5 Boundary Value Problems

5.1 General Comments

Many of the lectures so far have been concerned with the initial value problem

$$L[y] = f(x), \quad y(x_0) = \alpha,\ y'(x_0) = \beta, \tag{5.1}$$

where L is the differential operator

$$L[y] = \frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y. \tag{5.2}$$

From Picard's' theorem we know that, if a_1 and a_0 are smooth everywhere, then a unique solution of (5.1) exists everywhere. We have also developed an arsenal of methods for finding that solution.

In this section of the course we look at **boundary value problems**, where we solve a differential equation subject to conditions imposed at two different points $x = a$ and $x = b$. The most general boundary value problem we will consider is

$$L[y] = f(x), \quad B_a[y] = 0, \quad B_b[y] = 0, \tag{5.3}$$

where we have used the abbreviation

$$B_a[y] = \alpha_1 y(a) + \beta_1 y'(a) \quad \text{and} \quad B_b[y] = \alpha_2 y(b) + \beta_2 y'(b). \tag{5.4}$$

Choosing, for example, $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 1$ we obtain the condition that y vanishes at a and b. This boundary condition arises physically for example if we study the shape of a rope which is fixed at two points a and b. Choosing $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$ we obtain $y'(a) = y'(b) = 0$. The general conditions we impose at a and b involve both y and y'.

Unlike initial value problems, **boundary value problems do not always have solutions**, as the following example illustrates. Suppose we try to solve

$$y'' + y = f(x), \quad y(0) = y(\pi) = 0. \tag{5.5}$$

Multiplying the equation by $\sin x$ and integrating yields

$$\begin{aligned}
\int_0^\pi f(x)\sin x\,dx &= \int_0^\pi y''(x)\sin x\,dx + \int_0^\pi y(x)\sin x\,dx \\
&= y'(x)\sin x|_0^\pi - \int_0^\pi y'(x)\cos x\,dx + \int_0^\pi y(x)\sin x\,dx \\
&= -y(x)\cos x|_0^\pi - \int_0^\pi y(x)\sin x\,dx + \int_0^\pi y(x)\sin x\,dx \qquad (5.6)\\
&= 0. \qquad (5.7)
\end{aligned}$$

Thus a necessary condition for (5.5) to have a solution is

$$\int_0^\pi f(x)\sin x\,dx = 0 \tag{5.8}$$

This is not satisfied, for example, if $f(x) = x$.

Before, we present a general method for boundary value problems, let us first recall some facts about linearly independent functions and Wronskians. Two functions f and t are said to be **linearly dependent** on an interval I if there exist 2 constants k_1 and k_2, both non-zero, such that

$$k_1 f(t) + k_2 g(t) = 0 \tag{5.9}$$

for all $t \in I$. The functions are **linearly independent** if they are not linearly dependent - equivalently (5.9) holds for all $t \in I$ only if $k_1 = k_2 = 0$. Note that linear dependence implies that the two functions are proportional, i.e., we can write $f(t) = \lambda g(t)$ for a constant λ. Now consider the 2nd order linear homogeneous ODE $L[y] = 0$. It follows from Lemma 2.18, that two solutions $y^{(1)}$ and $y^{(2)}$ are linearly independent and hence form a fundamental solution set if and only if the Wronskian $W(y^{(1)}, y^{(2)}) \neq 0$ at one value of t, and hence all values of t.

5.2 Green's functions

After this revision of some of the necessary tools, we shall now explain how to find solutions to boundary value problems in the cases where they exist. Our main tool will be Green's functions, named after the English mathematician George Green (1793-1841). A Green's function is constructed out of two special choices of linearly independent solutions $y^{(1)}$ and $y^{(2)}$ of the homogeneous equation

$$L[y] = 0. \tag{5.10}$$

More precisely, let $y^{(1)}$ be the unique solution of the initial value problem

$$L[y] = 0, \quad y(a) = \beta_1, \quad y'(a) = -\alpha_1 \tag{5.11}$$

and $y^{(2)}$ be the unique solution of

$$L[y] = 0, \quad y(b) = \beta_2, \quad y'(b) = -\alpha_2. \tag{5.12}$$

These solutions thus satisfy

$$B_a[y^{(1)}] = 0 \quad \text{and} \quad B_b[y^{(2)}] = 0, \tag{5.13}$$

where we use the notation (6.47). In fact $y^{(1)}$ and $y^{(2)}$ are essentially the only solutions satisfying the boundary conditions at, respectively, a and b:

Lemma 5.1. *A function u satisfies*

$$L[u] = 0 \quad \textit{and} \quad B_a[u] = 0 \tag{5.14}$$

if and only if $u = \lambda y^{(1)}$ for some real number λ

Proof: Consider

$$B_a[u] = \alpha_1 u(a) + \beta_1 u'(a) = -y_1'(a)u(a) + y^{(1)}(a)u'(a) = W(y^{(1)}, u)(a). \quad (5.15)$$

Hence, $B_a[u] = 0 \Leftrightarrow W(y^{(1)}(a), u(a)) = 0$. $W(y^{(1)}(a), u(a)) = 0$ is in turn $\Leftrightarrow y^{(1)} = \lambda u$. $\square$

Clearly one can similarly prove that any solution u of $L[u] = 0$ and $B_b[u] = 0$ must be a multiple of $y^{(2)}$. It might of course happen that $y^{(1)}$ and $y^{(2)}$ are dependent. The following simple check follows directly from the fact that $B_a[y^{(2)}] = W(y^{(1)}, y^{(2)})$.

Corollary 5.2. *The solutions $y^{(1)}$ and $y^{(2)}$ are linearly independent if and only if $B_a(y^{(2)}) \neq 0$.*

For our construction of the Green's function, the starting point is two solutions $y^{(1)}$ and $y^{(2)}$ of $L[y] = 0$, which obey $B_a[y^{(1)}] = 0 = B_b[y^{(2)}]$ and are linearly independent. The next ingredient we require is a particular solution of the homogeneous equation

$$L[y] = f. \quad (5.16)$$

This is a problem we solved in section 2.5.2 using the method of variation of parameters. The particular solution constructed there is of the form

$$y_p(x) = c_1(x)y^{(1)}(x) + c_2(x)y^{(2)}(x) \quad (5.17)$$

with

$$\begin{aligned} c_1(x) &= -\int_a^x \frac{y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds \\ c_2(x) &= \int_a^x \frac{y^{(1)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds. \end{aligned} \quad (5.18)$$

Note, that we have deliberately chosen the lower limit if the integral to be a. Hence, we have the particular solution

$$y_p(x) = \int_a^x \frac{\left(y^{(1)}(s)y^{(2)}(x) - y^{(1)}(x)y^{(2)}(s)\right) f(s)}{W(y^{(1)}, y^{(2)})(s)} ds \quad (5.19)$$

with the property $y_p(a) = 0$. The next step is to differentiate to obtain

$$\begin{aligned} y_p'(x) &= \frac{y^{(1)}(x)f(x)}{W(y^{(1)}, y^{(2)})(x)}\, y^{(2)}(x) + \int_a^x \frac{y^{(1)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds\; y^{(2)\prime}(x) \\ &\quad - \frac{y^{(2)}(x)f(x)}{W(y^{(1)}, y^{(2)})(x)}\, y^{(1)}(x) - \int_a^x \frac{y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds\; y^{(1)\prime}(x) \\ &= \int_a^x \frac{(y^{(1)}(s)y^{(2)\prime}(x) - y^{(1)\prime}(x)y^{(2)}(s))f(s)}{W(y^{(1)}, y^{(2)})(s)} ds \end{aligned} \quad (5.20)$$

It follows that $y_p(a) = y_p'(a) = 0$ and hence it follows trivially that

$$B_a[y_p] = 0. \quad (5.21)$$

Thus we have a managed to find a particular solution that satisfies one boundary condition. On the other hand

$$\begin{aligned} B_b[y_p] &= \int_a^b \frac{(y^{(1)}(s)B_b[y^{(2)}] - B_b[y^{(1)}]y^{(2)}(s))f(s)}{W(y^{(1)}, y^{(2)})(s)}\, ds \\ &= -B_b[y^{(1)}] \int_a^b \frac{y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds \\ &\neq 0. \end{aligned} \tag{5.22}$$

Thus y_p satisfies the boundary condition at a but not at b. In order to satisfy the boundary condition at b we thus turn to the most general solution of $L[y] = f(x)$. According to the theory of inhomogeneous differential equations this is

$$y(x) = Ay^{(1)}(x) + By^{(2)}(x) + y_p(x). \tag{5.23}$$

It thus remains to determine the constants A and B so that the boundary conditions are satisfied. Since $B_a[y^{(1)}] = B_a[y_p] = 0$ but $B_a[y^{(2)}] \neq 0$ we have

$$B_a[y] = 0 \Rightarrow B = 0. \tag{5.24}$$

Similarly using $B_b[y^{(2)}] = 0$, $B_b[y^{(1)}] \neq 0$ and equation (5.22) we deduce

$$B_b[y] = 0 \Rightarrow A = \int_a^b \frac{y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds. \tag{5.25}$$

Inserting the values for A and B into (5.23) and using the form (5.19) for y_p we obtain the solution

$$\begin{aligned} y(x) &= \int_a^b \frac{y^{(1)}(x)y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds + \int_a^x \frac{(y^{(1)}(s)y^{(2)}(x) - y^{(1)}(x)y^{(2)}(s))f(s)}{W(y^{(1)}, y^{(2)})(s)} ds \\ &= \int_a^x \frac{y^{(1)}(s)y^{(2)}(x)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds + \int_x^b \frac{y^{(1)}(x)y^{(2)}(s)f(s)}{W(y^{(1)}, y^{(2)})(s)} ds. \end{aligned} \tag{5.26}$$

To write this solution in a convenient form, define the Green's function

$$G(x, s) = \begin{cases} \dfrac{y^{(1)}(s)y^{(2)}(x)}{W(y^{(1)}, y^{(2)})(s)} & \text{if} \quad a \leq s \leq x \leq b \\ \dfrac{y^{(1)}(x)y^{(2)}(s)}{W(y^{(1)}, y^{(2)})(s)} & \text{if} \quad a \leq x \leq s \leq b \end{cases} \tag{5.27}$$

so that (5.26) is

$$y(x) = \int_a^b G(x, s)f(s)\, ds. \tag{5.28}$$

In our derivation, the Green's function only appeared as a particularly convenient way of writing a complicated formula. The importance of the Green's function stems from the fact that it is very easy to write down. All we need is fundamental system of the homogeneous equation. Thus the quickest way of solving boundary problems like (5.3)is to proceed in the following **four steps**:

1. Find a fundamental system $\{u_1, u_2\}$ of $L[y] = 0$.

2. By taking suitable linear combinations of u_1 and u_2 find solutions $y^{(1)}$ and $y^{(2)}$ of $L[y] = 0$ satisfying $B_a[y^{(1)}] = 0$ and $B_b[y^{(2)}] = 0$ (often possible by inspection).

3. Define the Green's function G according to (5.27).

4. Compute the solution according to (5.28).

To illustrate the properties and use of the Green's function consider the following **examples**.

Example 5.3. *Find the Green's function for the following boundary value problem*

$$y''(x) = f(x), \quad y(0) = 0, \; y(1) = 0. \tag{5.29}$$

Hence solve $y''(x) = x^2$ *subject to the same boundary conditions.*

Solution: The homogeneous equation $y'' = 0$ has the fundamental solutions $u_1(x) = 1$ and $u_2(x) = x$. Take $y_1(x) = x$ and $y^{(2)}(x) = 1 - x$ to satisfy the boundary conditions $B_0[y] = y(0) = 0$ and $B_1[y] = y(1) = 0$ respectively. Then $W(y^{(1)}, y^{(2)})(x) = -1$ and therefore

$$G(x,s) = \begin{cases} s(x-1) & \text{if} \quad 0 \le s \le x \\ x(s-1) & \text{if} \quad x \le s \le 1 \end{cases} \tag{5.30}$$

Thus solve (5.29) with

$$y(x) = \int_0^x sf(s)\,ds\;(x-1) + \int_x^1 (s-1)f(s)\,ds\;x. \tag{5.31}$$

Inserting $f(s) = s^2$ and carrying out the integration yields

$$y(x) = \frac{1}{12}(x^4 - x). \tag{5.32}$$

Example 5.4. *Find the Green's function for the boundary value problem*

$$y''(x) + y(x) = f(x), \quad y(0) = 0, \; y'(1) = 0. \tag{5.33}$$

Solution: The equation $y'' + y = 0$ has the fundamental system $u_1(x) = \sin x$ and $u_2(x) = \cos x$. To satisfy $B_0[y] = y(0) = 0$ take $y^{(1)}(x) = \sin x$ and to satisfy $B_1[y] = y'(1) = 0$ take $y^{(2)}(x) = \cos(x-1)$. Then check that $W(y^{(1)}, y^{(2)})(x) = -\cos 1$ and find

$$G(x,s) = \begin{cases} -\dfrac{\sin s \cos(x-1)}{\cos 1} & \text{if} \quad 0 \le s \le x \\ -\dfrac{\sin x \cos(s-1)}{\cos 1} & \text{if} \quad x \le s \le 1. \end{cases} \tag{5.34}$$

Example 5.5. *Consider the Green's function found in Example 5.3. Show that*

$$\frac{\partial^2 G}{\partial x^2}(x,s) = \delta(x-s). \tag{5.35}$$

Solution: We have

$$G(x,s) = \begin{cases} s(x-1) & \text{if} \quad 0 \le s \le x \\ x(s-1) & \text{if} \quad x \le s \le 1 \end{cases} \tag{5.36}$$

Differentiating, we obtain

$$\frac{\partial G}{\partial x}(x,s) = \begin{cases} s & \text{if} \quad 0 \le s \le x \\ s-1 & \text{if} \quad x \le s \le 1 \end{cases} \tag{5.37}$$

which we can write in terms of the Heaviside function as

$$\frac{\partial G}{\partial x}(x,s) = s - 1 + u_s(x). \tag{5.38}$$

Then using the definition of the Dirac delta function as the derivative of the Heaviside function we obtain (5.35).

This result suggests a new way of understanding the fundamental formula (5.28). According to that formula, the solution of (5.29) in terms of the Green's function (5.30) is

$$y(x) = \int_0^1 G(x,s) f(s)\, ds \tag{5.39}$$

Differentiating twice with respect to x and using (5.35) we find immediately

$$\frac{d^2 y}{dx^2} = \int_0^1 \frac{\partial^2 G}{\partial x^2}(x,s) f(s) ds = \int_0^1 \delta(s-x) f(s) ds = f(x), \tag{5.40}$$

where we used theorem (4.7) about integrals involving the Dirac delta function.

The equation (5.35) also shows that we can view the Green's function as the response function to an instantaneous unit impulse at $x = s$. As we have seen, it then follows immediately that (5.39) solves the inhomogeneous equation (5.29). This point of view provides useful intuition when dealing with Green's functions and is important in the further development of the theory.

Example 5.6. *Show that the condition*

$$\int_0^\pi f(x) \sin x \, dx = 0 \tag{5.41}$$

is a sufficient condition to ensure that

$$y''(x) + y(x) = f(x), \quad y(0) = 0,\, y(\pi) = 0 \tag{5.42}$$

has a solution.

Solution: We know from our discussion at the beginning of this section that (5.41) is necessary for (5.42) to have a solution. In order to show that this condition is also sufficient we construct an explicit solution of (5.42). Using again the fundamental system $u_1(x) = \sin x$ and $u_2(x) = \cos x$ we find that $\sin x$ satisfies both $B_0[y] = y(0) = 0$ and $B_1[y] = y(\pi) = 0$. Thus $y^{(1)} = y^{(2)} = \sin x$ and we cannot construct a Green's function. In order to write down the solution, we therefore return to first principles. Using the method of variation of the parameters we seek a particular solution of the form

$$y_p(x) = c_1(x)\sin x + c_2(x)\cos x. \tag{5.43}$$

Inserting $y^{(1)}(x) = \sin x$ and $y^{(2)}(x) = \cos x$ into the formulae (5.18) we find (note that the Wronskian is -1 in this case):

$$\begin{aligned} c_1(x) &= \int_0^x f(s)\cos s\, ds \\ c_2(x) &= -\int_0^x f(s)\sin s\, ds. \end{aligned} \tag{5.44}$$

Thus we have the particular solution

$$y_p(x) = \int_0^x f(s)\cos s\, ds \quad \sin x - \int_0^x f(s)\sin s\, ds \quad \cos x \tag{5.45}$$

Clearly $y_p(0) = 0$, and

$$y_p(\pi) = -\cos\pi \int_0^\pi f(s)\sin s\, ds \tag{5.46}$$

which is zero by (5.41). Thus the condition (5.41) is sufficient to ensure that y_p solves the boundary value problem (5.42).