let $g(t) = e^{ct}f(t)$. Then $\bar{g}(s) = \bar{f}(s-c)$.*

Proof: This is a simple computation:

$$\bar{g}(s) = \int_0^\infty e^{-(s-c)t} f(t)dt = \bar{f}(s-c). \qquad \square \tag{4.22}$$

The following example illustrates this result:

$$\begin{aligned} \mathcal{L}[t] = \frac{1}{s^2} \quad &\Rightarrow \quad \mathcal{L}[te^{2t}] = \frac{1}{(s-2)^2} \\ \mathcal{L}[\cos(2t)] = \frac{s}{s^2+4} \quad &\Rightarrow \quad \mathcal{L}[e^t\cos(2t)] = \frac{(s-1)}{(s-1)^2+4} \end{aligned} \tag{4.23}$$

More importantly we can use the lemma (4.3) to invert Laplace transforms:

Example: Find $y(t)$ when $(i)\quad \bar{y}(s) = \dfrac{1}{s^2+2s+3}\quad (ii)\quad \bar{y}(s) = \dfrac{1}{s(s^2+2s+3)}$.

(i) Since $\bar{y}(s) = \dfrac{1}{(s+1)^2+2} = \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{2}}{(s+1)^2+2}$ we deduce $y(t) = \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$

(ii) Now use

$$\begin{aligned}\frac{1}{s(s^2+2s+3)} &= \frac{1}{3s} - \frac{1}{3}\frac{s+2}{(s^2+2s+3)} \quad \text{(by partial fractions)}\\ &= \frac{1}{3s} - \frac{1}{3}\frac{s+1}{(s+1)^2+2} - \frac{1}{3\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}.\end{aligned}$$

Thus we deduce

$$y(t) = \frac{1}{3} - \frac{1}{3}e^{-t}\cos(\sqrt{2}t) - \frac{1}{3\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

4.4 ODE's involving discontinuous functions

Let me start by presenting you with two 2nd order ODEs with direct physical meaning:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F(t).$$

This is the equation of a vibrating spring system, where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied force.

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}.$$

This is the equation for the current I in an electric circuit with inductance L, resistance R and capicitance C. V is the applied voltage.

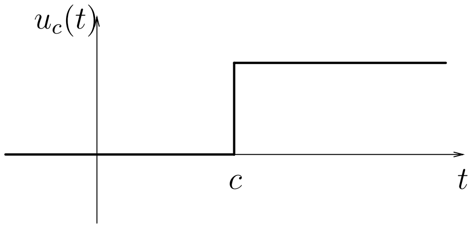
In both of these cases the inhomogeneous term on the right-hand-side represents an external applied force. In engineering, physics, and electronics one is unfortunately often interested in discontinous forces, that is, ones that you turn on by flicking a switch, or ones that you apply instantaneously by hitting your system with a hammer. In this section we shall see how to use Laplace transforms to tackle ODEs that involve such discontinuous functions.

4.4.1 The Step Function

The function defined by

$$u_c(t) = \begin{cases} 0 & \text{if} \quad t < c \\ 1 & \text{if} \quad t \geq c \end{cases} \tag{4.24}$$

is called the unit step function or Heaviside function (named after Oliver Heaviside, 1859-1925).



The step function is useful in describing situations where an external effect (a force, a voltage) is suddenly switched on. Suppose for example that an external force $f(t)$ acting on a spring is switched on at $t = c$. The force experienced by the spring is

$$F(t) = \begin{cases} 0 & \text{if} \quad t < c \\ f(t) & \text{if} \quad t \geq c \end{cases} \tag{4.25}$$

In terms of the step function we can simply write

$$F(t) = u_c(t) f(t). \tag{4.26}$$

Similarly, if an external force is switched off at $t = c$, i.e.

$$F(t) = \begin{cases} f(t) & \text{if} \quad t < c \\ 0 & \text{if} \quad t \geq c \end{cases} \tag{4.27}$$

we can express the resulting function in terms of the step function

$$F(t) = (1 - u_c(t)) f(t). \tag{4.28}$$

Since step functions arise naturally in a number of applications, we would like to be able to compute Laplace transforms of functions like (4.26) and (4.28). The following lemma is useful for that purpose:

Lemma 4.4. $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} \bar{f}(s)$.

Proof: This follows again by direct computation.

$$\mathcal{L}[u_c(t) f(t-c)] = \int_0^\infty u_c(t) f(t-c) e^{-st} dt = \int_c^\infty f(t-c) e^{-st} dt \tag{4.29}$$

Now change variables to $\tau = t - c$. Then

$$\mathcal{L}[u_c(t) f(t-c)] = \int_0^\infty f(\tau) e^{-s(\tau+c)} d\tau = e^{-cs} \int_0^\infty f(\tau) e^{-s\tau} d\tau = e^{-cs} \bar{f}(s). \qquad \square \tag{4.30}$$

Corollary 4.5. $\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}$.

In order to illustrate the application of the lemma to situations like (4.26) and (4.28) consider the following **examples**:

(i) Find the Laplace transform of

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 1 \\ t & \text{if} \quad t \geq 1 \end{cases} \tag{4.31}$$

We rewrite $y(t)$ as

$$y(t) = u_1(t)\,t = u_1(t)(t-1) + u_1(t). \tag{4.32}$$

Now we can apply Lemma (4.4) to the first term and its corollary to the second to find that

$$\bar{y}(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right). \tag{4.33}$$

(ii) If $\bar{y}(s) = \dfrac{e^{-2s}}{s^3}$, find $y(t)$.

Recall $\mathcal{L}[t^2] = \dfrac{2}{s^3}$ and hence write $\bar{y}(s)$ as $e^{-2s}\mathcal{L}[\frac{t^2}{2}]$. Then deduce from Lemma (4.4) that

$$y(t) = u_2(t)\frac{1}{2}(t-2)^2$$

.

(iii) Solve $y'' + 2y' + y = f(t), \quad y(0) = 1,\ y'(0) = 0$, where

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 3 \\ t-3 & \text{if} \quad t \ge 3 \end{cases}$$

We note that $f(t) = u_3(t)(t-3)$ and take the Laplace transform of the equation. Using the given initial data we find

$$\begin{aligned} (s^2+2s+1)\bar{y}(s) - s - 2 &= \frac{e^{-3s}}{s^2} \\ \Leftrightarrow \quad \bar{y}(s) &= \frac{s+2}{(s+1)^2} + \frac{e^{-3s}}{s^2(s+1)^2} \end{aligned} \tag{4.34}$$

Using

$$\frac{s+2}{(s+1)^2} = \frac{(s+1)+1}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \tag{4.35}$$

and the partial fraction expansion

$$\frac{1}{s^2(s+1)^2} = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2} \tag{4.36}$$

we have

$$\begin{aligned} \bar{y}(s) &= \frac{1}{s+1} + \frac{1}{(s+1)^2} + e^{-3s}\left(-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right) \\ &= \frac{1}{s+1} + \frac{1}{(s+1)^2} + e^{-3s}\mathcal{L}[-2 + t + 2e^{-t} + te^{-t}] \end{aligned} \tag{4.37}$$

and deduce from Lemma (4.4)

$$\begin{aligned} y(t) &= e^{-t} + te^{-t} + u_3(t)(-2 + (t-3) + 2e^{-(t-3)} + (t-3)e^{-(t-3)}) \\ &= e^{-t}(1+t) + u_3(t)(-5 + t + (t-1)e^{-(t-3)}). \end{aligned} \tag{4.38}$$

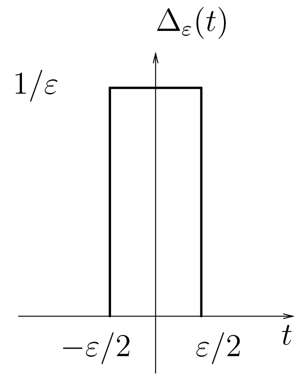
4.4.2 The Dirac delta function

Consider the situation where a mass is subjected to force $f(t)$. The **impulse** given to the mass is by definition

$$I(t) = \int_{-\infty}^{\infty} f(t)dt.$$

Often, we are interested in physical situations in which a large force is applied for a short time, for example when we hit a nail with a hammer. Let us suppose that a force is given by the function

$$\Delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } \ -\frac{\varepsilon}{2} \leq t \leq \frac{\varepsilon}{2} \\ 0 & \text{for } \ |t| > \frac{\varepsilon}{2} \end{cases}$$



Clearly, the impulse $I(t) = \int_{-\infty}^{\infty} \Delta_\varepsilon(t)\, dt$ due to this function is always equal to 1, independent of ε. The limit of $\varepsilon \to 0$ of this function has a special name:

Definition 4.6. *We define the Dirac delta function as:*

$$\delta(t) = \lim_{\varepsilon \to 0} \Delta_\varepsilon(t).$$

The function $\delta(t)$ is named after the English physicist Paul Maurice Adrien Dirac who lived 1902-1984. (This is the first time that this course enters the twentieth century!) The key properties of the Dirac delta function $\delta(t - t_0)$ are that it is zero everywhere except at t_0 where it is "infinite", and that

$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1.$$

It's clearly not your average function - in fact it is not a function at all in the normal sense, but an example of a generalised function.

In terms of the delta function, the differential equation for a particle of mass m on a spring with spring constant k immersed in a fluid with damping constant r which is subject to an impulse P at time $t = t_0$ is

$$m\frac{dy^2}{dt^2} + r\frac{dy}{dt} + ky = P\delta(t - t_0). \tag{4.39}$$

In order to solve equations of this form using Laplace transforms, we need to know the Laplace transform of the delta function. This will follow from the following important theorem.

Theorem 4.7. *For any continuous function* $f : \mathbb{R} \to \mathbb{R}$*,*

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0).$$

Proof: Expressing the delta function in terms of the limit (4.6) we have

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \Delta_\varepsilon(t-t_0)\, f(t)\, dt \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{t_0-\frac{\varepsilon}{2}}^{t_0+\frac{\varepsilon}{2}} f(t)\, dt \\ &= f(t_0), \end{aligned} \tag{4.40}$$

where we used the continuity of f in the last line. □

As an immediate consequence we have

Corollary 4.8. $\mathcal{L}[\delta(t-t_0)] = e^{-st_0}$ *provided* $t_0 \geq 0$*.*

Proof:

$$\mathcal{L}[\delta(t-t_0)] = \int_0^{\infty} e^{-st} \delta(t-t_0)\, dt = e^{-st_0}$$

if $t_0 \geq 0$. □

Note in particular that $\mathcal{L}[\delta(t)] = 1$. Armed with this corollary we can tackle the following example.

Example 4.9. *A mass of* 1 *kg oscillates on a spring with spring constant* 1 *N/m. Air resistance may be ignored. The mass is initially at rest in equilibrium and impulses of* $2N$ *sec and* $3N$ *sec are applied instantaneously at times* $t = 0$ *and* $t = 1$*. Describe the motion of the mass.*

Solution: The equation of motion is

$$y'' + y = 2\delta(t) + 3\delta(t-1), \quad y(0) = y'(0) = 0. \tag{4.41}$$

Taking Laplace transforms

$$(s^2+1)\bar{y}(s) = 2 + 3e^{-s} \tag{4.42}$$

so that

$$\bar{y}(s) = 2\frac{1}{s^2+1} + 3e^{-s}\frac{1}{s^2+1} = 2\mathcal{L}[\sin t] + 3e^{-s}\mathcal{L}[\sin t]. \tag{4.43}$$

Thus

$$y(t) = 2\sin t + 3u_1(t)\sin(t-1) = \begin{cases} 2\sin t & \text{if} \quad 0 \leq t < 1 \\ 2\sin t + 3\sin(t-1) & \text{if} \quad t \geq 1 \end{cases} \tag{4.44}$$

One final useful property of the Dirac delta function comes from the observation that we can write the $\Delta_\varepsilon(t)$ in terms of the step function as

$$\Delta_\varepsilon(t-c) = \frac{1}{\varepsilon}(u_{c-\varepsilon/2}(t) - u_{c+\varepsilon/2}(t)).$$

Using the property $u_c(t-\alpha) = u_{c+\alpha}(t)$ we may rewrite this as

$$\Delta_\varepsilon(t-c) = \frac{1}{\varepsilon}(u_c(t+\varepsilon/2) - u_c(t-\varepsilon/2)).$$

In the limit $\varepsilon \to 0$ the right-hand-side is the derivative of the step function. Thus we have

$$\delta(t-c) = u_c'(t). \tag{4.45}$$

In fact this expression (4.45) is sometimes taken as a definition of the Dirac delta function.

4.5 The convolution integral

Definition 4.10. *Let $f, g : [0,\infty) \to \mathbb{R}$. We define the convolution $f*g$ of the functions f and g as the function*

$$f * g\ (t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{4.46}$$

As an elementary example, consider $f(t) = \sin t$ and $g(t) \equiv 1$. Then

$$f * g\ (t) = \int_0^t \sin(t-\tau)d\tau = \int_0^t \sin v\, dv = 1 - \cos t, \tag{4.47}$$

where we have changed variables to $v = t - \tau$. A basic property of the convolution is that it is commutative:

Lemma 4.11. *Let $f, g : [0,\infty) \to \mathbb{R}$. Then*

$$f * g\ (t) = g * f\ (t). \tag{4.48}$$

Proof: Changing integration variables from τ to $v = t - \tau$ in the definition (4.46) we find

$$f * g\ (t) = -\int_t^0 f(v)g(t-v)dv = \int_0^t g(t-v)f(v)dv = g * f\ (t) \qquad \square. \tag{4.49}$$

A further remarkable property is the behaviour of the convolution under Laplace transforms. It is summed up in the **convolution theorem**:

Theorem 4.12. *(**The Convolution Theorem**) For two functions f and g*

$$\overline{f * g}\,(s) = \bar{f}(s)\bar{g}(s) \tag{4.50}$$

Thus the Laplace transform of a convolution of two function is the ordinary product of the Laplace transforms!
Proof:

$$\begin{aligned}\overline{f*g}\,(s) &= \int_0^\infty (f*g)(t)e^{-st}dt \\ &= \int_0^\infty \left(\int_0^t f(t-\tau)g(\tau)d\tau\right)e^{-st}dt \\ &= \int_0^\infty g(\tau)\left(\int_\tau^\infty f(t-\tau)e^{-st}dt\right)d\tau \qquad (4.51)\end{aligned}$$

where we have changed the order of integration in the last step. Note that the limits are such that we integrate over the region $0 \le \tau \le t < \infty$.

Next we change variables in the inner integral from t to $v = t - \tau$. Then

$$\begin{aligned}\overline{f*g}\,(s) &= \int_0^\infty g(\tau)\left(\int_0^\infty f(v)e^{-s(v+\tau)}dv\right)d\tau \\ &= \int_0^\infty f(v)e^{-sv}dv\int_0^\infty g(\tau)e^{-s\tau}d\tau \\ &= \bar{f}(s)\bar{g}(s) \qquad \square \qquad (4.52)\end{aligned}$$

This theorem is useful in a number of ways. For us, its prime use is in inverting Laplace transforms.

Example 4.13. *Find $y(t)$ when $\bar{y}(s) = \dfrac{6}{s^4(s^2+1)}$.*

Solution: It is useful to think of $\bar{y}(s)$ as the product of $\bar{f}(s) = 6/s^4$ and $\bar{g}(s) = 1/(s^2+1)$, for which we have $f(t) = t^3$ and $g(t) = \sin(t)$. Thus, by the convolution theorem we have

$$y(t) = \int_0^t \tau^3 \sin(t-\tau)d\tau = t^3 + 6\sin(t) - 6t.$$

You can check this by using partial fractions

$$\frac{6}{s^4(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E+Fs}{s^2+1} = \cdots = \frac{6}{s^4} + \frac{6}{s^2+1} - \frac{6}{s^2}.$$

However, you can see that the convolution theorem is somewhat easier in this case.

The convolution theorem is also useful for obtaining integral expressions for the solution of initial value problems, such as

$$y''(t) + y(t) = f(t), \quad y(0) = y'(0) = 0. \qquad (4.53)$$

Taking Laplace transforms

$$s^2\bar{y}(s) + \bar{y}(s) = \bar{f}(s) \qquad (4.54)$$

we deduce

$$\bar{y}(s) = \bar{f}(s)\frac{1}{s^2+1} = \mathcal{L}[f(t)]\mathcal{L}[\sin t]. \tag{4.55}$$

From the convolution theorem we deduce

$$y(t) = (f * \sin)(t) = \int_0^t f(t-\tau)\sin\tau d\tau. \tag{4.56}$$

This a convenient form of writing down a particular solution of the equation (4.53) for general $f(t)$.

4.6 Linear Systems of ODEs

Taking the Laplace transform changes a linear system of ODEs into a set of simultaneous equations.

Example 4.14. *Solve the initial value problem*

$$\dot{\boldsymbol{x}}(t) = A\,\boldsymbol{x}(t), \quad A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$$

Solution: Writing the equations

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) - 3x_2(t) \\ \dot{x}_2(t) &= -2x_1(t) + 2x_2(t) \end{aligned}$$

we may take the Laplace transform to give

$$\begin{aligned} s\bar{x}_1(s) - x_1(0) &= \bar{x}_1(s) - 3\bar{x}_2(s) \\ s\bar{x}_2(s) - x_2(0) &= -2\bar{x}_1(s) + 2\bar{x}_2(s) \end{aligned}$$

i.e.,

$$\begin{aligned} (s-1)\bar{x}_1(s) + 3\bar{x}_2(s) &= 0 \qquad (1) \\ 2\bar{x}_1(s) + (s-2)\bar{x}_2(s) &= 5 \qquad (2). \end{aligned}$$

(2) $\times (s-1) - 2 \times$ (1) gives $[(s-1)(s-2)-6]\bar{x}_2(s) = 5(s-1)$, i.e.,

$$\bar{x}_2(s) = \frac{5(s-1)}{s^2-3s-4} = \frac{5(s-1)}{(s-4)(s+1)} = \frac{3}{s-4} + \frac{2}{s+1}.$$

Hence $x_2(t) = 3e^{4t} + 2e^{-t}$.
(1) $\times (s-2) -$ (2) $\times 3$ gives $[(s-1)(s-2)-6]\bar{x}_1(s) = -15$. i.e.

$$\bar{x}_1(s) = \frac{-15}{(s-4)(s+1)} = \frac{-3}{s-4} + \frac{3}{s+1}.$$

Hence, $x_1(t) = -3e^{4t} + 3e^{-t}$.

Example 4.15. *Solve the system*

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + 5x_2(t), \quad x_1(0) = 2 \\ \dot{x}_2(t) &= -x_1(t) + 3x_2(t), \quad x_2(0) = 1\end{aligned}$$

Solution: Taking Laplace transforms gives

$$\begin{aligned}s\bar{x}_1(s) - x_1(0) &= -\bar{x}_1(s) + 5\bar{x}_2(s) \\ s\bar{x}_2(s) - x_2(0) &= -\bar{x}_1(s) + 3\bar{x}_2(s)\end{aligned}$$

i.e.,

$$\begin{aligned}(s+1)\bar{x}_1(s) - 5\bar{x}_2(s) &= 2 \qquad (1) \\ \bar{x}_1(s) + (s-3)\bar{x}_2(s) &= 1 \qquad (2).\end{aligned}$$

$(2) \times (s+1) - (1)$ gives $[(s+1)(s-3)+5]\bar{x}_2(s) = s+1-2$, i.e.,

$$\bar{x}_2(s) = \frac{(s-1)}{(s-1)^2+1}.$$

Hence $x_2(t) = e^t \cos(t)$.
$(1) \times (s-3) + (2) \times 5$ gives $[(s+1)(s-3)+5]\bar{x}_1(s) = 2(s-3)+5$. i.e.

$$\bar{x}_1(s) = \frac{2s-1}{(s-1)^2+1} = 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}.$$

Hence, $x_1(t) = 2e^t \cos(t) + e^t \sin(t)$.